AM205: Assignment 5

Question 1 [8 points] [Written question, no code required]

Use the first-order optimality conditions to show that $x^* = [2.5, -1.5, -1]^T$ is a critical point of the function

$$f(x_1, x_2, x_3) = x_1^2 - 2x_1 + x_2^2 - x_3^2 + 4x_3,$$

subject to

$$g(x_1, x_2, x_3) = x_1 - x_2 + 2x_3 - 2 = 0.$$

Question 2 [30 points]

A well known benchmark problem for optimization algorithms is minimization of Rosenbrock's function

$$f(x, y) = 100(y - x^2)^2 + (1 - x)^2,$$

which has a global minimum of 0 at $(x, y) = (1, 1)$. We shall apply three different optimization algorithms for this function; in each case you should terminate the optimization algorithm when the *absolute* step size falls below 10^{-8}.

(a) [10 points] Minimize Rosenbrock's function using steepest descent. You should try the three starting points $[-1, 1]^T$, $[0, 1]^T$, and $[2, 1]^T$, and report the number of iterations required for each starting point. Generate a plot (to be included in your written report) for each starting point that shows the contours of Rosenbrock's function, as well as the optimization path that is followed.

You may use a library function for the line search in steepest descent if you wish. Also, note that steepest descent may require a large number of iterations, so you should terminate the scheme when either the step size tolerance (indicated above) is satisfied, *or* once 2000 iterations have been performed.

(b) [10 points] Repeat (a), but with Newton's method (without line search) instead of steepest descent.

(c) [10 points] Repeat (a), but with BFGS instead of steepest descent. In your implementation of BFGS, set B_0 to the identity matrix.

Question 3 [30 points]

In this question we design a missile defense system that can detect an enemy missile, predict its path, and then shoot it down by launching an interceptor missile.[1]

The force on a missile at time $t \geq 0$ is given by the vector $F(t) \in \mathbb{R}^3$, where

$$F(t) = T \frac{v(t)}{\|v(t)\|_2} - K\|v(t)\|_2 v(t) + (0, 0, -Mg). \quad (1)$$

[1] This problem is inspired by the AM205 final project by D. Levary and L. Simozar from Fall 2012.

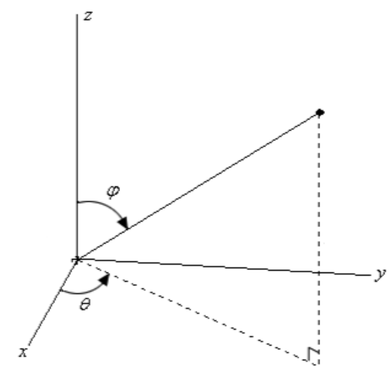


Figure 1: Definition of angles θ and ϕ.

In the equation above $T \in \mathbb{R}_{\geq 0}$ denotes thrust, $K \in \mathbb{R}_{>0}$ denotes the drag coefficient of the missile, $g = 9.81$ is the acceleration due to gravity, $M \in \mathbb{R}_{>0}$ is the missile's mass, and $x(t) \in \mathbb{R}^3$ and $v(t) = x'(t)$ are the position and velocity of the missile at time t, respectively. Also, assume that:

- M, T, and K remain constant throughout missile's motion.
- The missile is initially at rest, i.e. $v(t = 0) = (0, 0, 0)$.
- The missile is launched from the ground, so that $x(t = 0) = (x_0, y_0, 0)$ for some values x_0 and y_0.
- The initial direction of the missile is given by θ and ϕ, which are given in radians and are defined as in Figure 1. We consider $\theta \in [0, 2\pi]$, $\phi \in [0, \frac{\pi}{2}]$.

(We assume all quantities are in SI units in this question, but in order to simplify notation we do not state units explicitly.)

(a) [5 points] Use an ODE solver (e.g. a "built in" solver like `ode45` in MATLAB, or write your own solver if you prefer) to calculate the trajectory of a missile with $T = 8 \times 10^3$, $M = 15$, $K = 0.15$, $(x_0, y_0) = (0, 0)$ and $(\theta, \phi) = (\frac{\pi}{3}, \frac{\pi}{3})$. Let t^{impact} denote the time that the missile hits the ground. What is t^{impact}? What is $x(t^{\text{impact}})$? (For full credit, your values of t^{impact} and $x(t^{\text{impact}})$ should be correct to four significant digits.) Plot the trajectory of the missile for $t \in [0, t^{\text{impact}}]$.

(b) [10 points] Our radar system detects that an enemy has launched a missile at time $t = 0$. The radar then tracks the position of the enemy missile (with some measurement error) at snapshots in time. The missile position data is given in `radar_data.txt`. We know the type of missile that the enemy uses, hence we know that $T = 10^4$, $M = 20$, and $K = 0.1$.

Use nonlinear least-squares (e.g. `lsqnonlin` in MATLAB) to estimate x_0, y_0, θ and ϕ for the enemy missile, and report the 2-norm of the residual associated with the fit.

The enemy missile will destroy everything within a blast radius of 10 from the location where it hits the ground. Our military HQ is located at $x_{\text{HQ}} = (8000, 8000, 0)$. Assuming the enemy missile exactly follows the predicted trajectory, where will the enemy missile hit the ground? Will it destroy our military HQ?

(c) [5 points] In order to shoot down the enemy missile from (b), we launch an interceptor missile at $t = 2$ from our missile defense site at $(x_0, y_0) = (8000, 2000)$. The interceptor missile has mass $M = 30$ and drag coefficient $K = 0.05$, and it is capable of producing thrust in the range $T \in [0, 10^6]$.

We want to intercept the enemy missile at its point of maximum altitude (i.e. maximum z-value) in order to reduce the risk to our civilians. Let t^* denote the time at which the enemy missile reaches its maximum altitude. We need to choose T, θ and ϕ in order for the interceptor missile to be as close as possible to the predicted position of the enemy missile at $t = t^*$.

Write out a mathematical statement for an optimization problem for finding T, θ and ϕ. You should specify the objective function, as well as any constraints on the parameters.

(d) [10 points] Solve the optimization problem from part (c) using a "built in" constrained minimization algorithm (e.g. `fmincon` in MATLAB).

Report the optimal values of T, θ and ϕ. Plot the predicted enemy missile trajectory and the interceptor missile's trajectory for $t \in [0, t^*]$ on the same figure (use a different color for each trajectory).

We detonate our interceptor missile at $t = t^*$. On detonation, it will destroy anything within a blast radius of 10. Assuming the enemy missile exactly follows the predicted trajectory, do we successfully intercept it or not?[2]

(e) [Extra Credit: 10 points] In this question we develop a more advanced missile defense system that will intercept the enemy missile as far away as possible from our military HQ, which (recall from (b)) is located at $x_{\text{HQ}} = (8000, 8000, 0)$. Also, you should again assume that the interceptor missile is capable of producing thrust in the range $T \in [0, 10^6]$.

In this case, our interceptor missile will detonate at the moment that it is closest to the enemy missile (assuming the enemy missile exactly follows the predicted trajectory from (b)). Let us denote this detonation time as t^{det}, and let $x(t^{\text{det}})$ denote the position of our interceptor missile at t^{det}. Then our missile defense system must solve the following optimization problem: Maximize $\|x_{\text{HQ}} - x(t^{\text{det}})\|_2$, subject to the constraint that the interceptor missile is within the blast radius of 10 of the enemy missile at some point during its trajectory.

To solve this problem, you can again use a "built in" constrained minimization algorithm (e.g. `fmincon` in MATLAB), and in this case you will need to impose a nonlinear inequality constraint.

Report (i) the optimal values of T, θ and ϕ, (ii) t^{det}, (iii) $\|x_{\text{HQ}} - x(t^{\text{det}})\|_2$, and (iv) the distance between the interceptor missile and the enemy missile at t^{det}. Also, plot the predicted enemy missile trajectory and the interceptor missile trajectory for $t \in [0, t^{\text{det}}]$ on the same figure (use a different color for each trajectory).

Question 4 [22 points]

Consider the one-dimensional time-independent Schrödinger equation, which governs the behavior of a quantum particle in a potential well (for convenience we have normalized the equation by setting $\frac{\hbar^2}{2m} = 1$):

$$-\frac{\partial^2 \Psi(x)}{\partial x^2} + v(x)\Psi(x) = E\Psi(x), \tag{2}$$

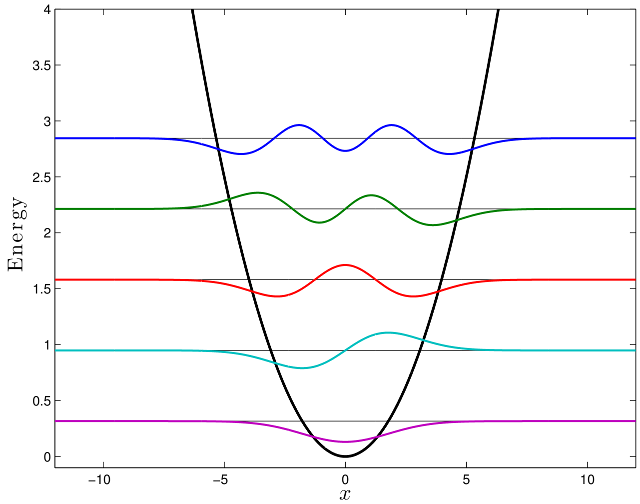
where $v(x) : \mathbb{R} \to \mathbb{R}$ is a real-valued potential function, $\Psi(x) : \mathbb{R} \to \mathbb{R}$ is the wavefunction, and $E \in \mathbb{R}$ is an eigenvalue (which corresponds to the energy of the system). Note: In general the wavefuntion is complex-valued, but in the time-independent case it is always possible to write it as a real-valued function.

The Schrödinger equation is posed on the infinite domain $(-\infty, \infty)$, and we require $\Psi(x) \to 0$ as $x \to \pm\infty$ so that the norm of Ψ is bounded. In this question, we shall consider the finite interval $[-12, 12] \subset \mathbb{R}$, which is a large enough domain for us to impose zero Dirichlet boundary conditions at the boundaries without compromising the accuracy of our results, i.e. we impose $\Psi(x = 12) = \Psi(x = -12) = 0$.

As an example of a solution of (2), in Figure 2 we show the first five eigenvalues and eigenmodes on $x \in [-12, 12]$ for the Schrödinger solution in the case that $v(x) = x^2/10$.[3]

[2] Of course, a more thorough analysis here would take into account the uncertainty in our prediction of the enemy missile's trajectory.

[3] The Schrödinger equation with a quadratic potential is known as a quantum harmonic oscillator, and it is one of the few cases where the Schrödinger equation can be solved analytically.



E_5	2.846013
E_4	2.213572
E_3	1.581127
E_2	0.948679
E_1	0.316227

Figure 2: The five lowest eigenvalues, and the corresponding eigenmodes, for $v(x) = x^2/10$. To plot the eigenmodes in a visually appealing way here we have plotted $y_i(x) = 3\Psi_i(x) + E_i$, $i = 1, \ldots, 5$, where Ψ_i is obtained from the output of MATLAB's `eigs` function.

(a) [12 points] Compute the five lowest eigenvalues and corresponding eigenmodes of (2) for the potentials:

(i) $v_1(x) = |x|$,

(ii) $v_2(x) = 12\left(\frac{x}{10}\right)^4 - \frac{x^2}{18} + \frac{x}{8} + 1.3$.

You should use a second-order accurate finite difference approximation of the Schrödinger equation with $n = 2001$ grid points on the interval $[-12, 12]$, and then employ an eigensolver (e.g. `eig` or `eigs` in MATLAB). Impose zero Dirichlet boundary conditions at $x = \pm 12$, as described above. Present your results using a figure and a table in the same way as in Figure 2.

(b) [10 points] Quantum mechanics tells us that if a particle has a wavefunction $\Psi(x)$, then the probability of finding it in a region $[a, b]$ is given by:

$$\frac{\int_a^b |\Psi(x)|^2 dx}{\int_{-\infty}^{\infty} |\Psi(x)|^2 dx}. \tag{3}$$

For $[a, b] \subset [-12, 12]$ we approximate this probablity on our finite grid as:

$$\frac{\int_a^b |\Psi(x)|^2 dx}{\int_{-12}^{12} |\Psi(x)|^2 dx}. \tag{4}$$

For each of the first five eigenmodes for the potential v_2, use the composite Simpson rule and (4) to compute the probability that the particle is in the region $x \in [0, 6]$ (i.e. specify five different probabilities, one corresponding to each eigenmode). When you use the composite Simpson rule here, you should use all grid points from (a) that are inside the interval of interest as quadrature points.