Applied Mathematics 205

Unit II: Numerical Linear Algebra

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Chapter II.1: Motivation

Motivation

Almost everything in Scientific Computing relies on Numerical Linear Algebra!

We often reformulate problems as $Ax = b$, e.g. from Unit I:

- Interpolation (Vandermonde matrix) and linear least-squares (normal equations) are naturally expressed as linear systems
- Gauss-Newton/Levenberg-Marquardt involve approximating nonlinear problem by a sequence of linear systems

Similar themes will arise in remaining Units (Numerical Calculus, Optimization, Eigenvalue problems)

Motivation

The goal of this Unit is to cover:

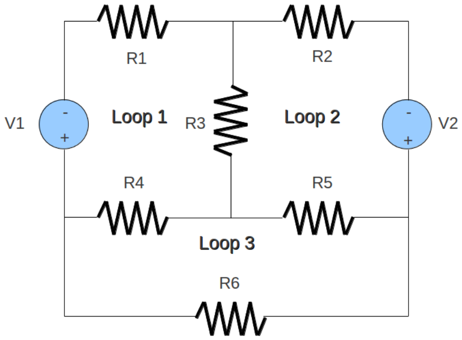
- key linear algebra concepts that underpin Scientific Computing
- algorithms for solving $Ax = b$ in a stable and efficient manner
- algorithms for computing factorizations of A that are useful in many practical contexts (QR, SVD)

First, we discuss some practical cases where $Ax = b$ arises directly in mathematical modeling of physical systems

Example: Electric Circuits

Ohm's Law: Voltage drop due to a current i through a resistor R is $V = iR$

Kirchoff's Law: The net voltage drop in a closed loop is zero



Example: Electric Circuits

Let i_j denote the current in "loop j"

Then, we obtain the linear system:

$$\begin{bmatrix} (R_1 + R_3 + R_4) & R_3 & R_4 \\ R_3 & (R_2 + R_3 + R_5) & -R_5 \\ R_4 & -R_5 & (R_4 + R_5 + R_6) \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ 0 \end{bmatrix}$$

Circuit simulators solve large linear systems of this type

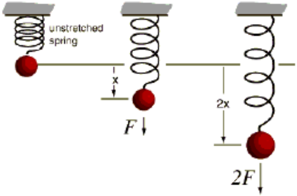
Example: Structural Analysis

Common in structural analysis to use a linear relationship between force and displacement, Hooke's Law

Simplest case is the Hookean spring law

$$F = kx,$$

- k: spring constant (stiffness)
- F: applied load
- x: spring extension



Example: Structural Analysis

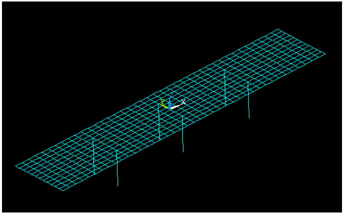
This relationship can be generalized to structural systems in 2D and 3D, which yields a linear system of the form

$$Kx = F$$

- $K \in \mathbb{R}^{n \times n}$: "stiffness matrix"
- $F \in \mathbb{R}^{n}$: "load vector"
- $x \in \mathbb{R}^{n}$: "displacement vector"

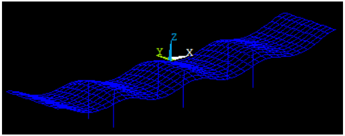
Example: Structural Analysis

Solving the linear system yields the displacement (x), hence we can simulate structural deflection under applied loads (F)



Unloaded structure

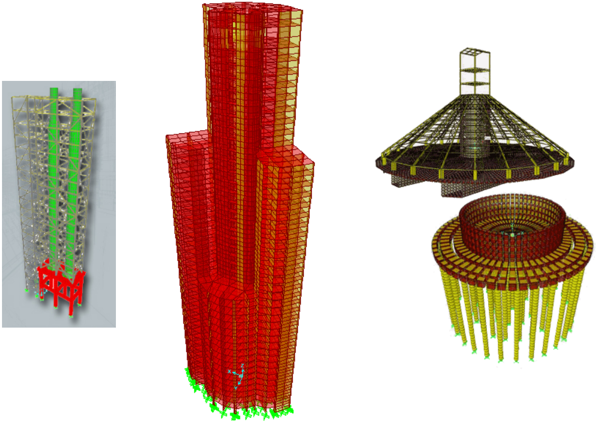
$$\xrightarrow{Kx=F}$$



Loaded structure

Example: Structural Analysis

It is common engineering practice to use Hooke's Law to simulate complex structures, which leads to large linear systems



(From SAP2000, structural analysis software)

Example: Economics

Leontief awarded Nobel Prize in Economics in 1973 for developing linear input/output model for production/consumption of goods

Consider an "economy" in which n goods are produced and consumed

- $A \in \mathbb{R}^{n \times n}$: a_{ij} represents amount of good j required to produce 1 unit of good i
- $x \in \mathbb{R}^n$: x_i is number of units of good i produced
- $d \in \mathbb{R}^n$: d_i is consumer demand for good i

In general $a_{ii} = 0$, and A may or may not be sparse

Example: Economics

The total amount of x_i produced is given by the sum of consumer demand (d_i) and the amount of x_i required to produce each x_j

$$x_i = \underbrace{a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n}_{\text{production of other goods}} + d_i$$

Hence $x = Ax + d$ or,

$$(\mathrm{I} - A)x = d$$

Solve for x to determine the required amount of production of each good

If we consider many goods (e.g. an "entire economy"), then we get a large linear system

Summary

Matrix computations arise all over the place!

Numerical Linear Algebra algorithms provide us with a toolbox for performing these computations in an efficient and stable manner

In most cases, can use these tools as "black boxes", e.g. use "backslash" in Matlab to solve $Ax = b$ for you

But it's important to understand what the linear algebra "black boxes" do:

- Pick the right algorithm for a given situation (e.g. exploit structure in a problem: symmetry, bandedness, etc)
- Understand how/when the "black box" can fail