Applied Mathematics 205

Unit V: Eigenvalue Problems

Lecturer: Dr. David Knezevic

Unit V: Eigenvalue Problems

Chapter $\mathrm{V}.1$: Motivation

Motivation: Eigenvalue Problems

A matrix $A \in \mathbb{C}^{n \times n}$ has eigenpairs $(\lambda_1, v_1), \ldots, (\lambda_n, v_n) \in \mathbb{C} \times \mathbb{C}^n$ such that

$$Av_i = \lambda v_i, \quad i = 1, 2, \ldots, n$$

(We will order the eigenvalues from smallest to largest, so that $|\lambda_1| \leq |\lambda_2| \leq \cdots \leq |\lambda_n|$)

It is remarkable how important eigenvalues and eigenvectors are in science and engineering!

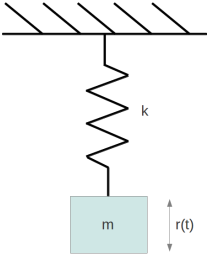
Motivation: Eigenvalue Problems

For example, eigenvalue problems are closely related to resonance

- Pendulums
- Natural vibration modes of structures
- Musical instruments
- Quantum mechanics
- Lasers
- Nuclear Magnetic Resonance (NMR)
- ...

Motivation: Resonance

Consider a spring connected to a mass



Suppose that:

- the spring satisfies Hooke's Law,[1] $F(t) = ky(t)$
- a periodic forcing, $r(t)$, is applied to the mass

[1] Here $y(t)$ denotes the position of the mass at time t

Motivation: Resonance

Then Newton's Second Law gives the ODE

$$y''(t) + \left(\frac{k}{m}\right) y(t) = r(t)$$

where $r(t) = F_0 \cos(\omega t)$

Recall that the solution of this non-homogeneous ODE is the sum of a homogeneous solution, $y_h(t)$, and a particular solution, $y_p(t)$

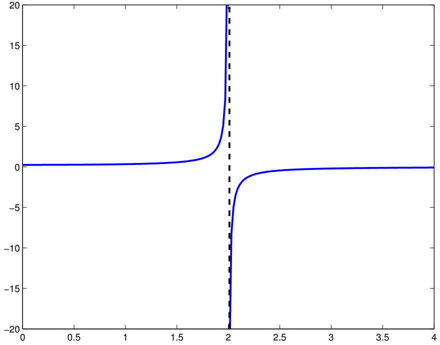
Let $\omega_0 \equiv \sqrt{k/m}$, then for $\omega \neq \omega_0$ we get:[2]

$$y(t) = y_h(t) + y_p(t) = C\cos(\omega_0 t - \delta) + \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos(\omega t),$$

[2] C and δ are determined by the ODE initial condition

Motivation: Resonance

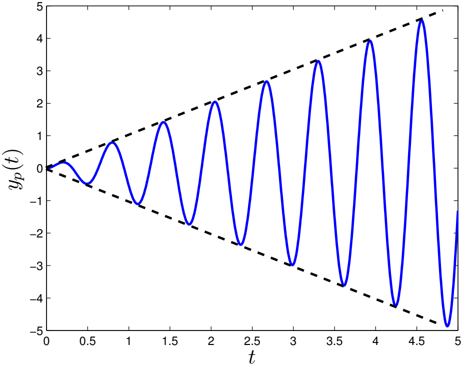
The amplitude of $y_p(t)$, $\frac{F_0}{m(\omega_0^2 - \omega^2)}$, as a function of ω is shown below



Clearly we observe singular behavior when the system is forced at its natural frequency, i.e. when $\omega = \omega_0$

Motivation: Forced Oscillations

If we solve the ODE in the case that $\omega = \omega_0$, we obtain $y_p(t) = \frac{F_0}{2m\omega_0} t \sin(\omega_0 t)$



The solution is unbounded as $t \to \infty$, this is resonance

Motivation: Resonance

In general, the natural frequency ω_0 is the frequency at which the unforced system has a non-zero oscillatory solution

To calculate ω_0 directly, we substitute an oscillatory "ansatz" into the unforced equation and solve for the frequency

For example, for the single spring-mass system we substitute[3]
$y(t) \equiv x e^{i\omega_0 t}$ into $y''(t) + \left(\frac{k}{m}\right) y(t) = 0$

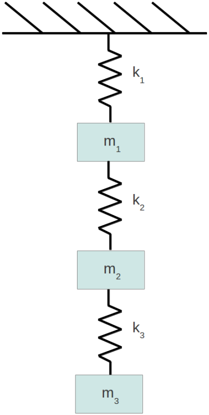
This gives a scalar equation for ω_0:

$$kx = \omega_0^2 m x \implies \omega_0 = \sqrt{k/m}$$

[3] Here x is the amplitude of the oscillatory solution

Motivation: Resonance

Suppose now we have a spring-mass system with three masses and three springs



Motivation: Resonance

In the unforced case, this system is governed by the ODE system

$$My''(t) + Ky(t) = 0,$$

where M is the mass matrix and K is the stiffness matrix

$$M \equiv \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}, \quad K \equiv \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}$$

We again seek a nonzero oscillatory solution to this ODE, i.e. set $y(t) = xe^{i\omega t}$, where now $y(t) \in \mathbb{R}^3$

This gives the algebraic equation

$$Kx = \omega^2 Mx$$

Motivation: Eigenvalue Problems

Setting $A \equiv M^{-1}K$ and $\lambda = \omega^2$, this gives the eigenvalue problem

$$Ax = \lambda x$$

Here $A \in \mathbb{R}^{3\times 3}$, hence we obtain natural frequencies from the three eigenvalues λ_1, λ_2, λ_3

Motivation: Eigenvalue Problems

The spring-mass systems we have examined so far contain discrete components

But the same ideas also apply to continuum models

For example, the wave equation models vibration of a string (1D) or a drum (2D)

$$\frac{\partial^2 u(x,t)}{\partial t^2} - c\Delta u(x,t) = 0$$

As before, we write $u(x,t) = \tilde{u}(x)e^{i\omega t}$, to obtain

$$-\Delta \tilde{u}(x) = \frac{\omega^2}{c}\tilde{u}(x)$$

which is a PDE eigenvalue problem

Motivation: Eigenvalue Problems

We can discretize the Laplacian operator with finite differences to obtain an algebraic eigenvalue problem

$$Av = \lambda v,$$

where

- the eigenvalue $\lambda = \omega^2/c$ gives a natural vibration frequency of the system
- the eigenvector (or eigenmode) v gives the corresponding vibration mode

Motivation: Eigenvalue Problems

We will use the Matlab functions `eig` and `eigs` to solve eigenvalue problems:

- `eig`: find all eigenvalues/eigenvectors of a dense matrix
- `eigs`: find a few eigenvalues/eigenvectors of a sparse matrix

Motivation: Eigenvalue Problems

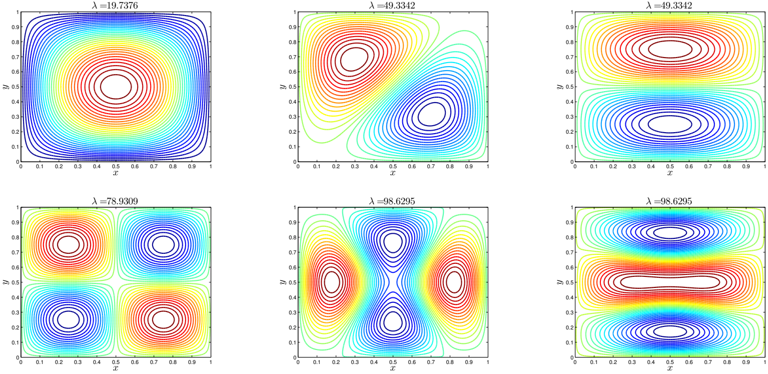
Matlab demo: Eigenvalues/eigenmodes of Laplacian on $[0, 1]^2$, zero Dirichlet boundary conditions

Based on separation of variables, we know that eigenmodes are $\sin(\pi i x)\sin(\pi j y)$, $i, j = 1, 2, \ldots$

Hence eigenvalues are $(i^2 + j^2)\pi^2$

i	j	$\lambda_{i,j}$
1	1	$2\pi^2 \approx 19.74$
1	2	$5\pi^2 \approx 49.35$
2	1	$5\pi^2 \approx 49.35$
2	2	$8\pi^2 \approx 78.96$
1	3	$10\pi^2 \approx 98.97$
$\vdots$	$\vdots$	$\vdots$

Motivation: Eigenvalue Problems



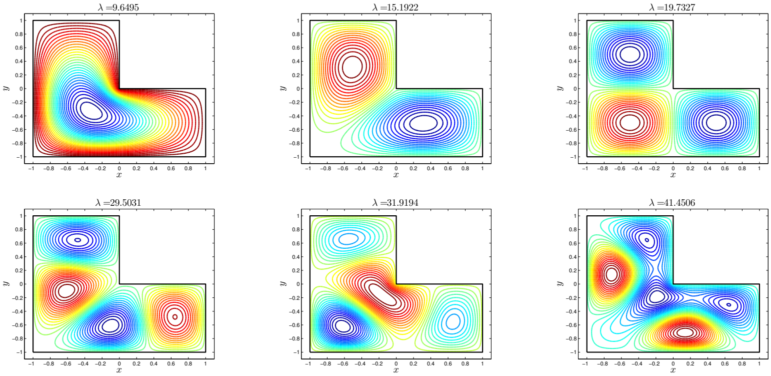
In general, for repeated eigenvalues, computed eigenmodes are linearly independent members of the corresponding eigenspace

e.g. eigenmodes corresponding to $\lambda = 49.3$ are given by

$$\alpha \sin(\pi x) \sin(\pi 2y) + \beta \sin(\pi 2x) \sin(\pi y), \quad \alpha, \beta \in \mathbb{R}$$

Motivation: Eigenvalue Problems

And of course we can compute eigenmodes of other shapes...



Motivation: Eigenvalue Problems

An interesting mathematical question related to these issues:
"Can one hear the shape of a drum?"[4]

The eigenvalues for a shape in 2D correspond to the resonant frequences that a drumhead of that shape would have

Therefore, the eigenvalues determine the harmonics, and hence the sound of the drum

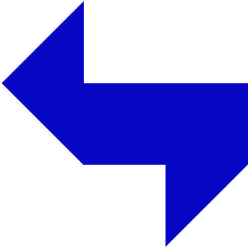
So in mathematical terms, this question is equivalent to: If we know all of the eigenvalues, can we uniquely determine the shape?

[4]Posed by Mark Kac in American Mathematical Monthly in 1966

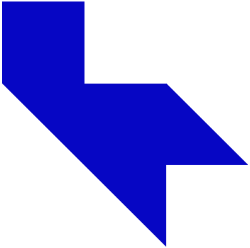
Motivation: Eigenvalue Problems

It turns out that the answer is no!

In 1992, Gordon, Webb, and Wolpert constructed two different 2D shapes that have exactly the same eigenvalues!



Drum 1



Drum 2

Motivation: Eigenvalue Problems

We can compute the eigenvalues and eigenmodes of the Laplacian on these two shapes using the algorithms from this Unit[5]

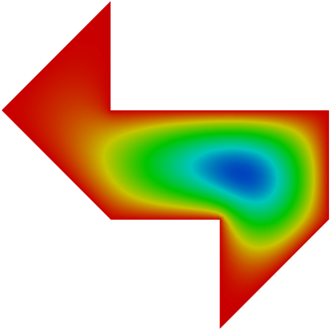
The first five eigenvalues are computed as:

	Drum 1	Drum 2
λ_1	2.54	2.54
λ_2	3.66	3.66
λ_3	5.18	5.18
λ_4	6.54	6.54
λ_5	7.26	7.26

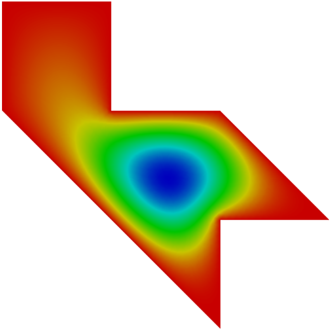
We next show the corresponding eigenmodes...

[5] Note here we employ the Finite Element Method (outside the scope of AM205), an alternative to F.D. that is well-suited to complicated domains

Motivation: Eigenvalue Problems

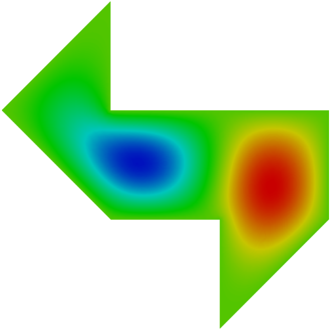


eigenmode 1

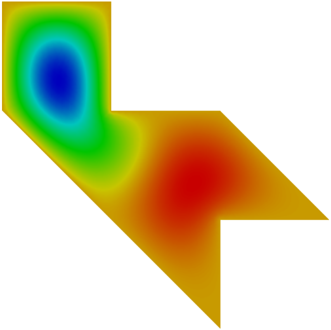


eigenmode 1

Motivation: Eigenvalue Problems

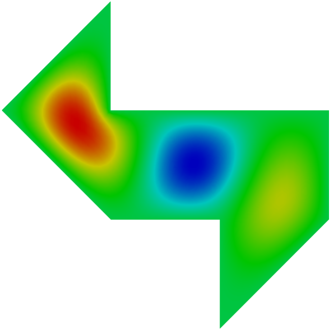


eigenmode 2

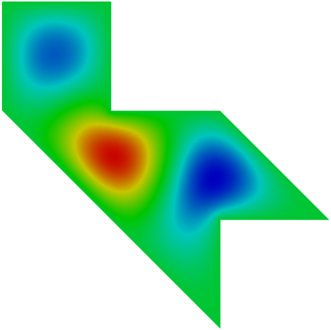


eigenmode 2

Motivation: Eigenvalue Problems

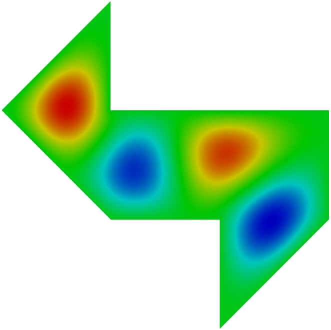


eigenmode 3

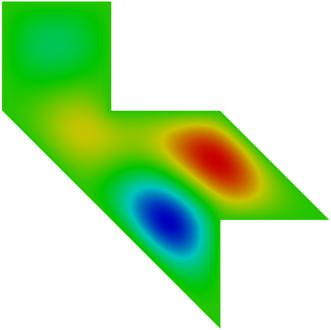


eigenmode 3

Motivation: Eigenvalue Problems

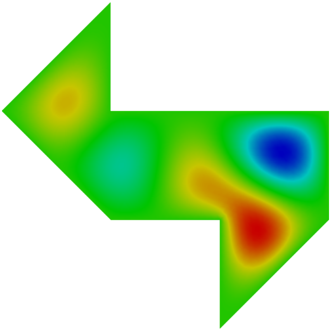


eigenmode 4

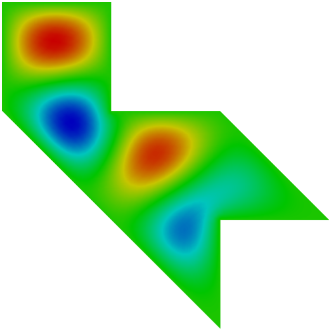


eigenmode 4

Motivation: Eigenvalue Problems



eigenmode 5



eigenmode 5

Summary

Eigenvalue problems have many interesting and important applications in science and engineering

In practice, the main challenge is often to formulate a problem as $Ax = \lambda x$

We can then employ reliable and efficient algorithms for computing the eigenvalues/eigenvectors (e.g. `eig`, `eigs` in Matlab)

In the remaining chapters of this Unit we explore the mathematical ideas that underpin algorithms for eigenvalue problems