AM205: Assignment 4

Question 1 [20 points]

Consider the Lax-Friedrichs scheme

$$\frac{U_j^{n+1} - \frac{1}{2}(U_{j+1}^n + U_{j-1}^n)}{\Delta t} + c\frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x} = 0,$$

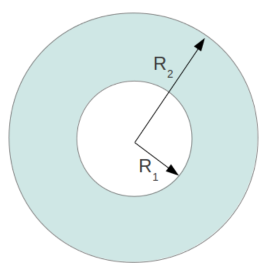
for the numerical solution of the advection equation $u_t + cu_x = 0$, where $c \in \mathbb{R}$.

(a) [10 points] [Written question, no code required] Use Fourier stability analysis to demonstrate that if $|\nu| \leq 1$ (where, as in lectures, $\nu \equiv c\Delta t/\Delta x$ is the CFL number), then this scheme is stable.

(b) [10 points] [Written question, no code required] Taylor expand about (t^n, x_j) to show that the Lax-Friedrichs scheme is first order accurate. In your analysis, you should suppose that ν is fixed to a value in $[-1, 1]$ in order to obtain a relationship between Δt and Δx.

Question 2 [36 points]

Suppose a metal pipe is heated in an industrial oven as part of a manufacturing process, and we wish to model the temperature distribution in the pipe as a function of time. The pipe's cross-section is shown below: it has an "inner radius" of R_1 and "outer radius" of R_2.



Due to the axis-symmetry of this problem, we can convert to polar coordinates (we suppose that the temperature in the pipe is uniform "into the page") to obtain the Initial Boundary Value Problem (IBVP):

$$\frac{\partial u}{\partial t} = \alpha\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right), \quad r \in [R_1, R_2],$$

for $t \in [0, t_f]$, with boundary conditions[1]

$$\text{BC}_1: \quad \left.\frac{\partial u}{\partial r}\right|_{r=R_1} = 0, \qquad \text{BC}_2: \quad \alpha\left.\frac{\partial u}{\partial r}\right|_{r=R_2} = -u(R_2).$$

[1]Note that for dimensional consistency we require a "heat transfer coefficient" in BC_2, but we omit this for the sake of simplicity (hence we effectively set this parameter to 1).

Here $u(r,t)$ is the "non-dimensional" temperature in the pipe, defined as

$$u(r,t) \equiv \frac{T(r,t) - T_{\text{oven}}}{T_0 - T_{\text{oven}}},$$

where $T(r,t)$ is the "dimensional" temperature, measured in degrees Farenheit, and T_0 and T_{oven} are the initial temperature in the pipe and the temperature in the oven, respectively.[2] As a result, the initial condition on u is $u_0(r) = 1$, and at steady state the pipe will be in thermal equilibrium with the oven so that $u(r,t) \to 0$ as $t \to \infty$. Also, α is the thermal diffusivity of the pipe. Thermal insulation on the interior of the pipe is modeled by BC_1, and BC_2 models heat transfer on the pipe's outer surface between the pipe and the air in the oven.

We will approximate this PDE using a finite difference method. Suppose we have spatial nodes $r_j = R_1 + (\Delta r)(j-1)$, $j = 1, 2, \ldots, n_r$, where $n_r = (R_2 - R_1)/\Delta r + 1$. Also, suppose we have discrete time-levels $t^n = (n-1)\Delta t$, $n = 1, 2, \ldots, n_t$, where $n_t = t_f/\Delta t + 1$.

(a) [4 points] [Written question, no code required] At time t^n and at an "interior node" r_j ($j \neq 1$, $j \neq n_r$), write down a Backward Euler (in time) and centered difference (in space) finite difference approximation for the above PDE.

(b) [4 points] [Written question, no code required] Derive a second-order finite difference approximation at time t^n of the left boundary condition, BC_1, at $r = R_1$. Use the "ghost node" approach described in lectures.

(c) [4 points] [Written question, no code required] Derive a second-order finite difference approximation at time t^n of the right boundary condition, BC_2, at $r = R_2$. Use the "ghost node" approach described in lectures.

(d) [14 points] For the parameters $R_1 = 1.7$, $R_2 = 3$, $\alpha = 0.3$, and with the initial condition $u(0,r) = 1$, use your answers to parts (a), (b) and (c) to compute an approximate solution to this IBVP for the time interval $t \in [0,2]$ (i.e. $t_f = 2$). Use a spatial step size of $\Delta r = 0.01$, and temporal step size $\Delta t = 0.01$, and plot the solution at $t = 0, 0.4, 0.8, 1.2, 1.6, 2$ in a single figure.

(e) [10 points] Use the composite trapezoid rule on the finite difference grid from (d) to compute the average (non-dimensional) temperature of the pipe at each time level in your calculation from (d), and plot the results as a function of time. Recall that the average value of a function f on a domain $\Omega \subset \mathbb{R}^2$ is $\bar{f} \equiv \left(\int_\Omega f \mathrm{d}x\mathrm{d}y\right)/|\Omega|$, where $|\Omega|$ denotes the area of Ω; you will need to account for the fact that we solved the PDE in polar coordinates.

The industrial heating process should be stopped at the time, t_{stop}, when the average (non-dimensional) temperature in the pipe is 0.5. Use your "average temperature" plot to estimate t_{stop}.

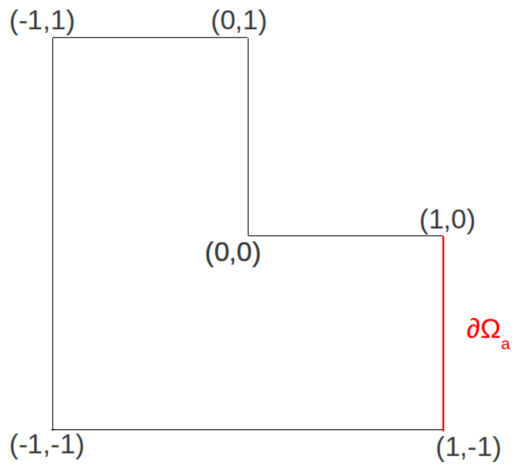
Question 3 [24 points]

Consider the Poisson equation

$$u_{xx} + u_{yy} = f(x,y),$$

on the "L-shaped" domain $\Omega = [-1,1]^2 \setminus (0,1]^2$. The boundary of the domain, $\partial\Omega$, is divided so that $\partial\Omega = \partial\Omega_a \bigcup \partial\Omega_b$, where $\partial\Omega_a = \{1\} \times [-1,0]$ (this will help us specify boundary conditions below). The domain Ω is illustrated below.

[2] Clearly there is a one-to-one mapping between $u(r,t)$ and $T(r,t)$; we will work with the non-dimensional temperature, $u(r,t)$, since in general it's more convenient not to have to worry about dimensional units.



Suppose that we introduce a uniform grid with n equally spaced nodes in each direction. Hence the node spacing in the x and y directions is given by $h = 2/(n-1)$, and $x_i = -1 + (i-1)h$ and $y_j = -1 + (j-1)h$. Furthermore, we suppose that n is an odd number (so that, for example, there is a node at $(0,0)$), i.e. $n = 2m + 1$ for some integer m. Then the mesh has $n_{\text{total}} = n^2 - m^2$ nodes in total.

(a) [6 points] Suppose we number the nodes from 1 to n_{total} starting with the "bottom row" (numbering "left to right" beginning with (x_1, y_1)), then the "second bottom row," etc. Let $\mathbb{U}$ denote our numerical solution vector. Write a Matlab function for the index mapping `index = index_function(i,j,n)`. The function `index_function` should provide the index $\mathcal{G}(i,j;n)$ (as discussed in lectures) such that $\mathbb{U}_{\mathcal{G}(i,j;n)} = U_{i,j}$, where $U_{i,j}$ is our finite difference approximation to $u(x_i, y_j)$ and $\mathbb{U} \in \mathbb{R}^{n_{\text{total}}}$. Your definition of `index_function(i,j,n)` should be valid for any odd value of n.

For $n = 11$, what are the indices $\mathcal{G}(i,j;n)$ of the nodes on the part of the boundary $\{0\} \times [0,1]$?

(b) [9 points] Let $f(x,y) = 5x + 5y$, and consider the Dirichlet boundary conditions $u = \sin(2\pi y)$ on $\partial\Omega_a$ and $u = 0$ on $\partial\Omega_b$. Use the "5-point stencil" finite difference scheme from lectures to compute an approximate solution to the above Poisson equation with $n = 81$. Make a filled contour plot of your approximate solution on Ω (with 30 levels and a colorbar) and report your approximation to $u(0.25, -0.75)$ to three significant digits. (You may find it helpful to use the function `index_function(i,j,n)` from part (a) in your code.)

(c) [9 points] A more general form of the Poisson equation allows for a non-uniform coefficient $a(x,y)$:

$$\operatorname{div}\big[a(x,y)\,\operatorname{grad}[u(x,y)]\big] = \nabla \cdot \big[a(x,y)\nabla u(x,y)\big] = f(x,y). \tag{1}$$

Use a second-order finite difference approximation to solve (1) on Ω (where again Ω denotes the "L-shaped" domain from above), with

$$a(x,y) = \exp\left[-\left(x + \tfrac{1}{2}\right)^2\right] \exp\left[-\left(y - \tfrac{1}{2}\right)^2\right], \qquad f(x,y) = y - x^2.$$

Impose the Dirichlet boundary condition $u = 0$ on all of $\partial\Omega$.

For $n = 81$ make a filled contour plot of the solution (with 30 levels and a colorbar) and report your approximation to $u(-0.25, 0.25)$ to three significant digits.

Question 4 [20 points]

Consider the nonlinear ODE BVP

$$u''(x) = e^{u(x)}, \qquad x \in (-1,1),$$

with zero Dirichlet boundary conditions $u(-1) = u(1) = 0$. We introduce an n-point grid $x_i = -1 + (i-1)h$, $h = 2/(n-1)$.

A finite difference approximation gives the nonlinear system $F(U) = 0$, where $U \in \mathbb{R}^{n-2}$ is our finite difference solution vector (where we have dropped the two boundary terms, $U_1 = 0$ and $U_n = 0$, since they are already known) and $F : \mathbb{R}^{n-2} \to \mathbb{R}^{n-2}$. The equations in our nonlinear system are as follows:

$$
\begin{aligned}
F_2(U) &\equiv \frac{U_3 - 2U_2}{h^2} - e^{U_2} = 0, \\
F_i(U) &\equiv \frac{U_{i+1} - 2U_i + U_{i-1}}{h^2} - e^{U_i} = 0, \quad i = 3, 4, \ldots, n-2, \\
F_{n-1}(U) &\equiv \frac{-2U_{n-1} + U_{n-2}}{h^2} - e^{U_{n-1}} = 0.
\end{aligned}
$$

(a) [6 points] [Written question, no code required] Derive the Jacobian $J_F \in \mathbb{R}^{(n-2)\times(n-2)}$ for the system $F(U) = 0$. Describe the sparsity pattern of J_F.

(b) [14 points] Use Newton's method to solve this nonlinear ODE BVP for $n = 101$, using the Jacobian matrix derived in (a). Start with an initial guess $U^0 = 0$. Terminate Newton's method once a relative step size, $\|\Delta U^k\|_2/\|U^k\|_2$, satisfies $\|\Delta U^k\|_2/\|U^k\|_2 \leq 10^{-10}$, where k refers to the Newton iteration count. Plot the approximate solution U (padded with U_1 and U_n) of the ODE BVP above on the n-point grid, and report your approximation to $u(0)$ to three significant digits.