The Response of a Zonally Symmetric Atmosphere to Subtropical Thermal Forcing: Threshold Behavior

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ABSTRACT

We consider the response of a zonally symmetric atmosphere to a thermal forcing that is localized in the subtropics. Specifically, the equilibrium temperature distribution has a local subtropical peak and is flat elsewhere, including at the equator. On the basis of inviscid steady-state theory, it is argued that the response to such forcing is one of two distinct types. Below a threshold forcing the atmosphere adopts a steady state of thermal equilibrium with no meridional flow. With supercritical forcing, this state breaks down and a strong meridional circulation is predicted. The threshold forcing value is that at which the absolute vorticity of the zonal flow (in gradient balance with the equilibrium temperatures) vanishes at the upper boundary. These inviscid predictions are tested in a zonally symmetric numerical model; while the model viscosity shifts the threshold and otherwise modifies the response, the threshold is clearly evident in the model behavior.

1. Background

Despite its fundamental role in the atmospheric general circulation, it is only relatively recently that our understanding of the idealized, axisymmetric, tropical Hadley cell has been put on a firm theoretical footing. Schneider (1977) and Held and Hou (1980) drew attention to the nonlinear, angular momentum–conserving nature of the near-equatorial response to equatorially symmetric heating and showed how, in the inviscid limit, this constraint determines the width and other characteristics of the Hadley cell. Thus, in response to a latitudinally broad heating distribution, symmetric about the equator, an axisymmetric, inviscid atmosphere adopts two regimes simultaneously: an angular momentum–conserving regime in the tropics with a strong, thermally direct meridional circulation and, poleward of this, a "thermal equilibrium" regime in which temperatures are in local thermal equilibrium and there is no meridional circulation. The two are separated by a sloping subtropical front.

Extension of these inviscid arguments to situations where the heating is not symmetric about the equator has been made by Lindzen and Hou (1988). They showed that as the heating maximum is skewed a little off the equator the meridional circulation changes rapidly from a configuration with two equal cells straddling the equator to one dominated by a single cell, with upwelling in the "summer" hemisphere and subsidence on the other side of the equator. Moreover, the intensity of the circulation increased dramatically, even for a very small displacement of the heating maximum off the equator.

In this paper, we use similar arguments to describe the more complex response of an axisymmetric atmosphere to an external thermal forcing that is localized off the equator. The inviscid theory is presented in section 2. All of the aforementioned cases had nonzero forcing at the equator (either the first or second derivative with respect to latitude of equilibrium temperature being nonzero there). The equator is in fact a singular location in the sense that the planetary vorticity—or, equivalently, the latitudinal gradient of planetary angular momentum—vanishes there. This is crucial to the dynamics of a steady, inviscid, meridional circulation, since the angular momentum budget of such a circulation is simply $\mathbf{u}\cdot\nabla M = 0$, where $\mathbf{u} = (v, w)$ is the meridional vector velocity and M is the absolute angular momentum density. In an inviscid steady state there can thus be no advection of angular momentum. This statement is of course central to the arguments of Schneider (1977) and Held and Hou (1980). A steady inviscid circulation may exist provided the induced relative vorticity (which is proportional to the relative angular momentum gradient) is large enough to cancel the weak planetary component.

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In low latitudes, where there is little planetary vorticity to be overcome, a circulation can be sustained with relatively weak forcing. In fact, in an axisymmetric inviscid atmosphere, thermal forcing that maximizes on the equator (specifically, one whose second derivative with respect to latitude is negative there) will, no matter how weak, always drive a meridional circulation in low latitudes (though the circulation becomes increasingly trapped near the equator as the forcing is reduced).

Contrast this behavior with that to be expected in response to localized thermal forcing some distance from the equator ("localized" here implying that the derivatives of the equilibrium temperature *do* vanish at the equator, so there is no equatorial forcing of the system). The planetary vorticity in the region of the forcing is now finite. Suppose first that the forcing is weak enough that the absolute angular momentum is dominated by the planetary component. Then, by the preceding argument, there can be no meridional circulation; the inviscid, steady response must be one of thermal equilibrium. Only if the forcing is sufficiently strong for the induced anticyclonic relative vorticity over the heated region to cancel the planetary vorticity can a meridional circulation exist. Thus, the inviscid theory predicts the existence of a threshold forcing amplitude at which the character of the forced flow will change.

This transition is the focus of this paper. In section 2, we present the inviscid arguments in detail and identify the threshold forcing magnitude for a given forcing structure. Of the two possible inviscid solutions, the thermal equilibrium solution is a legitimate approximation to the nearly inviscid solution only if the forcing is subcritical with respect to this threshold; the nonlinear, angular momentum–conserving solution is valid only in the supercritical case. The prediction of the inviscid theory, therefore, is that the response to such forcing will change (as a function of forcing magnitude) from a state with no meridional circulation in subcritical cases to one with a nonzero, angular momentum–conserving circulation for supercritical forcing.

The existence of such a transition is demonstrated explicitly in the results of numerical integrations to be discussed in section 3. Although the finite viscosity used in these experiments has a significant quantitative impact on the model solutions and on the actual value of the threshold forcing rate, the existence of such a threshold is easily identifiable in the model response and qualitatively in accord with the predictions of the inviscid theory. The transition marks a change from a state in which the dynamics is basically linear to behavior that is fundamentally nonlinear.

We note here that this transition (between thermal equilibrium and angular momentum conserving solutions) is quite distinct from that proposed by Schneider (1983) in connection with Martian dust storms; Schneider's transition is between two subsets—"local" and "global" forms—of the angular momentum-conserving solution. The distinction between these two transitions will be discussed further in what follows.

2. Theory

In axisymmetric flow on the sphere, forced by an imposed external thermal forcing (in units of temperature per unit time) $\alpha(T_e - T_0)$, where α is a relaxation rate, T_e an equilibrium temperature, and T_0 some reference background temperature, the relevant equations describe angular momentum conservation

$$\frac{\partial M}{\partial t} + \frac{v}{a}\frac{\partial M}{\partial \varphi} + w\frac{\partial M}{\partial z} = \mathcal{F}, \tag{1}$$

where $M = \Omega a^2 \cos^2\varphi + ua\cos\varphi$; continuity

$$\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi) + \frac{1}{p}\frac{\partial}{\partial z}(pw) = 0; \tag{2}$$

heat balance

$$\frac{\partial T}{\partial t} + \frac{v}{a}\frac{\partial T}{\partial \varphi} + wS = \alpha(T_e - T); \tag{3}$$

and, from hydrostatic and gradient balance,

$$\frac{\partial}{\partial z}\left[2\Omega u\sin\varphi + \frac{\tan\varphi}{a}u^2\right] = -\frac{g}{aT_0}\frac{\partial T}{\partial\varphi}. \tag{4}$$

Here (u, v, w) are the velocity components, p pressure, T temperature, Ω and a the earth's rotation rate and radius, φ latitude, and z log-pressure height. In (1), $\mathcal{F}$ is the viscous sink of angular momentum and, in (3), S is the static stability $\partial T/\partial z + \Gamma_{ad}$, Γ_{ad} being the adiabatic lapse rate. Note also that (3) includes a thermal relaxation of the system toward T_e; while we specify T_e, we do not specify the distribution of net diabatic heating, which is proportional to $\alpha(T_e - T)$. This fact is crucial to the behavior of the system, as the net diabatic heating is a part of the *response* to the forcing and, indeed, one possible inviscid solution is that it vanishes everywhere.

As noted by earlier authors, most explicitly by Held and Hou (1980), (1)–(4) have two steady solutions for zero viscosity:

(i) a thermal equilibrium (TE) solution

$$T = T_e; \quad v = w = 0;$$

with $u = u_e$, where

$$\frac{\partial}{\partial z}\left[2\Omega u_e\sin\varphi + \frac{\tan\varphi}{a}u_e^2\right] = -\frac{g}{aT_0}\frac{\partial T_e}{\partial\varphi}; \tag{5}$$

(ii) an angular momentum-conserving (AMC) solution for which $T \neq T_e$, $(v, w) \neq 0$ with M constant along streamlines of the meridional flow.

The TE solution is almost linear, since there is no meridional advection [the only nonlinearity appears in the balance condition (5) and is not of major importance whenever $u_e \ll \Omega a \cos\varphi$, which includes the cases we shall consider here]. In fact, it is approximately the solution, in the limit of small viscosity, of the problem linearized about a resting atmosphere. However, the TE solution is not necessarily a regular limit for vanishing viscosity. The condition for regularity—Hide's theorem (Hide 1969; Schneider 1977)—is that there can be no extrema of angular momentum except on the lower boundary. In physical terms, this statement rests on the observation that, if such an extremum exists and if momentum diffusion is dominated by vertical transport, the diffusive loss of angular momentum from the extremum cannot be balanced by advection of angular momentum (which necessarily sums to zero across a closed M contour surrounding the extremum) *no matter how small the coefficient of viscosity may be.* As this theorem is central to the arguments we present here, a formal mathematical demonstration of the nonregularity of the TE solution (5) under such circumstances is given in the Appendix.

Since the y and z components of absolute vorticity are

$$\eta_a = (a\cos\varphi)^{-1}\frac{\partial M}{\partial z};\quad \zeta_a = -(a^2\cos\varphi)^{-1}\frac{\partial M}{\partial \varphi}, \quad (6)$$

the regularity condition may be expressed as one for which the latitudinal and vertical components of vector absolute vorticity do not simultaneously vanish in the interior and the vertical component of absolute vorticity does not vanish on any upper, stress-free boundary (if such a boundary exists), unless the viscous force also vanishes at such locations.

Held and Hou (1980) discussed in detail the violation of this condition when T_e maximizes (with $\partial^2 T_e/\partial\varphi^2$ nonzero) on the equator, since then the balance condition in (5) predicts equatorial westerlies and an absolute maximum of angular momentum there. Then the nonlinear AMC circulation must exist near the equator under such circumstances. This is also true whenever T_e has nonzero gradient on the equator [as in the cases considered by Lindzen and Hou (1988) and by Dunkerton (1989)], since then the balance condition in (5) predicts infinite zonal wind there in the TE regime. On the other hand, the AMC solution would imply infinite velocities at the poles, so the TE solution must be selected at high latitudes.

In general, the TE solution may in fact be possible everywhere, depending on the structure of T_e. Consider a localized off-equatorial forcing of a form such as that shown in Fig. 1; note especially that T_e and its derivatives are zero at the equator. Now, the TE solution has a balanced flow given by (5). Integrating in the vertical from the lower boundary at $z = 0$ (where it is assumed that surface drag maintains a weak flow, which

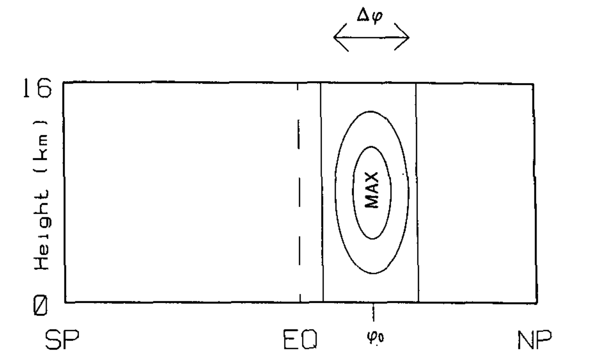

FIG. 1. Latitude–height distribution of equilibrium temperature in the model experiments. Contours are at 0, 1/3, and 2/3 of maximum value; T_e is zero outside the outer contour.

we neglect) to the upper boundary at $z = D$, assumed stress free, this may be rewritten

$$-a^2\frac{gD}{T_0}\frac{\cos^3\varphi}{\sin\varphi}\frac{\partial\hat{T}_e}{\partial\varphi} = M_D^2 - \Omega^2 a^4\cos^2\varphi,$$

where M_D is M evaluated at $z = D$ and

$$\hat{T}_e = \frac{1}{D}\int_0^D T_e dz$$

is the depth-averaged equilibrium temperature. Differentiating and using (6) gives an equation for the absolute vorticity at the top of the domain:

$$M_D\zeta_a = 2\Omega^2 a^2\sin\varphi\cos^2\varphi + \frac{1}{2}\frac{gD}{T_0}\frac{1}{\cos\varphi}\frac{\partial}{\partial\varphi}\left[\frac{\cos^3\varphi}{\sin\varphi}\frac{\partial\hat{T}_e}{\partial\varphi}\right]. \quad (7)$$

Provided M_D remains positive (which seems assured under all conceivable circumstances[1]) the sign of absolute vorticity is determined by the right-hand side of (7). For small forcing, the relative vorticity and the sign of ζ_a is determined by the planetary term; for sufficiently strong forcing, however, relative vorticity becomes significant and ζ_a may then, in places, vanish. Thus, the TE solution is regular provided (choosing signs for the Northern Hemisphere case)

$$-\frac{1}{2}\frac{gD}{T_0}\frac{1}{\cos\varphi}\frac{\partial}{\partial\varphi}\left[\frac{\cos^3\varphi}{\sin\varphi}\frac{\partial\hat{T}_e}{\partial\varphi}\right] < 2\Omega^2 a^2\sin\varphi\cos^2\varphi \quad (8)$$

everywhere; this will be satisfied if the forcing is weak, broad, and/or far from the equator. When (8) is violated, however, the TE solution is untenable, and it is expected that the response must take the form of an AMC solution somewhere in the region of the forcing (though not all the way to the poles).

[1] In fact, this follows from Schneider's (1977) proof that extrema in M must occur on the boundaries.

It might appear that an alternative prediction is that the AMC solution is always possible. However, in the Appendix we present a proof that this solution cannot be realized whenever the TE solution is regular. Therefore, we predict that only the TE solution can exist when (8) is satisfied and only the AMC solution when it is not. Thus, (8) represents a transition from TE to AMC behavior.

It should be noted that the existence of the TE response to a given forcing when the forcing is applied off the equator is simply a matter of the inertial rigidity of a fluid with nonzero absolute vorticity. The vanishing of planetary vorticity at the equator means that the medium has no dynamical rigidity there to restrict a meridional flow. When weak forcing is applied off the equator, however, the finite, local, inertial rigidity precludes any steady, inviscid, meridional flow. Increasing forcing reduces this rigidity (through induced anticyclonic relative vorticity); only if the forcing is strong enough to destroy ζ_a locally can an inviscid meridional circulation exist.

3. Model results

a. Model details

In order to test these ideas we have performed some experiments with a zonally symmetric model; the model grid points are evenly spaced in the vertical, with 32 increments between $z = 0$ and $D = 16$ km, and 60 increments evenly spaced in $\sin\varphi$ from pole to pole. The equilibrium temperature is specified as

$$T_e(\varphi, z) = \frac{\pi}{2}\Theta_e \sin\left(\pi \frac{z}{D}\right)\Phi(\varphi) + T_0,$$

where

$$\Phi(\varphi) = \begin{cases} \cos^2\left(\dfrac{\pi}{2}\dfrac{(\varphi - \varphi_0)}{\Delta\varphi}\right), \\ \qquad \varphi_0 - \Delta\varphi \leqslant \varphi \leqslant \varphi_0 + \Delta\varphi \\ 0, \quad \text{otherwise.} \end{cases} \tag{9}$$

The forcing is thus centered on $\varphi_0 = 25°$ and of half-width $\Delta\varphi = 15°$; the distribution of T_e is shown in Fig. 1. With the forcing (9), it is straightforward to show from (8) that the inviscid TE solution becomes irregular when $\Theta_e > T_c$, where $T_c = 3.45$ K.

The model was integrated with a second-order vertical diffusion (in most cases to be described, the diffusivity used was 2.5 $m^2 s^{-1}$), a fourth-order horizontal diffusion that has negligible effect on the large scales, and a thermal relaxation rate α of (10 days)$^{-1}$. A drag law boundary condition (with $c_d = 2.5 \times 10^{-3}$ m s^{-1}) was applied at $z = 0$, with a free-slip condition at the top. The corresponding thermal boundary conditions applied were $T = T_0$ at top and bottom. Several series of experiments were run; we concentrate here on those with infinite scale height (i.e., a Boussinesq system; results for a finite, realistic value of scale height are not very different) for which the only variable parameter is Θ_e, the magnitude of the forcing. The model is integrated to an almost steady state (which in some cases took 300 days or more to achieve).

b. *Results for $\nu = 2.5\ m^2\ s^{-1}$*

An example of results for weak forcing ($\Theta_e = 3.0$ K) is shown in Fig. 2. The relative vorticity is (except at the equator) substantially weaker than the planetary component, a fact evident in the rather weak distortion of the ζ_a contours in Fig. 2d. The meridional circulation is not zero, however, although it is localized near the forcing, relatively weak, and apparently viscously driven (Schneider and Lindzen 1977). It appears that we can relate the response in this case to the (viscously modified) TE regime; if so, the solution will be very similar to a linear solution, since (as noted previously) the only nonlinearity in the TE regime is in the balance condition (5) and in fact this nonlinear term is negligible here[2]. This interpretation is confirmed by Fig. 3, which shows results from a linearized version of the model for $\Theta_e = 12.1$ K; it is evident that the meridional circulation shown in the linear results is almost identical to that of the nonlinear case in Fig. 2 (apart, of course, from the expected factor of 4 difference in amplitude; note the factor of 4 difference in contour interval between Figs. 2a and 3a). Of course, the *total* fields of angular momentum and absolute vorticity shown in these figures cannot be compared in this way.

The existence of the meridional circulation ensures that the model temperature is not in thermal equilibrium, since the adiabatic cooling associated with this induced flow in the region of the T_e maximum counters the external diabatic heating. In fact, even at this relatively modest viscosity, the effect on T is substantial, with the maximum induced temperature being only about two-thirds of T_e. The effect on relative vorticity is even more marked (since the gradients of T are also weakened), so much so that the anticyclonic relative vorticity maximum near the maximum forcing is reduced to about one-half of the inviscid prediction. Consequently, the predicted transition out of the TE regime is modified substantially; according to the linearized, viscous model, zero absolute vorticity near the top boundary above the forcing will occur at $\Theta_e = 7.2$ K, rather than 3.45 K as predicted by inviscid theory. (The violation of the regularity condition in the linear solution for $\Theta_e = 12.1$ K is evident in the change of sign in absolute vorticity above the heated region in Fig. 3d.)

[2] This remains true under the forcing (9), even when magnitude of relative vorticity $|\zeta_r|$ becomes comparable with Coriolis parameter f, since $|\zeta_r|/f \approx U/fL$, where U and L are, respectively, velocity and length scales, whereas the relative contribution of the nonlinear term in (4) is U/fa, which is much smaller since $L \ll a$.

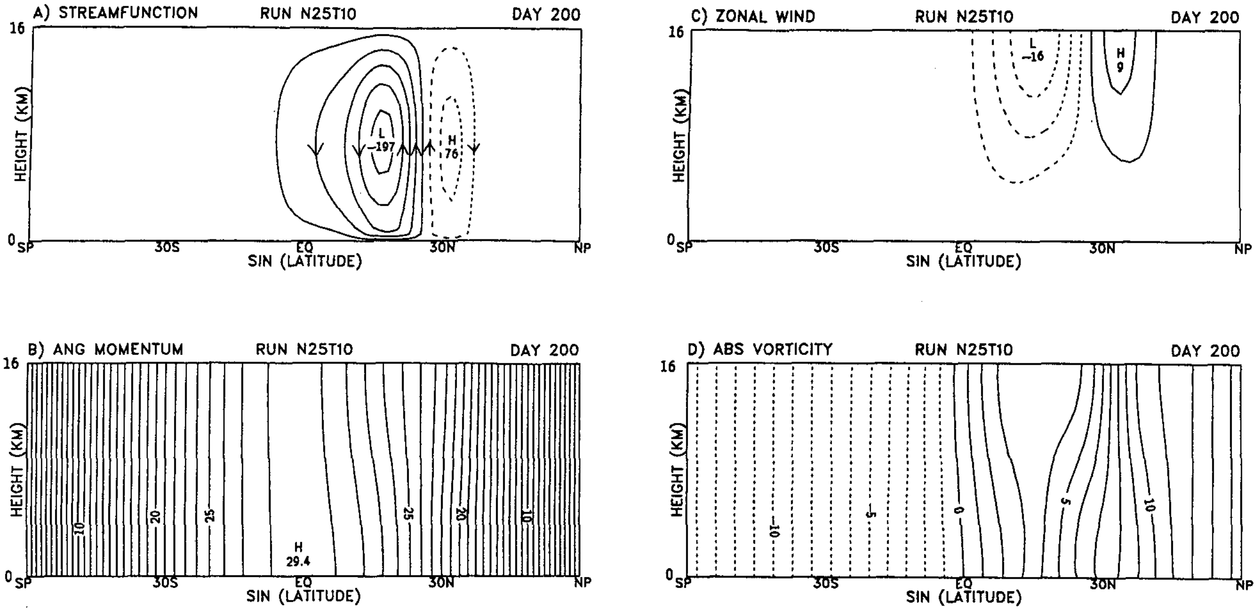

FIG. 2. Solution for $\Theta_e = 3.0$ K after 200 days of integration (by which time the flow is almost steady). Shown are (a) streamfunction χ ($m^2\ s^{-1}$; contour interval 40 $m^2\ s^{-1}$), (b) angular momentum density M ($10^8\ m^2\ s^{-1}$; contour interval $1 \times 10^8\ m^2\ s^{-1}$), (c) zonal wind u ($m\ s^{-1}$; contour interval 5 $m\ s^{-1}$) and (d) absolute vorticity ζ_a ($10^{-5}\ s^{-1}$; contour interval $1 \times 10^{-5}\ s^{-1}$). In (b) and (d), the contour values include zero, so the contours (interval Δ) are at 0, $\pm\Delta$, $\pm 2\Delta$, etc.; in (a) and (c), the zero line is not plotted, the contour values being $\pm\Delta/2$, $\pm 3\Delta/2$, etc.

This transition is evident near $\Theta_e = 7.2$ K in the model behavior; it shows up clearly in Fig. 4, which shows how $|\chi|_{max}$, the maximum value of the streamfunction of the meridional circulation (and thus the mass flow in the strongest, equatorward cell), varies with Θ_e in a series of numerical experiments. Predictions from the linearized viscous model are represented by the shallow-sloped dashed line; solutions are found to be close to this for Θ_e less than about 7 K, and these have the qualitative appearance of the TE response (in particular, ζ_a remains positive above the forcing region). For larger values of Θ_e, however, the character

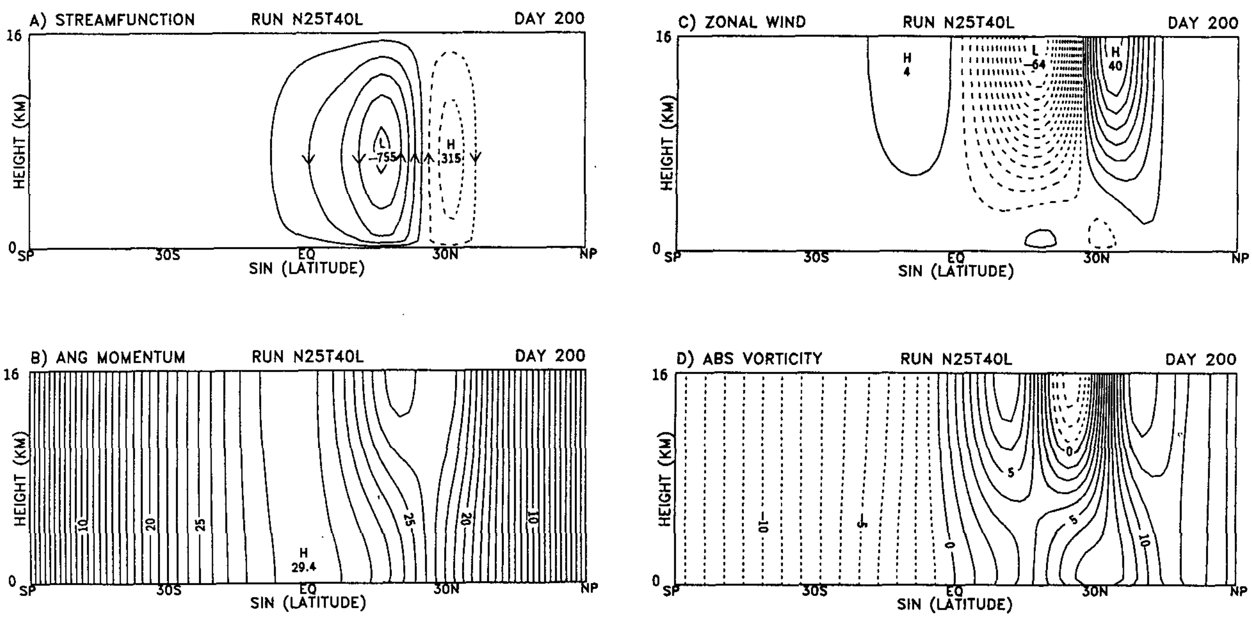

FIG. 3. Solution from the linearized model for $\Theta_e = 12.1$ K after 200 days of integration. Layout as for Fig. 2. Contouring is as for Fig. 2 except that the contour interval in (a) is 160 $m^2\ s^{-1}$ (i.e., four times that of Fig. 2).

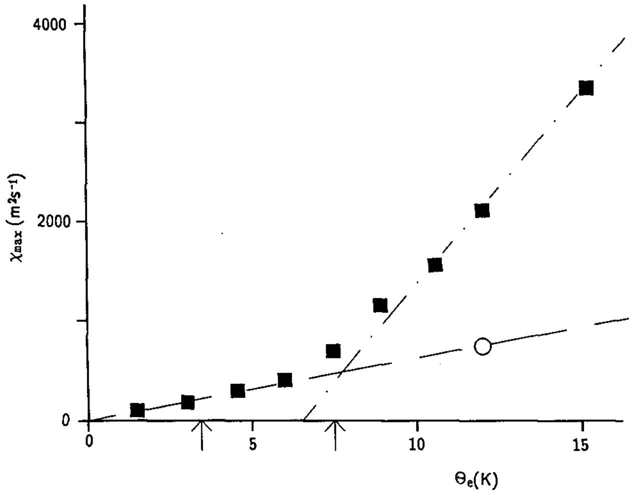

FIG. 4. Dependence of the maximum value of the steady streamfunction χ on the forcing amplitude Θ_e. The squares show points determined from results of the complete, nonlinear model. The circle shows a result from the linearized model, and the dashed line the linear dependence of χ_{max} on Θ_e. The steeper, dash–dot, line is drawn by eye and has no other significance. The two arrows show the theoretical value of Θ_e at which the TE solution becomes irregular; the left arrow is for the inviscid case, the right arrow for $\nu = 2.5$ m^2 s^{-1} according to the linear model results.

of the solution changes; the slope of the χ_{max}/Θ_e curve shown in Fig. 4 changes abruptly[3] and the induced circulation becomes broader; in fact, the streamfunction becomes more extensive, the descent region spreading into the tropics of the opposite hemisphere. An example of the solution in this regime ($\Theta_e = 12.1$ K) is shown in Fig. 5. This behavior is clearly nonlinear (e.g., note the distortion of the ζ_a contours; also compare the meridional circulation with the linear results in Fig. 3a for the same parameters). These results show other characteristics of the AMC solution: the absolute vorticity is close to zero (and angular momentum density relatively uniform) over and equatorward of the forced region at upper levels. The apparent nonconservation of angular momentum in the shallow cross-equatorial flow at upper levels is presumably an indication of the effects of viscosity there. Note that the upper-level easterly jet, rather than being confined to a narrow region on the equatorward flank of the forcing as in the subcritical cases, now extends right across the equator.

The association between the transition found in these experiments and that predicted theoretically in the inviscid limit seems clear, despite the effects of finite viscosity in the former. The apparent discontinuity in slope evident in Fig. 4 and the associated change in response occur close to the critical value of Θ_e predicted by the linearized, viscous model as the value at which ζ_a vanishes above the forcing. The association of the transition with the vanishing of ζ_a is also evident in the absolute vorticity structure of the response. However, a more direct, quantitative comparison with the inviscid theory is precluded by the impact of the viscously driven circulation on the modeled heat budget.

c. *Development of the flow*

In order to facilitate comparison of the model results with the steady, inviscid theory, we have focused the discussion thus far on the steady case. The evolution of χ_{max} for a selection of cases with $\nu = 2.5$ m^2 s^{-1} is shown in Fig. 6. In case (a), the subcritical case ($\Theta_e = 3.0$ K), the circulation spins up on a time scale of around 40 days before decaying to its steady value. The strong early circulation is not inhibited significantly by the constraint of angular-momentum conservation, since the state is far from steady at this stage: the angular momentum contours are advected by the flow. A strongly supercritical case ($\Theta_e = 12.1$ K)—curve (c)—initially evolves in the same way. However, the decay phase of the quasi-linear subcritical cases does not occur; rather, the circulation intensifies further (as ζ_a has become small) as it asymptotes to its steady value. Evolution is slower in cases close to the transition, as illustrated by curve (b).

d. *Relationship with Schneider's transition*

It is emphasized that the transition described here, between the quasi-linear TE solution and a nonlinear, approximately angular momentum–conserving regime, is quite distinct from that discussed by Schneider (1983). Schneider considered a δ-function profile in latitude for T_e, which corresponds to taking $\Delta\varphi \to 0$ in (8). Since the forcing amplitude at the predicted TE-to-AMC transition is proportional to $\Delta\varphi^2$, the TE regime vanishes as $\Delta\varphi \to 0$.

In the present experiments the supercritical response illustrated in Fig. 5 takes the form of (in Schneider's terminology) a "global" (i.e., cross-equatorial) AMC circulation. There is no indication in these results of Schneider's "local" AMC regime in which the circulation, though angular momentum conserving, remains localized in the forced hemisphere. We have in fact performed some experiments with a narrower forcing profile (with $\Delta\varphi = 10°$). In this case the results do appear to produce two transitions: first, a TE-to-local AMC transition and subsequently a local-to-global transition. Higher resolution, however, will be required to explore parameter space more fully than we have done thus far (with narrower forcing profiles).

4. Discussion

The results we have obtained appear to indicate that the quasi-linear TE regime breaks down at attainable forcing magnitudes. While the critical magnitude of

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[3] The dot-dashed line in Fig. 4 is fitted by eye to the points on the figure and has no other significance.

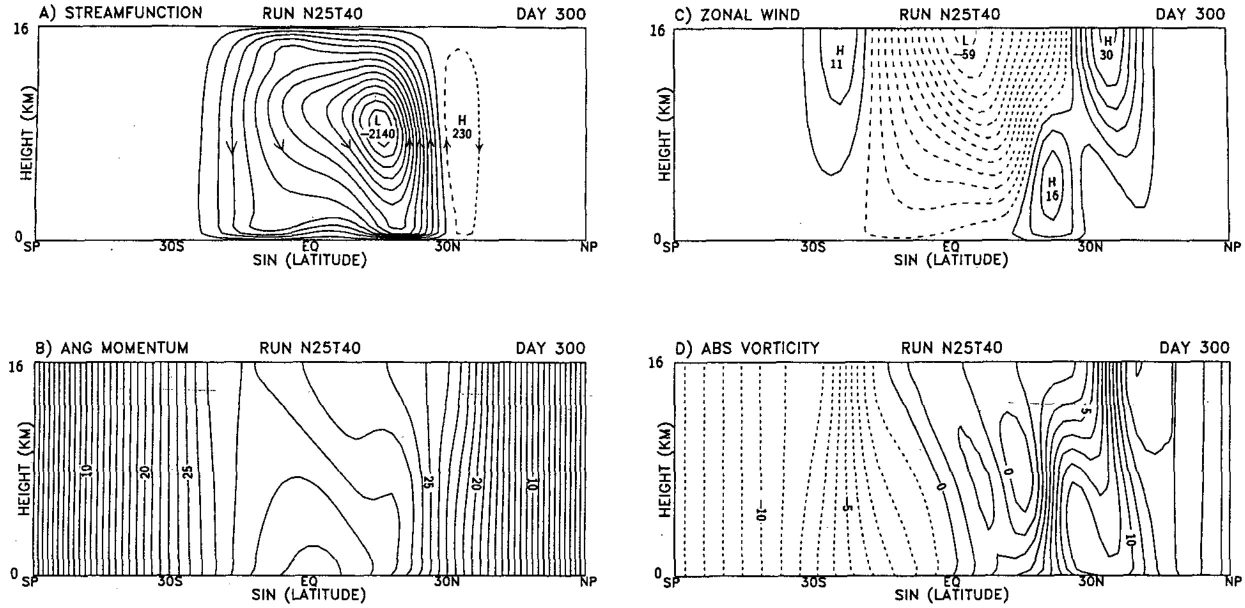

FIG. 5. Solution for Θ_e = 12.1 K after 300 days of integration (by which time the flow is almost steady). Layout as for Fig. 2. Contouring is as for Fig. 2 except that the contour interval in (a) is 160 m² s⁻¹ (i.e., four times that of Fig. 2a and the same as that of Fig. 3a).

7.2 K found in the "standard" case is quite substantial, note that in this case the width, 2 $\Delta\varphi$, of the forcing is 30° of latitude; since, from (7), T_c varies as $\Delta\varphi^2$, the critical value for a forcing of width 10° would be less than 1 K. There are reasons to suggest, therefore, that the AMC regime may be relevant to real thermally forced tropical flows other than the Hadley cell, monsoonal flows being perhaps the most obvious candidate. However, the study thus far has been confined—quite deliberately—to an investigation of the basic fluid dynamics of a highly simplified situation, in order to isolate the dynamical characteristics described here in the absence of any complicating factors such as moisture transport/latent heat feedback. Several issues need to

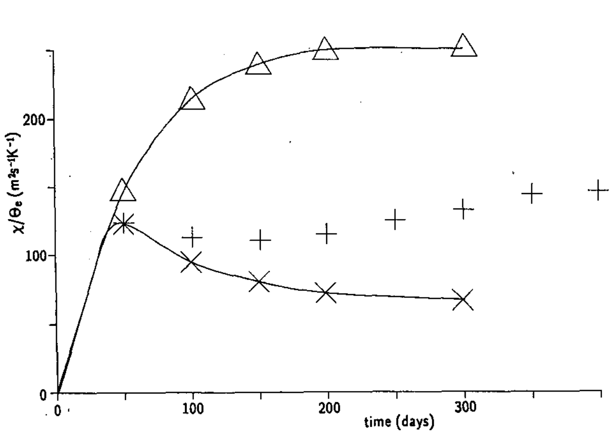

FIG. 6. Development of the circulation with time. Plot shows the streamfunction, χ_{max} scaled by Θ_e, as a function of time. Cases (values of Θ_e) are: (×) 3.0 K, (+) 7.5 K, (△) 12.1 K.

be resolved before its implications for realistic cases can be assessed.

One such issue is the restriction to zonal symmetry. With the constraint of angular momentum conservation being fundamentally a property of zonally symmetric flows, one might think that these results will have no application to the more general case. However, it can be shown (Schneider 1987) that similar criteria (in particular, the importance of vanishing absolute vorticity) apply in the three-dimensional case. There is some interest, therefore, in studying the response to thermal forcing that is localized in both latitude and longitude to see whether any similar threshold exists.

In order to facilitate comparison of the model results with the steady, inviscid theory, we have focused on discussion of the steady circulation. However, as remarked earlier, in some cases evolution to steadiness takes as much as 400 days, that is, much longer than the time available to most thermally forced tropical circulations. There are three basic time scales in the problem at hand: the thermal adjustment time α^{-1} (10 days in all cases described here), the viscous dissipation time, and the turnover time of the circulation cell (both of these being around 100 days in these experiments). These latter two appear to determine the long adjustment time of the entire system. On the whole, however, the general character of the solution is established more rapidly than this in most cases. The characteristics of these steady circulations may therefore be of relevance to real flows with subseasonal time scales, but this is yet to be demonstrated.

A third limitation of this system is that the thermal forcing is prescribed (through the equilibrium temperature) as an external parameter. In practice, of course, much of the tropical forcing arises from latent heat release associated with deep convection, which in turn is strongly influenced by the circulation; the implied feedback processes are not represented in these calculations. This is a major limitation in any attempt to relate these results to such phenomena as monsoon onset, since there the thermal forcing is a part of the onset process itself. Indeed, model studies such as those of Webster and Chou (1980) indicate that the moisture feedback may be important in determining the general characteristics of these circulations. Furthermore, convective heating may necessarily be accompanied by convective angular momentum transport (e.g., Emanuel 1983), which is not accommodated in this model and could impact on the arguments we have presented. Nevertheless, the results presented here suggest that there may be a purely dynamical "switch" that can control the nature of the response to any forcing; it will be of interest to investigate the possible occurrence of this phenomenon in more realistic models that explicitly incorporate such effects.

Acknowledgments. We thank Richard Lindzen for drawing our attention to the work of Schneider (1983), and Tim Dunkerton and Ed Schneider for helpful reviews. The numerical model described in section 3 was originally developed with the assistance of Kevin Hennessy of CSIRO, Australia.

APPENDIX

1. *Condition for regularity of the TE solution*

It is assumed that the viscous force per unit mass in (1) may be represented by vertical viscosity:

$$\mathcal{F} = \nu a \cos\varphi \frac{1}{p}\frac{\partial}{\partial z}\left(p\frac{\partial u}{\partial z}\right) = \nu \frac{1}{p}\frac{\partial}{\partial z}\left(p\frac{\partial M}{\partial z}\right). \quad \text{(A1)}$$

For weak viscosity ν, we seek a regular expansion in Ekman number $E = \nu/(\Omega D^2)$:

$$[M, v, w, T] = \sum_{n=0}^{\infty} [M_n, v_n, w_n, T_n]E^n. \quad \text{(A2)}$$

The TE solution (5) has

$$v_0 = 0; \quad w_0 = 0; \quad T_0 = T_e;$$

and $M_0 = M_e$, where

$$M_e^2 = \Omega^2 a^4 \cos^2\varphi + a^2 D \frac{\cos^3\varphi}{\sin\varphi}\frac{\partial \hat{T}_e}{\partial \varphi}. \quad \text{(A3)}$$

Now consider the next order in the expansion. The angular momentum equation (1) gives, using (A3),

$$\mathbf{u}_1 \cdot \nabla M_0 = \Omega D^2 \frac{1}{p}\frac{\partial}{\partial z}\left(p\frac{\partial M_0}{\partial z}\right), \quad \text{(A4)}$$

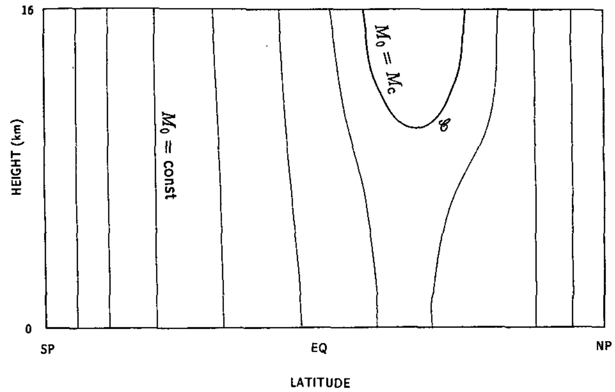

FIG. A1. Schematic diagram of distribution in the latitude–height plane of angular momentum density M_0, showing a region of maximum M_0 and the contour $\mathcal{C}$ on which $M_0 = M_c$.

where $\mathbf{u}$ is the meridional velocity vector (v, w). The regularity of the zero-viscosity TE solution (A3) for $E \to 0$ rests on the convergence of the expansion (A2). Consider, however (following Schneider 1977 and Held and Hou 1980), (A4) in the vicinity of an extremum in M_0; a case with M_0 maximizing on a rigid, stress-free, upper boundary is illustrated in Fig. A1. Integration of $p \times$ (A4) within the region bounded by the contour $\mathcal{C}$ (on which $M_0 = M_c$, a constant) and using the divergence theorem gives

$$M_c \oint_{\mathcal{C}} p\mathbf{u}_1 \cdot \mathbf{n} ds = \Omega D^2 \oint_{\mathcal{C}} p(\mathbf{k}\cdot\mathbf{n})(\mathbf{k}\cdot\nabla M_0)ds, \quad \text{(A5)}$$

where ds is the line element along $\mathcal{C}$, $\mathbf{n}$ the outward unit normal to $\mathcal{C}$ (and thus antiparallel to ∇M_0), and $\mathbf{k}$ the unit vertical vector. Now, the integral on the lhs of (A5) is simply the net mass flux across $\mathcal{C}$, which is of course zero in steady state. On the other hand, the integrand on the right-hand side is zero along the stress-free upper boundary and negative definite elsewhere. Therefore, the right-hand side of (A5) is negative and the expansion (A2) cannot be satisfied with any finite $\mathbf{u}_1$ whenever M_0 has an extremum. (The same is true for an interior extremum, since the right-hand side is negative both above and below.) Under these circumstances, then, the inviscid TE solution (5) is not a regular solution in the limit of small viscosity.

2. *Nonexistence of the AMC solution when the TE solution is regular*

The nonlinear balance equation (5) may be written, with $\mu = \sin\varphi$,

$$-Da^2 \frac{\partial \hat{T}}{\partial \mu} = \frac{\mu M^2}{(1-\mu^2)^2} - \Omega^2 a^2 \mu, \quad \text{(A6)}$$

where $\hat{T}$ is the vertically averaged temperature and $M(\mu)$ the total angular momentum at the top $z = D$.

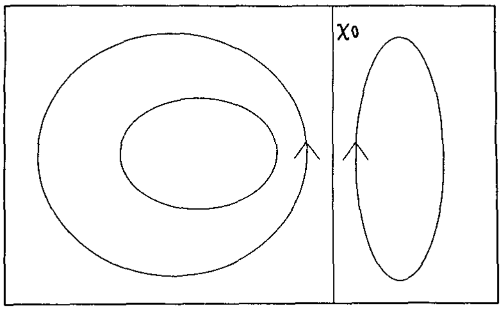

FIG. A2. Schematic figure of the meridional streamfunction, showing the dividing streamline χ_0. See text for discussion.

It has been assumed here, following Held and Hou (1980), that boundary-layer friction ensures that u may be assumed to be zero at $z = 0$. If the AMC solution has constant angular momentum M_a at $z = D$ and vertically averaged temperature $\hat{T}_a$, then

$$-Da^2 \frac{\partial \hat{T}_a}{\partial \mu} = \frac{\mu M_a^2}{(1-\mu^2)^2} - \Omega^2 a^2 \mu, \quad \text{(A7)}$$

while the angular momentum of the TE solution at $z = Z$ is $M_e(\mu)$, where

$$-Da^2 \frac{\partial \hat{T}_e}{\partial \mu} = \frac{\mu M_e^2}{(1-\mu^2)^2} - \Omega^2 a^2 \mu. \quad \text{(A8)}$$

[Note that we may define $M_e(\mu)$ in this way even when the TE solution is not valid.]

Now, consider the AMC solution and, in particular, the region of thermally forced upwelling depicted schematically in Fig. A2. Within the AMC region, $\hat{T} = \hat{T}_a$; elsewhere, $\hat{T} = \hat{T}_e$. If two circulation cells exist, as seems usual in such solutions (Lindzen and Hou 1989), we label the dividing streamline χ_0[4]. Following Held and Hou (1980) we assume that angular momentum is conserved everywhere outside of the boundary layer. At, $\mu = \mu_0$, the streamline χ_0, which has angular momentum M_a, carries boundary-layer angular momentum up to the upper levels. Within the boundary layer it is assumed that friction ensures a weak zonal flow such that the relative angular momentum is negligible there and therefore $M_a = \Omega a^2(1 - \mu_0^2)$. From (A7) it then follows that

$$\frac{\partial \hat{T}_a}{\partial \mu}(\mu_0) = 0. \quad \text{(A9)}$$

Note further that, in order for the circulation to maintain itself against surface drag, the region of vertically averaged rising motion must be *warmer* than average (since there must be a conversion from potential to kinetic energy to balance frictional losses), so $\hat{T}_a$ must be a *maximum* at μ_0.

Consider now the temperature structure in the region of μ_0, sketched in Fig. A3. Note that, since we know from (3) and (A9) that $wS = \alpha(T_e - T)$ for the steady flow and $\hat{T} = \hat{T}_a$ within the AMC region, $\hat{T}_e$ must exceed $\hat{T}_a$ in the vicinity of μ_0. If there are two cells, then somewhere on each side of μ_0, the flow is subsiding and the opposite is true, $T_e < T_a$. Therefore, there are two locations, $\mu = \mu_1, \mu_2$, where $\hat{T}_e = \hat{T}_a$.

Now, it is evident that

$$\frac{\partial \hat{T}_e}{\partial \mu}(\mu_1) > \frac{\partial \hat{T}_a}{\partial \mu}(\mu_1); \quad \frac{\partial \hat{T}_e}{\partial \mu}(\mu_2) < \frac{\partial \hat{T}_a}{\partial \mu}(\mu_2). \quad \text{(A10)}$$

Since both curves are continuous, there must be some point in $\mu_1 < \mu < \mu_2$, say at $\mu = \mu_s$, where the slopes are equal:

$$\frac{\partial \hat{T}_e}{\partial \mu}(\mu_s) = \frac{\partial \hat{T}_a}{\partial \mu}(\mu_s), \quad \text{(A11)}$$

and where, therefore, from (A7) and (A8),

$$M_a = M_e(\mu_s). \quad \text{(A12)}$$

Now, consider the region poleward of μ_s (i.e., $\mu > \mu_s$). From (A11) and (A12) and the second of (A10), it follows that, *somewhere* in $\mu_s < \mu < \mu_2$, the negative curvature of the $\hat{T}_e$ curve must exceed that of the $\hat{T}_a$ curve; that is

$$\frac{\partial^2 \hat{T}_a}{\partial \mu^2} - \frac{\partial^2 \hat{T}_e}{\partial \mu^2} > 0 \quad \text{somewhere in } \mu_s < \mu < \mu_2. \quad \text{(A13)}$$

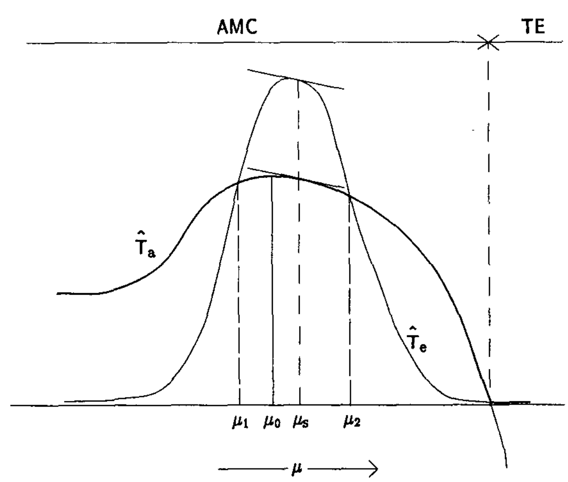

FIG. A3. Schematic figure showing vertically averaged temperature $\hat{T}$ as a function of $\mu = \sin\varphi$. Curves shown are $\hat{T}_a$, the AMC solution, and $\hat{T}_e$, the TE solution; the actual $\hat{T}$ is shown heavy. See text for discussion.

[4] It is possible, at least in principle, that only one cell exists. In that case the AMC/TE transition occurs at μ_2 in Fig. A3; the argument is otherwise unchanged.

Therefore, (A13) is a necessary condition that the AMC solution be physically acceptable.

Now, from (A8)

$$Da^2\left[\frac{\partial^2\hat{T}_a}{\partial\mu^2}-\frac{\partial^2\hat{T}_e}{\partial\mu^2}\right] = \lambda(\mu)(M_e^2 - M_a^2) + \frac{2\mu M_e}{(1-\mu^2)^2}\frac{\partial M_e}{\partial\mu}, \quad \text{(A14)}$$

where

$$\lambda(\mu) = \frac{1+3\mu^2}{(1-\mu^2)^3} > 0.$$

Now, in the regime in which the TE solution is regular, $\mu\partial M_e/\partial\mu < 0$ everywhere. From this fact and (A12) it follows that $M_e < M_a$ in $\mu > \mu_s$. But then both terms on the rhs of (A14) are negative poleward of μ_s, and therefore, the acceptability condition (A13) cannot be satisfied. Therefore, *the AMC solution cannot exist when the TE solution is regular.*

REFERENCES

Dunkerton, T., 1989: Nonlinear Hadley circulation driven by asymmetric differential heating. *J. Atmos. Sci.,* **35,** 2325–2333.

Emanuel, K. A., 1983: The Lagrangian parcel dynamics of moist symmetric instability. *J. Atmos. Sci.,* **40,** 2368–2376.

Held, I. M., and A. Y. Hou, 1980: Nonlinear axially symmetric circulations in a nearly inviscid atmosphere. *J. Atmos. Sci.,* **37,** 515–533.

Hide, R., 1969: Dynamics of the atmospheres of the major planets with an appendix on the viscous boundary layer at the rigid boundary surface of an electrically conducting rotating fluid in the presence of a magnetic field. *J. Atmos. Sci.,* **26,** 841–853.

Lindzen, R. S., and A. Y. Hou, 1988: Hadley circulations for zonally averaged heating centered off the equator. *J. Atmos. Sci.,* **45,** 2416–2427.

Schneider, E. K., 1977: Axially symmetric steady-state models of the basic state for instability and climate studies. Part II: Nonlinear calculations. *J. Atmos. Sci.,* **34,** 280–297.

——, 1983: Martian great dust storms: Interpretive axially symmetric models. *ICARUS,* **55,** 302–331.

——, 1987: A simplified model of the modified Hadley circulation. *J. Atmos. Sci.,* **44,** 3311–3328.

——, and R. S. Lindzen, 1977: Axially symmetric steady-state models of the basic state for instability and climate studies. Part I: Linearized calculations. *J. Atmos. Sci.,* **34,** 263–279.

Webster, P. J., and L. C. Chou, 1980: Seasonal structure of a simple monsoon system. *J. Atmos. Sci.,* **37,** 354–367.