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A THEORY OF GLACIER SURGES

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Abstract. We propose a model of glacier flow that is capable of explaining temperate glacier surges. The laws of conservation of mass and momentum are supplemented by the prescription of a sliding law that gives the basal shear stress τ as a function of the basal velocity u and the effective pressure N. The effective drainage pressure N is determined by a simple study of the subglacial hydraulic system. Following Röthlisberger, we determine $N = N_R$ for the case of drainage through a single subglacial tunnel. Alternatively, following Kamb, we find that the corresponding theory for a linked-cavity drainage system yields $N = N_K < N_R$. Furthermore, the stability of each drainage system depends on the velocity u, such that for large enough u, there is a transition from tunnel to cavity drainage. Consequently, one can write N = N(u). We then find that the sliding law $\tau = \tau(u)$ is multivalued, and hence so also is the flux/depth relation Q = Q(H). An analysis of the resulting system of equations is sketched. For large enough accumulation rates, a glacier will undergo regular relaxation oscillations, resembling a surge. The surge is triggered at the point of maximum stress; from this point two hydraulic transition fronts travel up and down glacier to calculable boundary points. The speed of propagation is the order of 50 metres an hour. At these fronts, the tunnel drainage system collapses, and a high water pressure cavity drainage system is installed. This activated zone has high velocities and quickly relaxes (surges) to a quasi-equilibrium state. This relaxation is much like opening a sluice gate, in that a large wave front propagates forward. Behind this wave front, the velocity can decay oscillatorily, and thus the flow can be compressive. We conclude with some discussion of the effects of seasonal variation and of prospects for the current theory's applicability to soft-bedded glaciers.

Introduction

Surging glaciers exhibit large-scale relaxational periodic motions. There is typically a long, quiescent phase (~20-100 years) when the glacier is overextended, thin, and slow moving. During this phase the glacier retreats and thickens. At some critical thickness, a surge is triggered, in which ice in a large "reservoir" region begins to move rapidly downslope. This fast phase typically lasts a year or two and is equally abruptly terminated, with the glacier again overextended and thin. The cycle then repeats itself.

In effect, glacier surges have been monitored for at least a hundred years, and a major conference (see Canadian Journal of Earth Sciences, vol. 6, pp. 807-1018, 1969) was devoted to their study. Meier and Post [1969] summed up many of their characteristics and outlined directions for future research.

It is apparent that the only way of achieving the sorts of velocities encountered in surges (kilometres per year) is by sliding at the glacier bed, and thus theoretical explanations have centred on the prescription of a physically realistic sliding law, relating basal shear stress τ to the basal velocity u. The notion that sliding was a key process led Robin [1955] to suggest (essentially) that surges were triggered by a thermal instability at the glacier bed. This idea was taken up by Clarke and others [Clarke and Goodman 1975; Jarvis and Clarke, 1975; Clarke, 1976; Clarke and Jarvis, 1976], who showed that some surging glaciers in the Yukon were cold but partially temperate at their base, and Clarke [1976] suggested that it was possible that the surging behaviour was thermally controlled. Subsequently, Clarke et al. [1977], Cary et al. [1979], Paterson et al. [1978], and Yuen and Schubert [1979] related the proposed thermal instability mechanism to that of thermal runaway, which is a catastrophic instability characteristic of stress-driven flows with temperature-dependent viscosity in fixed domains [Gruntfest, 1963]. However, in the glaciological context, one can show [Fowler, 1980; Fowler and Larson, 1980 a, b] that the free boundary nature of the problem renders the solution (in one approximate limit) both unique and linearly stable. Whether thermal runaway is viable is thus not clear. Moreover, surges require large sliding velocities, which requires some further sliding instability mechanism. As suggested by Clarke [1976], it may be that the thermal regime in subpolar surging glaciers exerts more of a regulatory effect than a causative one.

Evidently, an explanation of surges is intimately bound up with a realistic sliding law at the glacier bed. Almost all the theory that has been done on this problem has considered the glacier bed to be hard (i.e., rigid, impermeable) rather than soft (i.e., ice resting on permeable till). Hard beds have been studied by many authors (for reviews, see Lliboutry [1979] and Weertman [1979]); a treatment for soft beds has been sketched by Jones [1979] and Boulton and Jones [1979]. Many glaciologists now believe soft beds to be the norm, for example, under Trapridge Glacier [Clarke et al., 1984]. This paper will consider hard beds, which may be a potentially reasonable assumption for Variegated Glacier [Kamb et al., 1985]. However, we will offer some observations about soft beds in the concluding section.

The idea has been around for some time that a multiple-valued sliding law (u a triple-valued function of τ) could lead to surging behaviour

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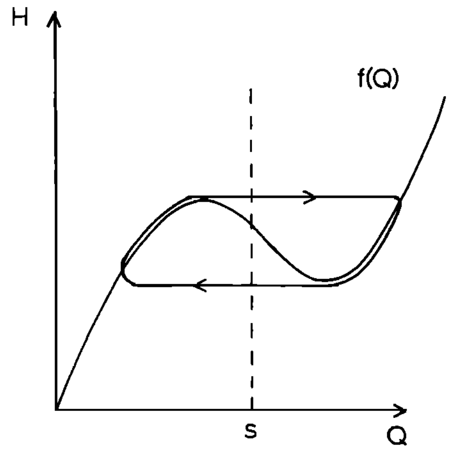

Fig. 1. Phase portrait of the relaxation oscillation of (1).

[Lliboutry, 1969; Hutter, 1982; Johnson and McMeeking, 1986], and Lliboutry [1968] derived such a law in the case of a bed consisting of four superimposed sine waves. Although Lliboutry emphasized the importance of the effective pressure N in his sliding law, it plays a passive role in the multivaluedness of $\tau(u,N)$; specifically he finds $\tau(u,N)$ can be nonmonotonic at constant N, as u varies.

It is an easy concept to anyone familiar with nonlinear relaxation oscillators (e.g., the van der Pol oscillator) that such a multivaluedness will produce in the system a periodic solution of relaxational type, i.e., a surge [Kevorkian and Cole, 1981], and such an idea seems to lie behind various efforts over the years to understand surges [Palmer, 1972; Budd, 1975; Johnson and McMeeking, 1986]. As an analogy, consider the simple pair of ordinary differential equations

$$\dot{H} = s - Q$$
$$\varepsilon\dot{Q} = H - f(Q) \qquad (1)$$

where one should think of H as depth, Q as flux, and s as mass balance. If $f(Q) > 0$ and is cubic (so f^{-1} is S-shaped), then it is easy to do a phase plane analysis. If f has turning points at $Q = Q_1$ and $Q = Q_2$ (see Figure 1), then if $Q_1 < s < Q_2$, the system (1) has a steady periodic relaxation oscillation, if ε is small. The analogy with surging (and it is only an analogy) is that basal stress τ is proportional to depth H, so that if $\varepsilon \to 0$, (1) gives approximately a "sliding law"

$$Q \approx f^{-1}(H) \qquad (2)$$

which is multivalued.

As we shall see, the analogy is quite a good one. The question is, can we derive a spatially variant model that will have the same behaviour; in particular, how does a space-varying relaxation oscillator actually work, and can we compare it in quantitative and qualitative detail with observations? Notice, in particular, that it is important for (1) that ε, if small, not actually be zero.

The present theory is an attempt to flesh out the discussion above. We first seek to establish a realistic physical process whereby the sliding law can be multivalued. Having elucidated this, we proceed to sketch the analysis of the resultant equations, which shows the detailed evolution of a surge. Much of the present work is based on pioneering theories of glacier sliding by Lliboutry [1968], subglacial hydrology [Röthlisberger, 1972], and particularly on extraordinarily well-documented observations of the 1982-1983 surge of Variegated Glacier by Kamb et al. [1985]. Theoretical offshoots of these observations, currently in the process of publication, focus primarily on the detailed, even day-to-day, sequences of events during a surge. Particular emphasis has been laid on the water pressure fluctuations and on the short-term time variability of the surge (due to seasonal variations in accumulation/rainfall, etc.). This paper takes the larger view that seeks to understand the long-term periodic repetition of surges in a predictive manner, which, however, neglects such short-term detail. Thus the approach embodied here is a complementary one. It is my belief that seasonal effects are indeed, in the present context, a detail.

Sliding Law With Cavitation

In this section we summarize results from two other papers [Fowler, 1986, 1987] on the sliding law. Fowler [1986] studied the two-dimensional Nye-Kamb [Nye, 1969, 1970; Kamb, 1970] sliding problem when cavities are included, and also regelation is ignored (i.e., small length scales are excluded). He corroborated Lliboutry's [1968] finding that for a regular array of identical humps, the stress first increases with velocity and then decreases to zero. Lliboutry pointed out that this result depends on the simultaneous drowning of the entire bed at large u. The precise form of the sliding law takes the form

$$\tau = Nf(u/N^n) \qquad (3)$$

where N is the effective pressure and n is the exponent in Glen's law. For n = 1, this is rigorous, for n > 1, it is a reasonable, but a heuristic, generalization.

For a more realistic hard bed (i.e., consisting of many superimposed bumps on varying scales), the problem is nonlinear (even if n = 1) because of the free-boundary nature of the cavities, and one must resort to some modeling procedure. The method we have adopted is that of superimposing bumps of asymptotically distinct scales, so that in a formal solution they do not interfere with each other. This is the procedure adopted by Lliboutry [1968, 1979] and Weertman [1964]. With this formal procedure in mind, we are able to derive a renormalization scheme, which predicts that cavities extend to all scales. Using the (approximate) solution to this, we can sum the contributions to the stress from the various bumps and compute a sliding law. On the assumption of a self-similar bed (the amplitude γ of bumps of wavelength λ is given by $\gamma \sim \lambda^{\alpha/2}$), we find that the dimensionless sliding law is given by

$$\tau^* = k(\mu_2 N^*)^a (u^*/\mu_1)^{(1-a)/n} \qquad (4)$$

where $k = O(1)$ is a measure of the bed profile; τ^*, N^*, u^* are dimensionless stress, effective pressure, velocity defined relative to their dimensional counterparts τ^*, N, u by

$$\tau/\tau^* = \rho g d\beta \equiv [\tau]$$
$$N = [\tau]N^*/\beta \qquad (5)$$
$$u/u^* = A[\tau]^n d$$

and here d, β would be (typical) depth and slope of a glacier, A is the multiplicative constant in Glen's law, and

$$\nu = y_0/x_0$$
$$\sigma = x_0/d \qquad (6)$$

determine dimensionless measures of roughness. Here x_0 and y_0 are a typical length and amplitude scale for the rough bed. The exponents in (4) are determined by

$$a = 1 - n\left(\frac{\alpha - 2}{4 - \alpha}\right) \qquad (7)$$

and the dimensionless parameters μ_1 and μ_2 in (4) are given by

$$\mu_1 = \sigma/\nu^{n+1}, \quad \mu_2 = \nu/\beta \qquad (8)$$

The laborious definition of (4) is made so as to be convenient for later analysis: it tells us when sliding should be important, as we can see from (5) that we expect, for a glacier, $\tau^* \sim O(1)$, $N^* \lesssim 1$, and $u^* \sim O(1)$, unless shearing is negligible. For example, if x_0 = 5m, y_0 = 1m, d = 100m, then $\sigma \sim 1/20$, $\nu \sim 1/5$, and if $\beta \sim 0.1$, then $\mu_1 \sim 30$, $\mu_2 \sim 2$, so that if a = 1/4, u = 3, then $u \sim 15^{1/4} \approx 2$; thus sliding should be significant in that case.

The sliding law (4) is a generalization of Weertman's [1957] law and, as such, appealingly simple. It has been used (with no a priori justification) to test laboratory and field sliding data by Budd et al. [1979], with values n = 2 and α = 2.5, and Bindschadler [1983], who found similar values (though he forced his data on to values a = 1/(n + 1)).

The physical reason for the more realistic (4) being monotonic with u is because of the observation that for simple monochromatic bumps, the stress continues to increase with velocity until the cavity attached to one bump begins to reach the next bump. It is only because this happens simultaneously with all bumps that the stress can decrease. In reality, however, if bumps at one wavelength begin to be drowned, the larger bumps with longer wavelength will not be and will take up the extra stress. In consequence, stress will continue to increase as long as the bed cannot be completely flooded. This is likely to be the general case.

It may be objected that the neglect of regelation renders the above results suspect. In Nye's theory, the crucial importance of regelation is to allow a convergence of the stress for self-similar bedrocks. In particular, for the amplitude $\gamma(\lambda)$ (defined before (4)) given by

$$\gamma \sim \lambda^{\alpha/2} \qquad (9)$$

convergence at small wavelengths (with n = 1) requires only that $\alpha > 2$, whereas this is $\alpha > 4$ if regelation is suppressed. However, one can show that, when cavitation is considered, convergence of the resulting stress at small wavelengths is assured if $\alpha > 2$ (in the absence of regelation). It follows that, so long as $\alpha > 2$, regelation may be consistently ignored, since the regelative drag contribution will be uniformly small [Fowler 1986, 1987].

We have already mentioned the requirement that $\alpha > 2$. Convergence at long wavelengths then requires the more complete

$$2 < \alpha < 2\left(\frac{n + 2}{n + 1}\right) \qquad (10)$$

For n = 3, this is $2 < \alpha < 2.5$. It is either interesting or important that values of around 2.4 have been quoted [Benoist, 1979; Hallet, 1979]. If we put α = 2.4 and n = 3 into (7), we find (1 - a)/n = 1/4 = a, so that (4) is

$$\tau^* \propto (N^* u^*)^{1/4} \qquad (11)$$

not radically different from Budd et al's [1979] and Bindschadler's [1983] results (corresponding to α = 2.5, n = 2).

Hydraulic Drainage

Subglacial water drainage typically occurs through one or more ice tunnels at the bed. The mechanism of this drainage was described by Röthlisberger [1972]. (See also Nye [1976].) The salient features are the following. A single cylindrical channel in temperate ice tends to close because of viscous contraction of the ice, if the ice overburden pressure p_i is greater than the channel water pressure p_w. This tendency is counteracted by melting of the channels due to turbulent energy dissipation by the flowing water. A dynamical steady state can exist (in the absence of external variations in the water flux to the channel) that is stable, and in which the effective pressure $N = p_i - p_w$ is approximately given by

$$N = N_R = \left[\left(\frac{v_i \rho_w g_s}{KL}\right)\left(\frac{\rho_w g_s}{N_T}\right)^{3/8}\right]^{1/n} Q^{1/4n} \qquad (12)$$

Here $v_i = 1/\rho_i$ is the specific volume of ice, ρ_w is the density of water, K is a viscosity constant proportional to A in (5), L is latent heat, $g_s = g\beta$ is the downslope acceleration, and N_T is a turbulent drag coefficient [Nye, 1976]. The single assumption involved in (12) is that the downslope water pressure gradient $\partial p_w/\partial x \ll \rho_w g_s$, which is tantamount to assuming $d/\ell \ll \beta$, where d and ℓ are depth and length scales for the flow. Thus d/ℓ is the aspect ratio, typically of $O(10^{-2})$, whereas β is more like $O(10^{-1})$. In many cases (12) will be quite realistic, sufficiently far from the snout. Further, the water flux Q varies with distance due mainly to input from the surface via crevasses, etc. However, since n = 3, the variation of $Q^{1/4n}$ is very slow, and to a good approximation we can take Q in (12) equal to its outlet value at the terminus. Thus drainage may be considered to be prescribed. With these two assumptions, N_R in (12) is a constant.

Notice, in particular, that Q increases if p_w decreases, that is, $\partial Q/\partial p_w < 0$. It follows from this that a network of channels is unstable and will break down to form a single main channel. This may be seen by the following argument [Röthlisberger, 1972]. If two neighbouring channels have fluxes Q_1, Q_2 and corresponding pressures p_1 and p_2 (at some x), then in equilibrium $p_1 = p_2$, $Q_1 = Q_2$ (different pressures cannot be maintained since the glacier bed is not watertight [Lliboutry, 1968]). Now if $p_1 > p_2$, water flows from conduit one to conduit two and thus Q_2 increases, Q_1 decreases, and so $p_1 - p_2$ increases; this positive feedback ensures that coexisting channels are not stable.

It was pointed out by Kamb et al. [1985], on the basis of observations of Variegated Glacier, that an alternative (linked-cavity) drainage system is possible. Here large tunnels are absent, and drainage takes place through cavities at the bed, linked by joints and striae at the bed. However, a calculation shows that small striae on their own will not suffice to carry a reasonable drainage, and so some of the inter-cavity connections must enlarge (by viscous heating) to form miniature Röthlisberger channels. One thus has drainage through cavities connected by a network of small Röthlisberger channels. One can carry out the same analysis as before. The result that is obtained depends on what one assumes about the hydraulic potential gradient in the passageways [Walder, 1986]. If we suppose that the mean hydraulic gradient is $\rho_w g_s$ as for the Röthlisbrger channel, then we find [Fowler, 1987] that N for this case is much lower and, concomitantly, the degree of cavitation much higher. The result is that

$$N = N_K = \delta N_R \tag{13}$$

where

$$\delta = s n_K^{-1/4n} \tag{14}$$

where n_K is the number of cavities across the width of the glacier (or, more precisely, the number of passageways across the width), and s is the area fraction of bedrock that is cavity-free. If we take $n_K \sim 10^3$, n = 3, and s = 1/2, we have

$$\delta \approx 1/4 \tag{15}$$

Increased tortuosity of the passageways may lead to lower values of δ [Walder, 1986]. Thus the linked-cavity drainage system operates at much higher water pressure. This has already been explained by Kamb et al. [1985] and was the subject of a talk by Kamb [1985]. Further, one finds that in this case, $\partial Q/\partial p_w > 0$, so that the linked-cavity system is a stable network.

It remains to ascertain the stability of a combined system. This is easily done using a reservoir model, and one finds that tunnel drainage is stable if

$$\Lambda = \frac{\nu u}{x_0 A N^n} < \Lambda_C = (3nS_R/A^*)^{(4-\alpha)/\alpha} \tag{16}$$

where A^* is essentially of the order of the total cavity cross-sectional area. A typical estimate is $\Lambda_C \approx 1/4$. If $\Lambda > \Lambda_C$, a central tunnel will collapse and linked-cavity drainage takes over. A typical theoretical value of N_R is ≈30 bars. (Values of 15 bars are more appropriate for Variegated Glacier [Kamb et al., 1985].) By comparison, a glacier of depth 100 m has p_i = 9 bars. If $N_R > p_i$, the Röthlisberger tunnel cannot be filled, and thus $p_w = p_a$, atmospheric pressure, and in this case, $N \approx p_i$. Thus more generally we have

$$N = \min(p_i, N_R) \tag{17}$$

(tunnel) and similarly,

$$N = \min(p_i, \delta N_R) \tag{18}$$

(cavity).

Finally, we render these relations dimensionless. For simplicity, suppose that $p_i > N_R$, as will be the case for a sufficiently deep glacier (≳300 m). Then Λ in (16) is given in terms of the dimensionless variables introduced in (5) and (8) by

$$\Lambda = u^*/\mu_1 \mu_2^n N^{*n} \tag{19}$$

and (17) and (18) give N^* implicitly as a function of u^* by

$$N^* = N_R^* \quad \Lambda < \Lambda_C \qquad N^* = \delta N_R^* \quad \Lambda > \Lambda_C \tag{20}$$

where $N_R^* \sim O(1)$. We denote the functional dependence of N^* on Λ as

$$N^* = g(\Lambda) \tag{21}$$

depicted in Figure 2. In reality, the transition at $\Lambda = \Lambda_C$ will take time: observations on Variegated Glacier are consistent with a relaxation time of ≲1 day. As a simple model of this relaxation, we thus put

$$\varepsilon^* \frac{\partial N^*}{\partial t^*} + N^* = g(\Lambda) \tag{22}$$

Here t^* is a dimensionless time, scaled with a convective time scale t_{conv} ~ 20-100 y. Then if t_{tunnel} is the tunnel collapse time ≲1 day, we have

$$\varepsilon^* = t_{tunnel}/t_{conv} \lesssim 10^{-5} \tag{23}$$

equation (22) is a model that deliberately obscures the actual physics of transition, in order to give a simple reflection of what the end result is. It is worthwhile to point out the similarity of (22) and the second equation in (1).

The discussion above follows Kamb's work quite closely. In writing (22), we have assumed that the transition from linked-cavity to tunnel system is at the same value of $\Lambda = \Lambda_C$ at which the transition from linked-cavity to tunnel occurs. This is not at all obvious and is made here purely for convenience. That transition has been studied by Kamb [1985] and a fusion of the two complementary viewpoints would be useful.

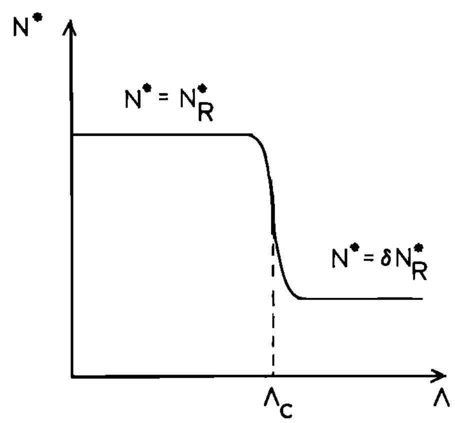

Fig. 2. Schematic representation of dependence of dimensionless effective pressure N* on the flow parameter Λ. At $\Lambda = \Lambda_c$ there is a transition from tunnel drainage to cavity drainage.

Flow Dynamics

When the drainage law (21), Figure 2, is combined with the generalized Weertman law (4), we find that the sliding law is multivalued, as indicated in Figure 3. As a consequence the ice flux, Q is a multivalued function of depth H, and we can anticipate the kind of periodic relaxational motions (surges) discussed in the introduction. This section aims to provide a summary of how the governing equations can predict surges.

Dimensionless equations that govern two-dimensional motions of glaciers over flat (but rough) sloping beds have been derived by Fowler [1982]. With the present sliding law, these take the form

$$H_t + (Hu)_x = s'(x) \qquad (24a)$$

$$H(1 - \mu H_x) + \alpha[Hu_x|u_x|^{(1/n)-1}]_x = Nf(\Lambda) \qquad (24b)$$

$$\epsilon N_t + N = g(\Lambda) \qquad (24c)$$

$$\Lambda = u/N^n \qquad (24d)$$

These equations are written for dimensionless variables representing depth H, ice velocity u, which is assumed to be predominantly by basal sliding, and effective pressure N. We have thus dropped the asterisks that previously distinguished dimensionless from dimensional variables. In (24), subscripts represent partial differentiation, and the equations represent conservation of mass (24a), with the balance term $s'(x) = ds/dx$ representing accumulation if $s' > 0$, ablation of $s' < 0$. The second equation (24b) is the sliding law (4), or more generally (3). The left-hand side is the shear stress. The term $H(1 - \mu H_x)$ corresponds exactly to the more familiar $\tau = \rho g h \sin\theta$, and the term in α represents approximately a small correction due to longitudinal stress that is important in highly compressive/extensional regions, i.e., where $\partial u/\partial x$ is large. This term is the same as the term G, derived by many authors [see Paterson, 1981, p. 100]. Its derivation in this context involves the formal assumption that $1 \ll u_{sliding}/u_{shearing} \ll 1/\delta$, where $u_{sliding}$ and $u_{shearing}$ are the basal and shearing components of motion, and δ is the aspect ratio d/ℓ of the glacier, where d and ℓ are representative depth and length scales. Typically $\delta \sim 10^{-2}$, and this assumption will form a reasonably realistic basis for study. For higher sliding velocities, further analysis is necessary, and this has been examined by McMeeking and Johnson [1985].

The function $g(\Lambda)$ is represented in Figure 2. The function $f(\Lambda)$ is typically monotonically increasing, and the generalized Weertman law (4) is represented by

$$f(\Lambda) = c\Lambda^{1/m} \qquad (25)$$

where

$$m = n/(1 - a) = \frac{4 - \alpha}{\alpha - 2} > n \qquad (26)$$

as follows from the right-hand inequality of (10). We shall henceforth take $f(\Lambda)$ as defined by (25).

The parameters α, μ, ϵ are typically small. Typical estimates (see Fowler [1982], also (23)) are

$$\alpha \sim 10^{-2} \qquad \mu \sim 10^{-1} \qquad \epsilon \sim 10^{-5} \qquad (27)$$

Consequently, the simplest procedure is to ignore them altogether, in a first approximation. Then we have only to solve the reduced system

$$H_t + Q_x = s'(x) \qquad Q \simeq F(H) \qquad (28)$$

where H ($\propto \tau$) as a function of Q (= Hu) is essentially the same as Figure 3, and so F(H) is as shown in Figure 4. This is a nonlinear first-order wave equation, but the multivaluedness renders it rather peculiar. Nonlinear waves are discussed by Whitham [1974] and in the glaciological context by Fowler and Larson [1980c] and Hutter [1983]. It is important to

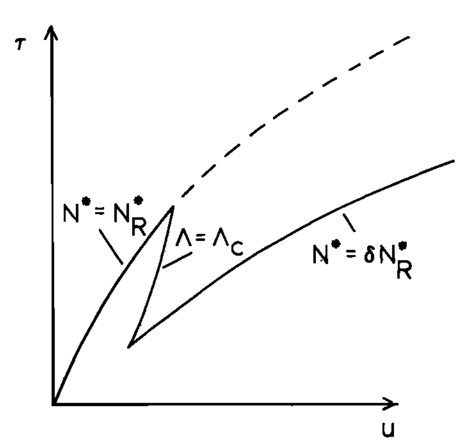

Fig. 3. Multivalued sliding law, taking into account the dependence of N* on u*.

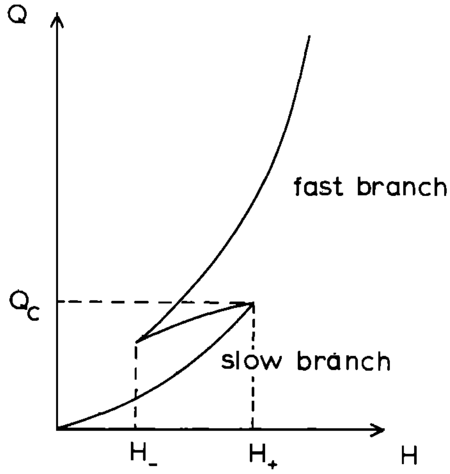

Fig. 4. Multivalued flux/depth relation, corresponding to the sliding law in Figure 3.

realize that linearization techniques, as for example in Nye [1960] and Lliboutry [1969], are incapable of correctly analyzing such systems.

We begin our analysis of (28) by sketching the evolution of a surge pictorially. First, note that there is generally a steady state profile

$$Q = s(x) \qquad (29)$$

where s(x) is a concave profile [Fowler and Larson, 1978]. Examining Figure 4, we see that if $s_{max} < Q_c$, this steady state is perfectly sensible, and no anomalous behaviour will occur. However, if $s_{max} > Q_c$, then the steady state would consequently be (most probably) very unstable; time-dependent behaviour is thus expected in this case.

To see what happens, suppose that $Q < Q_c$ everywhere initially (but $Q_c < s_{max}$); this corresponds to the quiescent phase of a surge. The situation is represented in Figure 5, where the dashed profile for H indicates the steady state that would be obtained if the lower branch of Q versus H on the right were continued beyond H_+. Thus as time evolves, H relaxes toward its (apparent) steady state. However, after some time (of O(1) on a convective time scale, e.g., 20 years), the maximum value of H reaches H_+. The profile is still increasing in depth but can no longer do so continuously because either a jump in H or Q would then form. What happens is shown in Figure 6. Two activation waves propagate very rapidly both up and down the glacier to the points where $H = H_-$. The passage of these waves indicates the collapse of the tunnel drainage system and transfers the ice flux from the lower branch of the Q, H curve. The activation waves are halted at $H = H_-$, where the upper (fast) branch Q no longer exists. During the passage of these waves, H is virtually unaltered.

In order to justify the above description, we have to show that such waves are consistent with solution of the full system (24) and that they have features consistent with observation. This has been done, but the detailed analysis will be reported separately. The results are that such waves can exist, travelling both up and down glaciers. Their dimensionless speed (in terms of a typical, nonsurging flow velocity) is given by V, where

$$V \sim \frac{1}{\varepsilon}\left(\frac{\alpha}{\delta^p}\right)^{\frac{n}{n+1}} \qquad p = \frac{(m-n)^2}{m(m+1)} \qquad (30)$$

If we take m = 4, n = 3, then p = 1/20, so the δ term is not important. With n = 3, $\alpha = 10^{-2}$, $\varepsilon = 10^{-5}$, we have V ~ 3000. For a presurge velocity of ~0.4 m d^{-1} [Kamb et al., 1985], this corresponds to an actual velocity of propagation of 50 m h^{-1}, although this should only be taken as an order of magnitude. Although activation waves as such were not documented by Kamb et al. [1985], "minisurges", in which large fluctuations in velocity occurred, were observed and had similar propagation speeds. The depth change accompanying propagation of these waves is a drop of ~1/V; in a 400 m-thick glacier, this corresponds to 12 cm. The width of the activation front is essentially $\sim\alpha^{n/(n+1)}$. For $\alpha \sim 10^{-2}$, n = 3, this is $\sim 3 \times 10^{-2}$. For a 20-km-long glacier, that is ~600 m. At a fixed

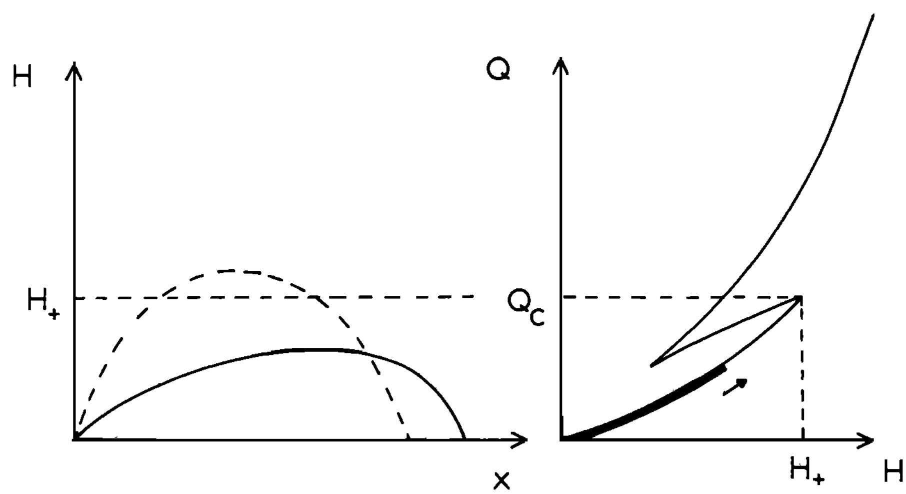

Fig. 5. Quiescent phase of surge: H on subcritical branch, $H < H_+$.

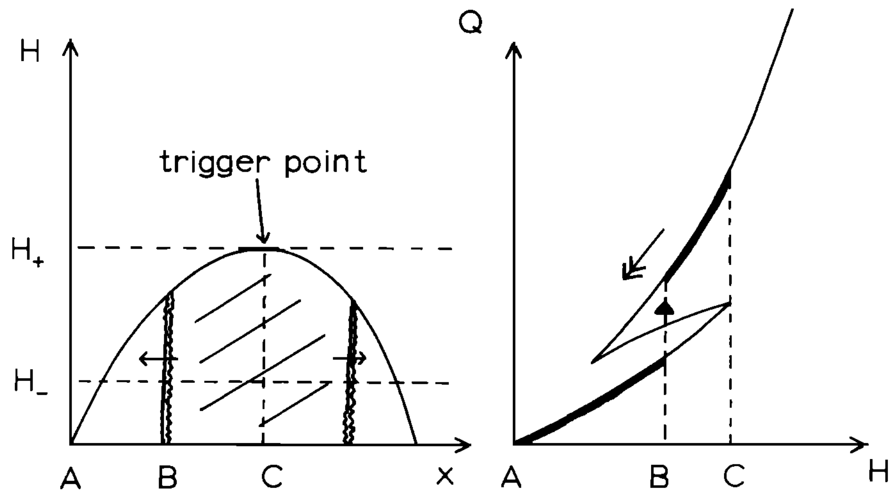

Fig. 6. Surge activation. H reaches H_+ (at point C); hydraulic transition fronts propagate rapidly up and down glacier and in so doing transfer the ice between them to the upper branch of the Q-H curve.

value of x, the rise in velocity would thus occur on a time scale of a few hours.

Once the activation waves have activated the reservoir region ($H > H_-$), the surge proper begins. (It is important to realize that the present description is for absolutely static external conditions; there are no seasonal effects of snow cover, ablation, or rainfall; these can in principle be added but not until the overall description is seen to be satisfactory.) The activated zone now moves rapidly, while the slow zones $H < H_-$, still on the lower branch, move slowly. If Q is large enough (δ is small enough), then the slow zones are essentially static, while the fast ice in the reservoir region rushes downstream. This is portrayed in Figure 7. Accumulation/ablation is irrelevant on a fast time scale, and so the mass balance equation for the fast ice is approximately

$$H_t + Q_x \approx 0 \qquad (31)$$

where Q(H) is on the upper branch. Thus the surge front propagates forward as a shock wave into the quasi-stagnant downstream ice until the entire activated zone is reduced to $H = H_-$. At this point, shown in Figure 8, the surge terminates, although the shock wave at the front of the activated region continues to propagate forwards as an ordinary shock wave on the lower branch [Fowler and Larson, 1980c]. The entire glacier reverts to the lower branch by the propagation from either end of de-activation waves, similar in type to the activation waves. We have not studied these in detail. After this the surge cycle begins again (Figure 5).

The propagation of an ordinary shock wave occurs at a speed

$$\frac{dx_s}{dt} = \frac{[Q]}{[H]} \qquad (32)$$

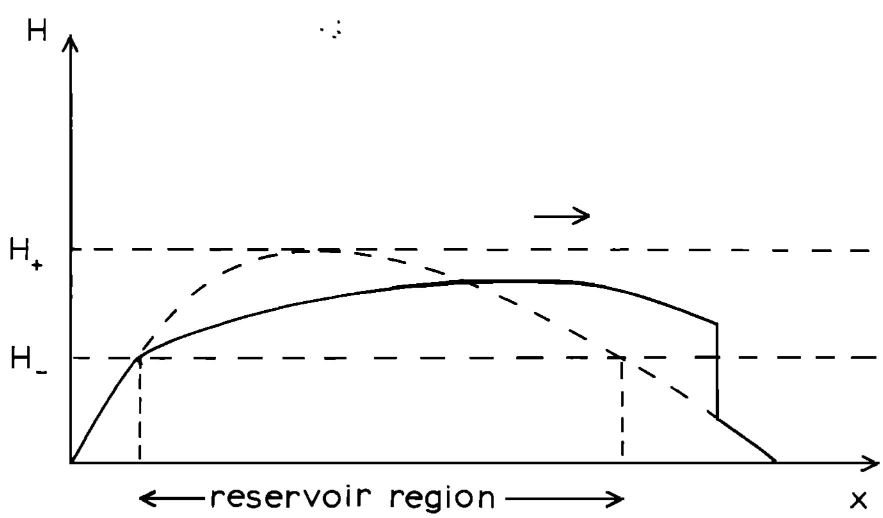

Fig. 7. The surge. The activated zone moves (slumps) rapidly forward until it reaches a quasi-equilibrium at the left-hand end of the upper branch of the Q-H curve.

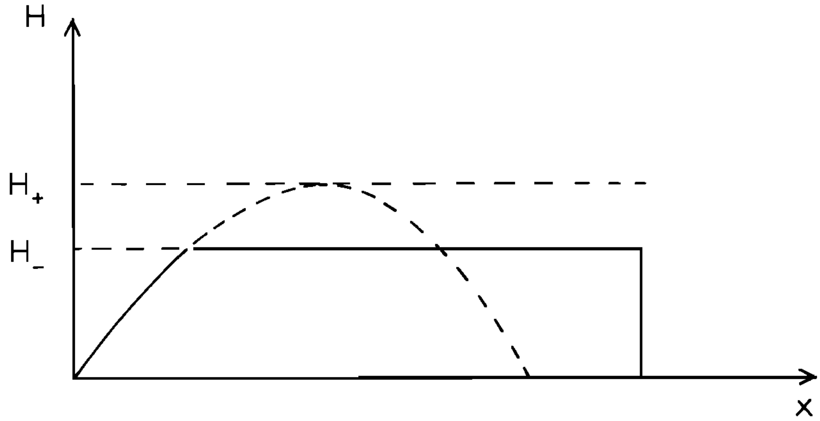

Fig. 8. The end of a surge. The ice "falls" off the upper branch, back to the lower branch of the Q-H curve, presumably by passage of "deactivation" waves, which restore the tunnel drainage. The quiescent phase resumes.

where x_s is the shock position and [Q] and [H] are the jumps across the shock of flux and depth, respectively [Fowler and Larson, 1980c]. For the Variegated surge front [Kamb et al., 1985], we have $H_+ \sim 150$ m, $H_- \sim 250$ m, $u_+ \sim 0$, $u_- \sim 50$ m d^{-1}, whence $dx_s/dt \sim 80$ m d^{-1}, with an apparent shock structure width of ~2 km [Kamb et al., 1985, figure 4]. We can analyze the shock structure of the surge front, using the full equations (24). We find the shock speed is indeed given by (32), and the dimensionless shock thickness λ is given by

$$\lambda \sim \alpha^{n/(n+1)}\delta^{-q} \qquad (33)$$

where q depends on m and n, and is small. With $\alpha = 10^{-2}$, $\delta = 10^{-1}$, m = 4, n = 3, we find $\lambda \sim 0.03$, corresponding to a length scale of 600 m on a 20 km-long glacier.

The variation of velocity u behind the front depends not only on μ and α, but on the depths H^+ and H^- in front of, and behind it. A complete analysis of this has not yet been done, but for the Newtonian case (n = 1), oscillatory decay of the velocity behind the front is likely to occur (see Figure 4 of Kamb et al. [1985]); a similar analysis by Fowler [1982] showed the existence of oscillatory behaviour in so-called transcritical shock propagation and consequently alternate compressive and extensive zones behind the front (see Figure 5 of that paper).

Thus the description of surges based on the dynamics resulting from the theoretically derived sliding law bears a useful resemblance, both qualitatively and quantitatively, to at least one well-documented surge, that of Variegated Glacier. It should be emphasized once again that because rainfall, etc., varies on the same fast time scale over which the surge occurs, it is unrealistic to expect the average description given here to portray the same time variability as occurs in practice (e.g., surge velocity decrease during winter); nevertheless, it is felt that, so far as the model goes, this is a detail, even if a substantial one.

Finally, we may compare the predicted increase in velocity with that observed, typically 10-100 (say, 100 for Variegated). With the sliding law (4), and (20), the jump in u across an activation wave is such that stress (or depth) is essentially constant. Then if u_-, u_+ are the quiescent and surging velocities, respectively, we have

$$N_R^a u_-^{(1-a)/n} = \delta^a N_R^a u_+^{(1-a)/n} \qquad (34)$$

whence

$$u_+/u_- \sim (1/\delta)^{na/(1-a)} = (1/\delta)^{[2(n+2)-(n+1)\alpha]/(\alpha-2)} \qquad (35)$$

In particular, for n = 3, this is $u_+/u_- \sim \delta^{-(10-4\alpha)/(\alpha-2)}$. For $\alpha = 2.4$, we have u_+, $u_- \sim \delta^{-1}$. For $\alpha = 2.2$, this is $u_+/u_- \sim \delta^{-6}$. If $\delta \sim 1/4$, ratios of 10-100 are easily attainable. The corresponding sliding laws are $\tau \propto N^{0.25}u^{0.25}$ and $\tau \propto N^{2/3}u^{1/9}$, respectively.

An interesting inference from (35) is the effect of bed erosion. Low values of α correspond to "jagged" beds. One might reasonably expect α to be lower for geologically younger beds. Then $u_+/u_- \gg 1$ and surges (if they occur) would be spectacular. However, as repeated surges erode the bed, α increases, so that u_+/u_- decreases, and "surges" would become very tame affairs. One would thus envisage a kind of main sequence for glaciers, whose active surging behaviour is in their youth. This idea can be compared with the last paragraph of Clarke et al's [1984] paper.

Discussion

This paper offers an explanation of surges based on the following ingredients. The sliding law is a function of velocity and effective pressure, and $\partial\tau/\partial u > 0$, $\partial\tau/\partial N > 0$. There are (at least) two possible types of drainage, each stable on its own, but such that a preference for one or the other depends on the ice flow rate: the stable drainage mechanism at higher ice velocity must have lower effective pressure. These ingredients combine to yield a multivalued sliding law, and such a law explains surges. However, not all glaciers surge, because (1) one needs a high enough accumulation ($s_{max} > Q_c$), i.e., the glacier must become deep enough; (2) the bed must be rough enough: as $\alpha \to 2.5$

(smoother bed), (35) indicates that the drainage switches but no surge occurs; (3) even if $\alpha < 2.5$, no rapid ice advance will take place unless u_+/u_- is large: from (25), we thus require δ to be "small"; and (4) also the parameter Λ in (19) must be able to reach Λ_c given by (16). From (8), we have

$$\Lambda \sim \nu\beta^n/\sigma \qquad (36)$$

with typical value ($\nu = 0.2$, $\beta = 0.1$, $n = 3$, $\sigma = 10^{-2}$) $\Lambda \sim 0.02$, compared with a typical estimated value $\Lambda_c \sim 1/4$. So there are complicated parametric restrictions that will ensure that not all glaciers will surge but the possibility exists.

So if we could produce the above ingredients as a possibility for all glaciers, a uniform explanation of surges might be possible. The major complication with this aim is that the assumption behind the sliding law and drainage envisaged here is that we visualize a hard bed and subglacial drainage through the ice. Trapridge Glacier [Clarke et al., 1984] is a well-documented example of a surging glacier for which neither of these assumptions is true. It is not even temperate. Nevertheless, it is plausible that exactly the same qualitative features may be involved in the theoretical explanation of this glacier's surging behaviour. The following argument is based on that of Clarke et al. [1984]. Water arrives at the bed of the glacier, where it flows away through a permeable subglacial till. In this till, the water fraction w affects the effective pressure $N = p_i - p_w$, as usual in soil mechanics. Specifically, $N = N(w)$, where N decreases from a very large value as $w \to 0$, to zero when w approaches saturation. Also, w affects the permeability $K = K(w)$, K increasing with w. The process of piping involves the formation of preferential flow paths by washing out fines. In this way, one can envisage drainage being focussed into a few large channels within the permeable till. These channels survive by balancing the processes of washing away of their side walls, which depends on the vigour of the flow through them, with the tendency for the till to close up because of its relative overpressure. When there are a few such channels, the flow through the till elsewhere will be relatively low, the water fraction will be low, and the effective pressure N will be high. In fact, these channels may be the permeable till version of Röthlisberger channels; we might call them Clarke channels. Evidently, their dynamics is virtually identical to that of Röthlisberger channels, with the same stability properties.

If such channels did not exist, then drainage would have to be through the till. Since the flux through the till would then be higher, the permeability would be higher, i.e., w would be higher, and so N would be lower. Further, an increase in water flux in this system leads to increased K, increased w, and decreased N, i.e., increased p_w. Thus exactly the same drainage switch for this case seems possible as for the hard bed.

The other ingredient in the sliding law is the dependence of τ on u and N. The simplest prescription [Jones, 1979] has

$$\tau = \eta(w)u/h \qquad (37)$$

where h is the till layer thickness. The effective viscosity for till deformation would have $\partial\eta/\partial w < 0$, so that with $N = N(w)$, $\partial N/\partial w < 0$, we have

$$\tau = \eta(N)u/h \qquad (38)$$

with $\partial\eta/\partial N > 0$. Hence $\partial\tau/\partial N$, $\partial\tau/\partial u > 0$, which would give the same qualitative dependence as the adumbrated here.

In conclusion, we offer the opinion that, whereas the physics of surging glacier beds may be extremely important (e.g., as to whether the spatial distribution of surging glaciers is geologically controlled [Clarke et al., 1984]), the theory that explains such surges may be mathematically similar for both hard and soft beds and for subpolar or temperate glaciers.

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References

Benoist, J.-P., The spectral power density and shadowing function of a glacial micro-relief at the decimetre scale, J. Glaciol., 23, 57-66, 1979.

Bindschadler, R., The importance of pressurized subglacial water in separation and sliding at the glacier bed, J. Glaciol., 29, 3-19, 1983.

Boulton, G.S., and A.S. Jones, Stability of temperate ice caps and ice sheets resting on beds of deformable sediment, J. Glaciol., 24, 29-43, 1979

Budd, W.F., A first simple model for periodically self-surging glaciers, J. Glaciol., 14, 3-21, 1975.

Budd, W.F., P.L. Keage, and N.A. Blundy, Empirical studies of ice sliding, J. Glaciol., 23, 157-170, 1979.

Cary, P.W., G.K.C. Clarke, and W.R. Peltier, A creep instability analysis of the Antarctic and Greenland ice sheets, Can. J. Earth Sci., 16, 182-188, 1979.

Clarke, G.K.C., Thermal regulation of glacier surging, J. Glaciol., 16, 231-250, 1976.

Clarke, G.K.C., and R.H. Goodman, Radio echo soundings and ice-temperature measurements in a surge-type glacier, J. Glaciol., 14, 71-78, 1975.

Clarke, G.K.C., and G.T. Jarvis, Post-surge temperatures in Steele Glacier, Yukon Territory, Canada, J. Glaciol., 16, 261-268, 1976.

Clarke, G.K.C., U. Nitsan, and W.S.B. Paterson, Strain heating and creep instability in glaciers and ice sheets, Rev. Geophys., 15, 235-247, 1977.

Clarke, G.K.C., S.G. Collins, and D.E. Thompson,

Flow, thermal structure, and subglacial conditions of a surge-type glacier, Can. J. Earth Sci., 21, 232-240, 1984.
Fowler, A.C., The existence of multiple steady states in the flow of large ice masses, J. Glaciol., 25, 183-184, 1980.
Fowler, A.C., Waves on glaciers, J. Fluid Mech., 120, 283-321, 1982.
Fowler, A.C., A sliding law for glaciers of constant viscosity in the presence of subglacial cavitation, Proc. R. Soc. London, Ser. A, 407, 147-170, 1986.
Fowler, A.C., Sliding with cavity formation, J. Glaciol., in press, 1987.
Fowler, A.C., and D.A. Larson, On the flow of polythermal glaciers, I, Model and preliminary analysis, Proc. R. Soc. London, Ser. A, 363, 217-242, 1978.
Fowler, A.C., and D.A. Larson, The uniqueness of steady state flows of glaciers and ice sheets, Geophys. J. R. Astron. Soc., 63, 333-345, 1980a.
Fowler, A.C., and D.A. Larson, Thermal stability properties of a model of glacier flow, Geophys. J. R. Astron. Soc., 63, 347-359, 1980b.
Fowler, A.C., and D.A. Larson, On the flow of polythermal glaciers, Part II, Surface wave analysis, Proc. R. Soc. London, Ser. A, 370, 155-171, 1980c.
Gruntfest, I.J., Thermal feedback in liquid flow, plane shear at constant stress, Trans. Soc. Rheol., 7, 195-207, 1963.
Hallet, B., [Discussion], J. Glaciol., 23, 396, 1979.
Hutter, K., Dynamics of glaciers and large ice masses, Ann. Rev. Fluid Mech., 14, 87-130, 1982.
Hutter, K., Theoretical Glaciology, D. Reidel, Hingham, Mass., 1983.
Jarvis, G.T., and G.K.C. Clarke, The thermal regime of Trapridge Glacier and its relevance to glacier surging, J. Glaciol., 14, 235-250, 1975.
Johnson, R.E., and R.M. McMeeking, On the mechanics of surging glaciers, J. Glaciol., 32, 120-132, 1986.
Jones, A.S., The flow of ice over a till bed, J. Glaciol., 22, 393-395, 1979.
Kamb, W.B., Sliding motion of glaciers: theory and observation, Rev. Geophys., 8, 673-728, 1970.
Kamb, W.B., An observationally based mechanism of glacier surging, paper presented at meeting on glacier hydraulics, International Glaciological Society, Interlaken, Switzerland, 1985.
Kamb, W.B., C.F. Raymond, W.D. Harrison, H. Engelhardt, K.A. Echelmeyer, N. Humphrey, M.M. Brugman, and T. Pfeffer, Glacier surge mechanism: 1982-1983 surge of Variegated Glacier, Alaska, Science, 227, 469-479, 1985.
Kevorkian, J., and J.D. Cole, Perturbation Methods in Applied Mathematics, Springer-Verlag, New York, 1981.
Lliboutry, L.A., General theory of subglacial cavitation and sliding of temperate glaciers, J. Glaciol., 7, 21-58, 1968.
Lliboutry, L.A., Contribution à la theorie des ondes glaciaires, Can. J. Earth Sci., 6, 943-953, 1969.
Lliboutry, L.A., Local friction laws: A critical review and new openings, J. Glaciol., 23, 67-94, 1979.
McMeeking, R.M., and R.E. Johnson, On the analysis of longitudinal stress in glaciers, J. Glaciol., 31, 293-302, 1985.
Meier, M.F., and A. Post, What are glacier Surges?, Can. J. Earth Sci., 6, 807-817, 1969.
Nye, J.F., The response of glaciers and ice-sheets to seasonal and climatic changes, Proc. R. Soc. London, Ser. A, 250, 559-584, 1960.
Nye, J.F., A calculation on the sliding of ice over a wavy surface using a Newtonian viscous approximation, Proc. R. Soc. London, Ser. A, 311, 445-467, 1969.
Nye, J.F., Glacier sliding without cavitation in a linear viscous approximation, Proc. R. Soc. London, Ser. A, 315, 381-403, 1970.
Nye, J.F., Water flow in glaciers: Jökulhlaups, tunnels and veins, J. Glaciol., 17, 181-207, 1976.
Palmer, A.C., A kinematic wave model of glacier surges, J. Glaciol., 11, 65-72, 1972.
Paterson, W.S.B., The Physics of Glaciers, Pergamon, New York, 1981.
Paterson, W.S.B., U. Nitsan, and G.K.C. Clarke, An investigation of creep instability as a mechanism for glacier surges, Mater. Glyatsiologicheskikh Issled. Khronika Obsuzhdeniya, 32, 201-209, 1978.
Robin, G. de Q., Ice movement and temperature distribution in glaciers and ice sheets, J. Glaciol., 2, 523-532, 1955.
Röthlisberger, H., Water pressure in intra- and subglacial channels, J. Glaciol., 11, 177-203, 1972.
Walder, J.S., Hydraulics of subglacial cavities, J. Glaciol., 32, 439-445, 1986.
Weertman, J., On the sliding of glaciers, J. Glaciol., 3, 33-38, 1957.
Weertman, J., The theory of glacier sliding, J. Glaciol., 5, 287-303, 1964.
Weertman, J., The unsolved general glacier sliding problem, J. Glaciol., 23, 97-115, 1979.
Whitham, G., Linear and Nonlinear Waves, John Wiley, New York, 1974.
Yuen, D.A., and G. Schubert, The role of shear heating in the dynamics of large ice masses, J. Glaciol., 24, 195-212, 1979.

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