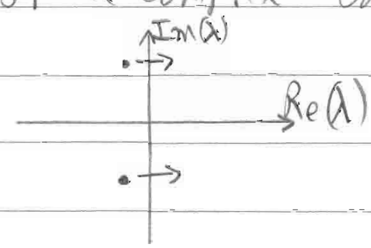
* bifurcations in 2d: [chapter 8, strogatz]

+ As mentioned previously, the 1d bifurcations (saddle-node, transcritical, pitchfork) can occur in higher dimensional systems, along the center manifold. [p. 85–91 in first notebook].
+ There are also bifurcations that are inherently 2d:

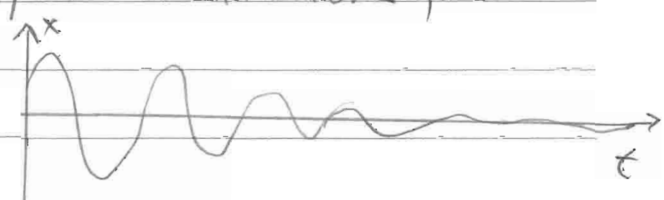
* Hopf bifurcations:

+ bifurcations occur when $Re(\lambda)$ passes through zero. The 1d bifurcations mentioned above occure when a real eigenvalue (in a possibly n-dim system) passes through zero.
+ consider now the case of a complex conjugate pair of eigenvalues:

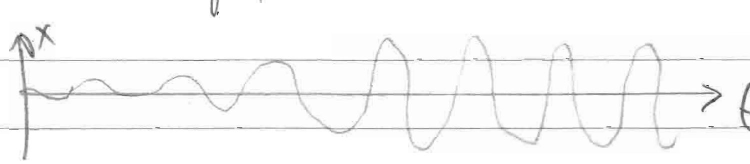


* Super critical Hopf:

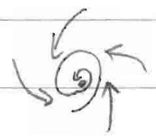
When $Re(\lambda) < 0$, we have damped oscillations, or what we called a spiral:



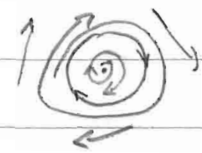
we also saw (weakly nonlinear van der pol), a case where an unstable spiral saturates at a limit cycle.



+ Hopf bifurcation in phase space:



$\mu < \mu_c$



$\mu > \mu_c$

+ generic example: $\left.\begin{aligned} \dot r &= \mu r - r^3 \\ \dot\theta &= \omega + b r^2 \end{aligned}\right\}$ (#)

$\mu < 0 \Rightarrow \left\{\begin{matrix}\text{stable}\\ \text{origin}\end{matrix}\right\}$; $\mu > 0 \Rightarrow$ unstable origin.

ω = frequency at $0 < \mu \ll 1$

b = nonlinear correction to frequency.

+ linearize in cartesian coordinates: $\dot x = f(x,y)$; $\dot y = g(x,y)$.

$x = r\cos\theta$; $y = r\sin\theta$. $\Rightarrow r^2 = x^2 + y^2$

$\Longrightarrow \dot x = \dot r\cos\theta - r\dot\theta\sin\theta$; $\dot y = \cdots$

subst this in (#):

$$\dot x = (\mu - [x^2+y^2])x - (\omega + b[x^2+y^2])y$$

$$\Rightarrow \quad \dot x = \mu x - \omega y + O(x^3, y^3)$$

$$\dot y = \omega x + \mu y + O(x^3, y^3)$$

$$\Rightarrow \quad J = \begin{pmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{pmatrix} = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}$$

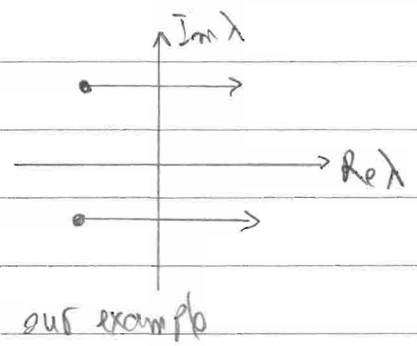
$$|J - \lambda I| = 0 \quad \Rightarrow \quad \underline{\lambda = \mu \pm i\omega}$$

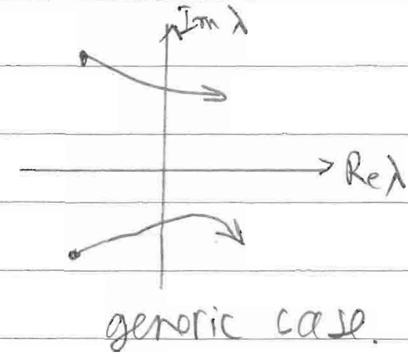
+ conclusions:

1. size of limit cycle is prop to $(\mu-\mu_c)^{1/2}$ for $|\mu-\mu_c| \ll 1$.
2. frequency of limit cycle is $Im(\lambda)$. period is given by $T = 2\pi / Im(\lambda) + O(\mu-\mu_c)$.

+ Also:

1. As μ varies, λ moves horizontally in the complex λ plane in this example. In most cases it does not:





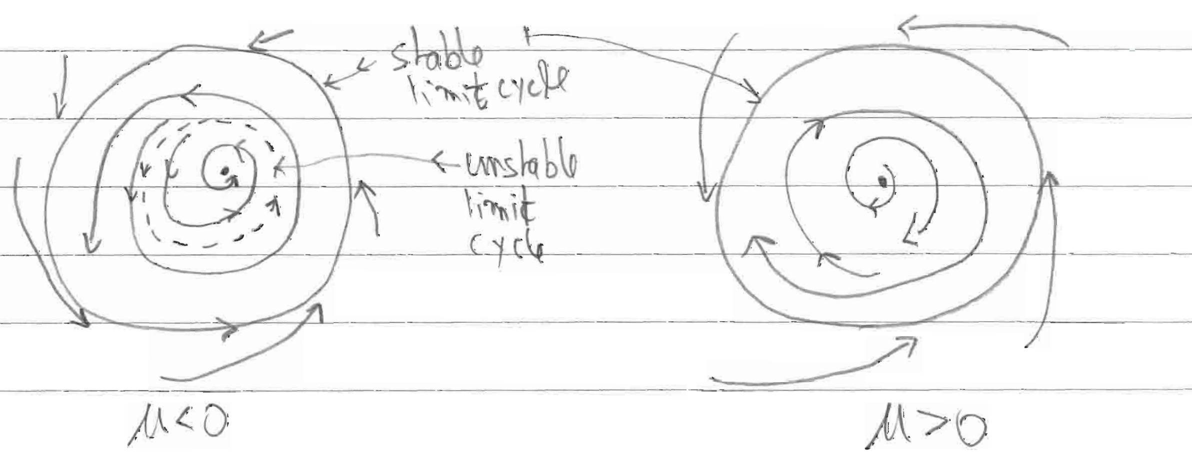
* subcritical Hopf bifurcation:

+ consider $\dot{r} = \mu r + r^3 - r^5$; $\dot{\theta} = \omega + b r^2$

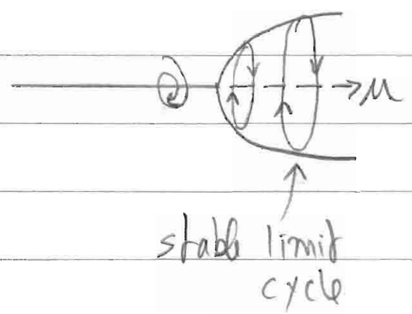
new..

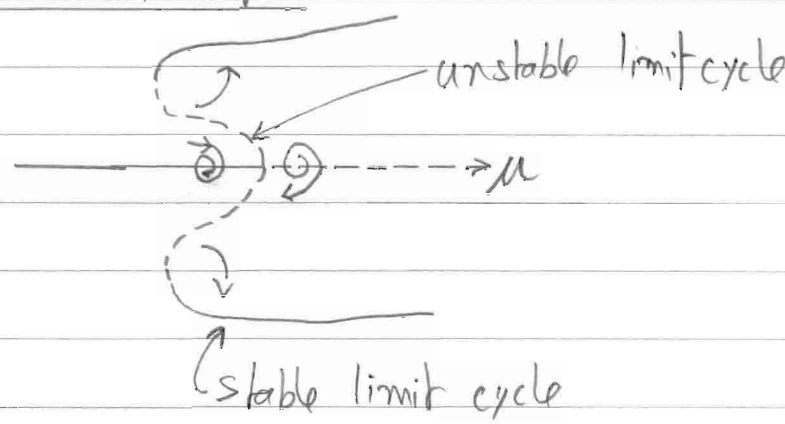
$\Rightarrow$ the r^3 term is destabilizing at the origin.

+ for $\mu<0$: origin is stable, $\mu>0$: unstable



* super critical vs. subcritical Hopf:





+ Note that the $-r^5$ term in our model system for subcritical Hopf bifurcation is not typical of all such bifurcations. Beyond μ_c, the system jumps to some distant attractor. may be a fixed pt, infinity, another limit cycle (as with $-r^5$), or chaos.

+ Note the hysteresis in the sub-critical case, as evident in above figure.

+ The super- & sub- versions may be differentiated by the appearance of a small, growing like $\sqrt{\mu}$ limit circle in the super-case, vs immediate appearance of finite-amplitude limit cycle in sub-case.

* Global bifurcations of cycles: [Strogatz 8.4]

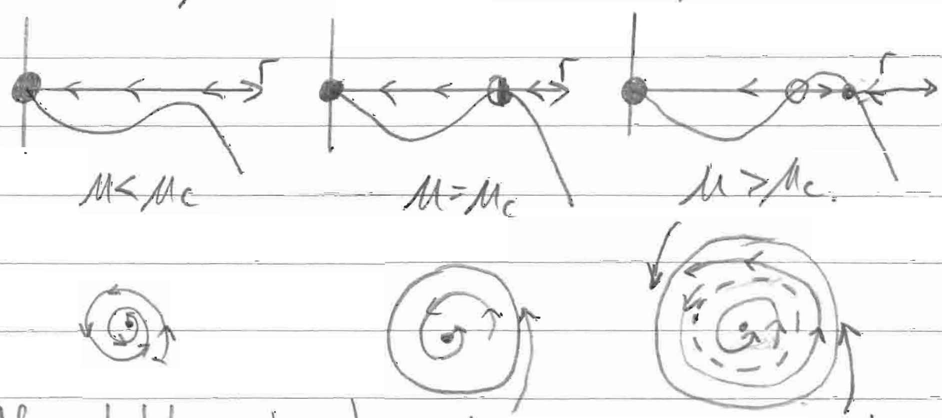
+ In addition to Hopf bifurcation, there are three more bifurcations that may lead to the appearance of limit cycles in 2d systems.
+ While Hopf is local: limit cycle appears near a f.p., the other 3 are global: involve larger areas of phase plane:

* Saddle node bifurcation of cycles:

+ Two limit cycles colliding, coalescing & annihilating each other:
+ consider:

$$\dot{r} = \mu r + r^3 - r^5$$
$$\dot{\theta} = \omega + b r^2 .$$

previously we looked at $\mu = 0$, subcritical Hopf. consider now $\mu_c = -1/4$.



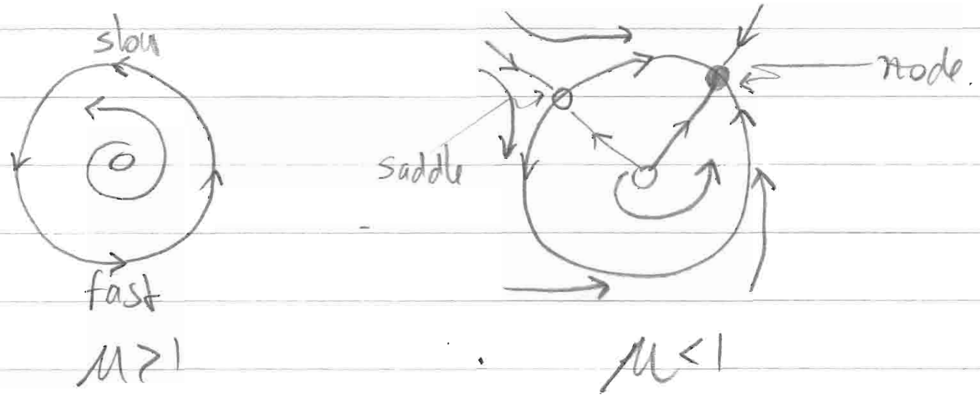
+ At μ_c, the half-stable limit cycle appears at full amplitude out of no where.
+ it then splits into stable & unstable cycles gradually.
+ origin is always stable & does not participate.

* Infinite-period bifurcation:

$\dot{r} = r \cdot (1 - r^2)$; $\dot{\theta} = \mu - \sin\theta$. theta=[0,2*pi]

+ radially, all trajectories starting at $r \neq 0$ go to $r = 1$.
+ For $\mu > 1$, motion is counterclockwise everywhere.

+ for $\mu=1$, there is a f.p. at $r^*=1$, $\theta^*=\cancel{0}$ pi/2
+ for $\mu<1$, there are two f.p. at $r^*=1$ & at the two θ^* satisfying $\mu=\sin\theta^*$:



$\Rightarrow$ as μ increases, the saddle & node colide, annihilate each other & disappear to leave a limit cycle.

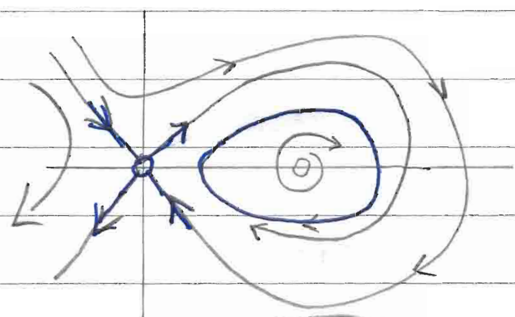
+ at $\mu=1$, there is a half-stable f.p. at $\theta=0, r=1$, & the period of the osillation is ∞.
for larger μ, the period decreases like $(\mu-1)^{-1/2}$ (#).
[see strogatz section 4.3, or my notes, other notebook, p. 73, for derivation of (#).]

+ the amplitude of the oscillation is $O(1)$ throughout this bifurcation, & only the period varies.

* Homoclinic bifurcation:

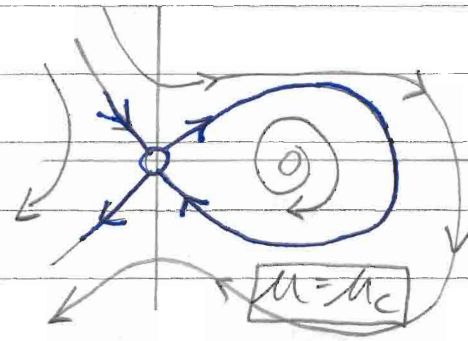
+ Another infinite-period bifurcation involving the creation or destruction of a limit-cycle. This time due to the colision of a limit cycle and a saddle point.
+ A numerical example:

$$\begin{Bmatrix} \dot{x} = y \\ \dot{y} = \mu \cdot y + x - x^2 + x \cdot y \end{Bmatrix}$$

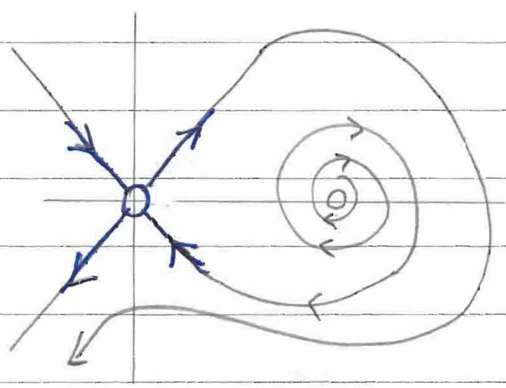


$\boxed{\mu < \mu_c}$

saddle + limit cycle.



homoclinic orbit



$\mu_c = -0.8645...$

$\boxed{\mu > \mu_c}$

no limit cycle, only saddle pt.

+ limit cycle appears again with $O(1)$ radius.
+ the period of the limit cycle in this case: $O(\ln(\mu - \mu_c))$.

* Degeneracies, pathologies

+ degenerate Hopf bifurcations: $\ddot{x} + \mu\dot{x} + \sin x = 0$
(nonlinear pendulum, bifurcation parameter is friction coefficient).
at $\mu = 0$ the origin is a center, not an isolated limit cycle $\Rightarrow$ not an actual hopf bif.

+ van der pol: $\ddot{x} + \varepsilon\dot{x}(x^2-1) + x = 0$
at $\varepsilon = 0$, $\lambda = \pm i$, $Re(\lambda) = 0$ like in Hopf bif. but:
as $0 < \varepsilon \ll 1$, we know that the limit cycle has $O(1)$ amplitude, not $O(\sqrt{\varepsilon})$ like in Hopf. [Note: in subcritical Hopf, unstable limit cycle has amplitude of $\sqrt{\mu}$, in supercritical, stable cycle's amplitude is again $\sqrt{\mu}$!].
$\Rightarrow$ strange.

+ reason: at $\varepsilon = 0$, nonlinearity vanishes. $\Rightarrow$ degenerate case.
if we define: $u^2 = \varepsilon x^2$, we find
$\ddot{u} + u + u^2\dot{u} - \varepsilon\dot{u} = 0$
& for $0 < \varepsilon \ll 1$ we have $u(t,\varepsilon) \approx (2\sqrt{\varepsilon})\cos t$
$\Rightarrow$ standard supercritical Hopf.

* Example: driven damped pendulum:

...with both infinite-period & Homoclinic bifurcations.

Eq'n is $\phi'' + \alpha\phi' + \sin\phi = I$ (3)

acceleration; friction; gravity; constant torque;

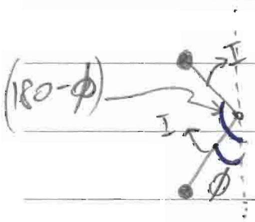
OR, in 2d: $\left\{ \begin{array}{l} \phi' = y \\ y' = I - \sin\phi - \alpha y \end{array} \right\}$ (4)

+ fixed pts: zero velocity, & torque balancing gravity:

$y^* = 0, \quad I = \sin\phi.$

$\Rightarrow$ two such pts if $I<1$, none if $I>1$.

of the two pts, one is stable (sink) & one unstable (saddle)

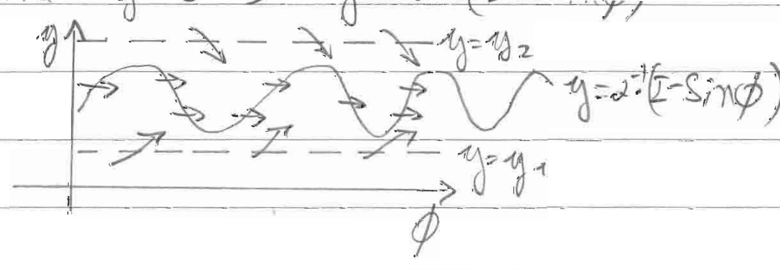


The two f.p.; lower one is stable, upper unstable.

As $I \to 1$ from below, the two pts merge at $\phi = \cancel{0}$ pi/2 via a saddle-node bifurcation of f.p. $\Rightarrow$ birth of limit cycle (which occurs for $I>1$) via infinite-period bifurcation. [limit cycle may also occur for $I<1$! see later...]

+ closed orbits: show now that there is one & only one limit cycle for $I>1$:

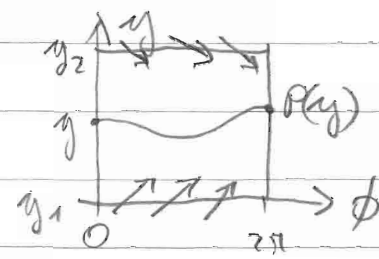
consider the nullcline $y' = 0 \Rightarrow y = \alpha^{-1}\cdot(I - \sin\phi)$:



Note that:

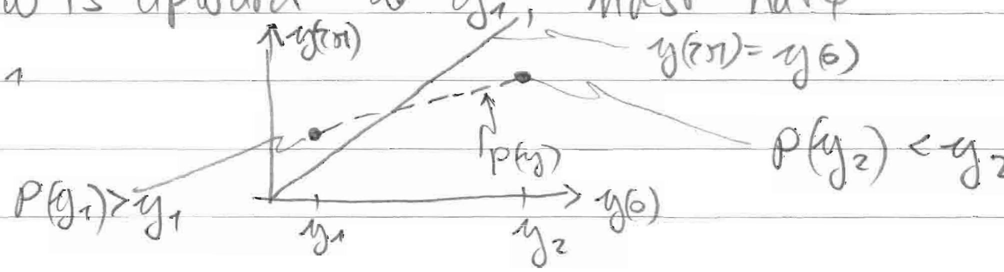
+ at $y=y_2$ (a line strictly above the nullcline), flow is downward (because it is above nullcline).
+ at $y=y_1$, flow is upward (similarly...).
+ on nullcline, flow is horizontal ($y'=0$).
+ in entire strip, flow has $\phi'>0$ (because $\phi'=y>0$).

+ consider a map from $\phi=0$ to $\phi=2\pi$, defined such that if y is entry point of trajectory at $\phi=0$, then $P(y)$ is location of trajectory at $\phi=2\pi$:



clearly if $y^* = P(y^*)$, we have a limit cycle, as $\phi=0$, & $\phi=2\pi$ are the same pt, so that $y^*=P(y^*)$ means that trajectory returned to its original starting pt $\Rightarrow$ closed orbit.

+ because flow is downward at y_2, we must have $P(y_2) < y_2$
+ because flow is upward at y_1, must have $P(y_1) > y_1$



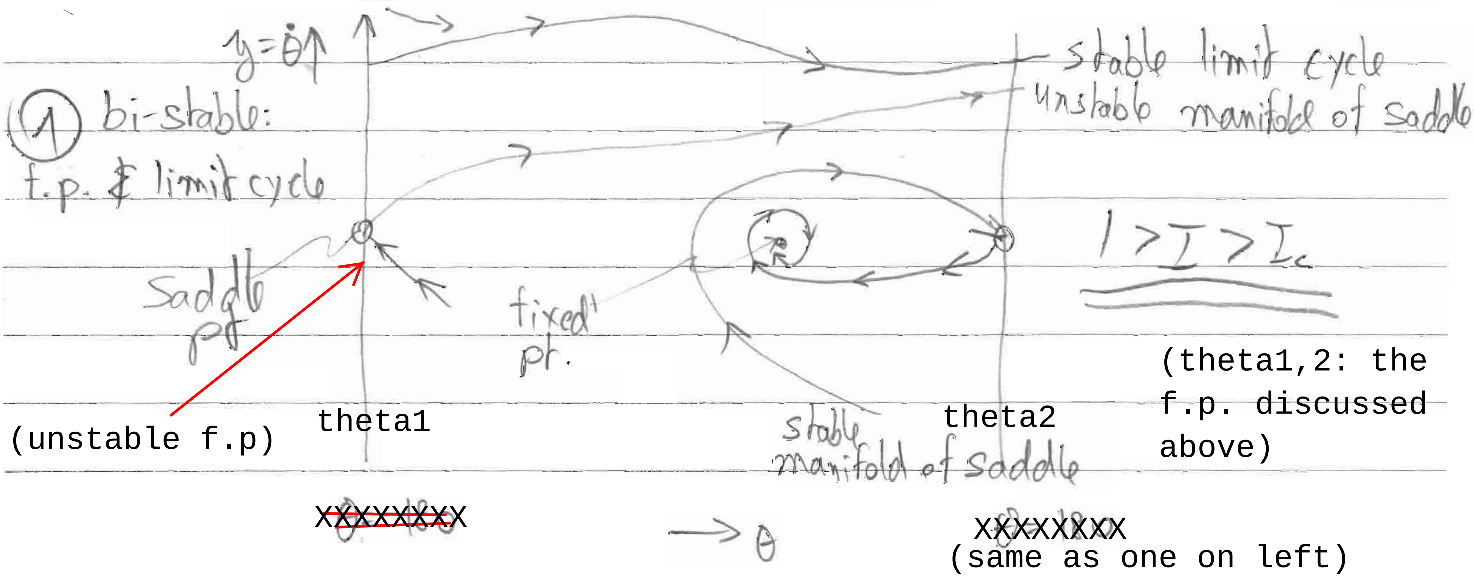
but $P(y)$ is continuous, so must cross the slope 1

line at a point y^*, at which we have
$P(y^*) = y^*$ $\Rightarrow$ limit cycle.
Note that $P(y)$ is monotonic, or else different trajectories may cross. can also show that the limit cycle for $I>1$ is unique... [may have not been unique had the $P(y)$ line merged for some interval with the slope 1 line]

+ Homoclinic bifurcation:

Decrease I from $I>1$. what happens?
initially, there is a limit cycle: pendulum rotates in a periodic way. decreasing below $I<1$, rotation continues (if friction α is sufficiently small. at some $0<I_c<1$, rotation cannot be sustained by force I against friction & gravity, & pendulum goes back to a fixed pt, as calculated above, with $\theta = \text{const}$, $\dot\theta = 0$.

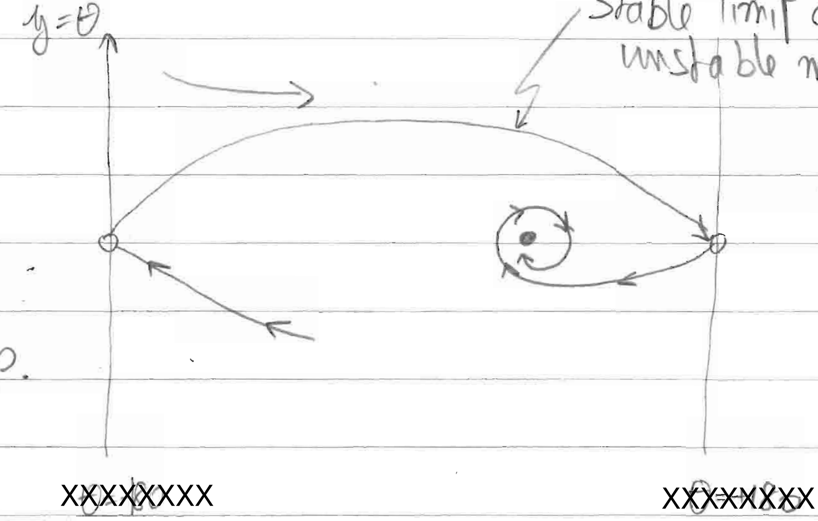
How does this happen: at $I = I_c$, pendulum takes forever... to reach the $\theta = \pm 180$ pt (top pt on its rotation), & then continues falling back. This is a homoclinic bifurcation:



(2) homoclinic orbit & f.p.

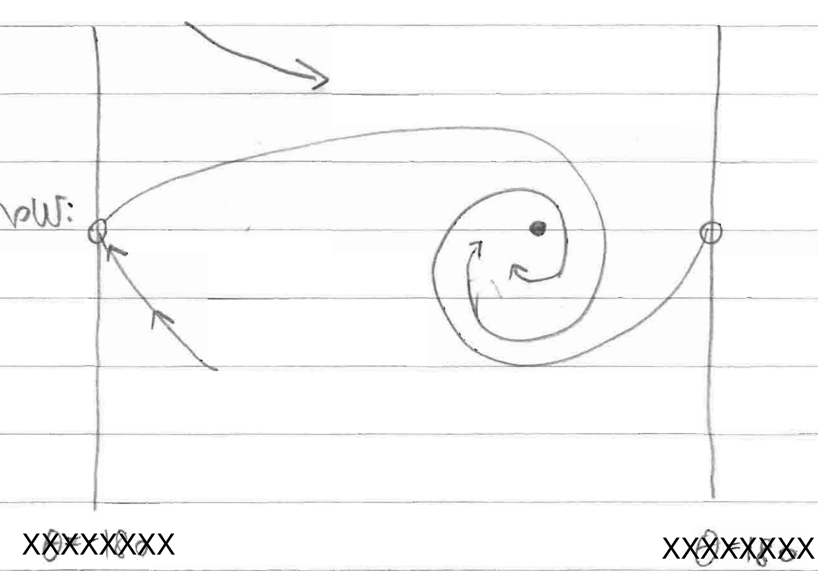
stable limit cycle merged with unstable manifold = homoclinic orbit.

$\underline{\underline{I = I_c < 1}}$

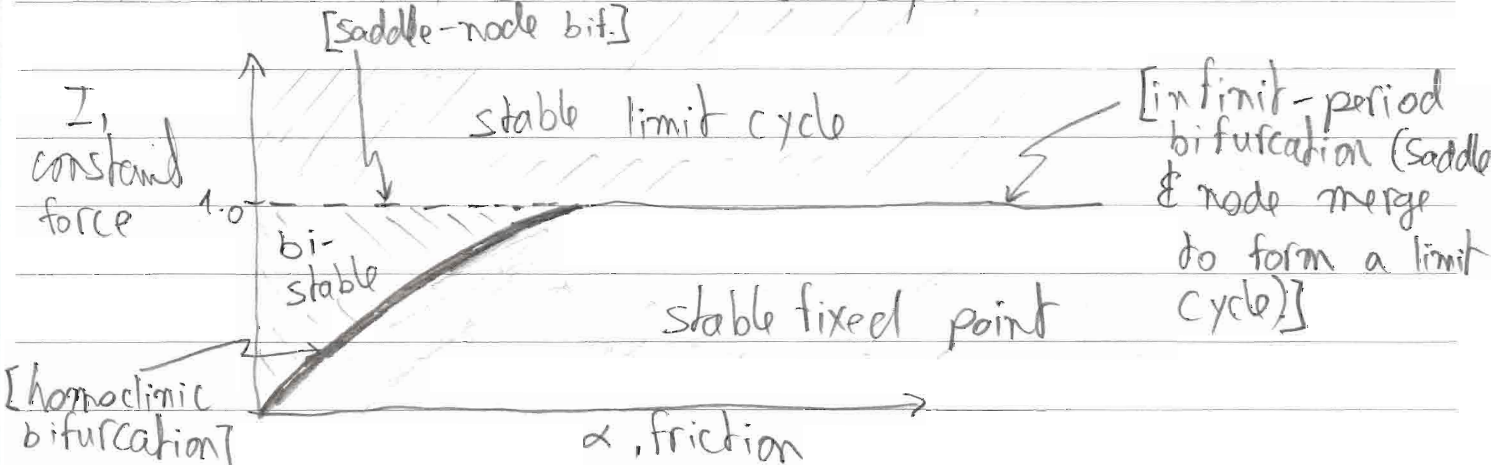


(3) only stable fixed pt, now:

$\underline{\underline{I < I_c < 1}}$



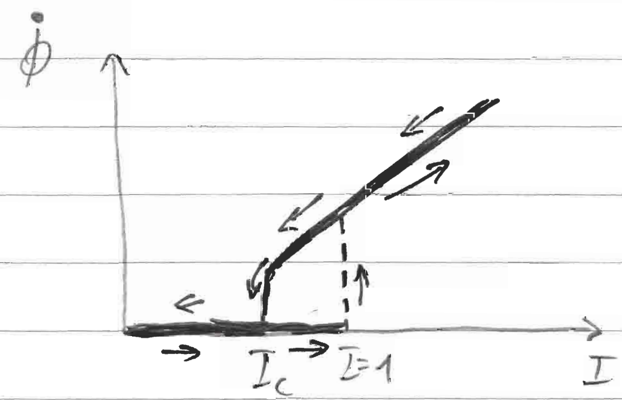
+ All this may be summarized by:



+ Hysteresis:

The bi-stable regime implies an hysteresis as the torque strength I is slowly varied:

- small I: only a fixed pt at $\theta = \text{const}$, $\dot{\theta} = 0$
- increase I to $I > 1$ $\Rightarrow$ a limit cycle.
- decrease I, to below $I = 1$ but at $I > I_c$: still a limit cycle (rotation).
- decrease I below I_c $\rightarrow$ stable fixed pt again



* Finally:

Later we'll make the forcing periodic in time, which will make the system non autonomous. It will therefore become 3d in effect, which will allow it to become chaotic.

This will demonstrate the "quasi-periodicity route to chaos"...