EPS131, 04: waves, part I

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Contents

1 Inertial oscillations

Starting from the momentum balance $F = ma$,
acceleration=pressure gradient+Coriolis+friction+gravity
Assume that all terms except for
acceleration=Coriolis
Are negligible. The equations are,

$$
\begin{aligned}
u_t - fv &= 0 \\
v_t + fu &= 0
\end{aligned}
$$

substitute the second into the first,

$$v_{tt} = -f^2 v$$

try exponential solution $v = e^{at}$ to find $a^2 = -f^2$ or $a = \pm if$. The solution is therefore,

$$v = A'e^{ift} + B'e^{-ift},$$

or, equivalently,

$$
\begin{aligned}
v &= A'(\cos(ft) + i\sin(ft)) + B'(\cos(ft) - i\sin(ft)) \\
&= (A' + B')\cos(ft) + i(A' - B')\sin(ft)).
\end{aligned}
$$

Letting $A = (A' + B')$ and $B = i(A' - B')$ this may be written as,

$$v = A\sin(ft) + B\cos(ft).$$

Using $u = -v_t/f$, we therefore also find that

$$u = -A\cos(ft) + B\sin(ft).$$

Now consider specific initial conditions of $v(0) = v_0$ and $u(0) = 0$ to solve for the constants: $A = 0$, $B = v_0$ so that,

$$
\begin{aligned}
v &= v_0\cos(ft) \\
u &= v_0\sin(ft).
\end{aligned}
$$

These are oscillations! what does the trajectory of a fluid particle look like? Let its coordinates be x, y and they satisfy $dx/dt = u$, $dy/dt = v$, or

$$
\begin{aligned}
\frac{dx}{dt} &= v_0\sin(ft) \\
\frac{dy}{dt} &= v_0\cos(ft).
\end{aligned}
$$

Given initial conditions of $x(0) = x_0$ and $y(0) = y_0$, these equations are integrated [hint: $\int_0^t \frac{dx}{dt} dt = x(t) - x(0)$ and $\int_0^t \sin(ft) dt = -\frac{1}{f}(\cos(ft) - \cos(0))$] to find,

$$x(t) = x_0 + \frac{v_0}{f}(1 - \cos(ft))$$
$$y(t) = y_0 + \frac{v_0}{f}\sin(ft).$$

(substitute these solutions in the above equations to verify that both equations and initial conditions are indeed satisfied). Note that

$$(x - x_0 - v_0/f)^2 + (y - y_0)^2 = \left(\frac{v_0}{f}\right)^2 = \text{constant}$$

which is the equation for circular motion with a radius v_0/f. The larger the initial velocity (excited by the passage of some storm, say) the larger is the radius of motion. The frequency of the oscillation/ circular motion is given by $f = 2\Omega \sin\theta$.

What is the center of the circular trajectory? Rationalize this. How does the radius change with the initial velocity and with the Coriolis parameter? What is the period of inertial oscillations (in days) at 30N? 20N? 40N?

2 Wave basics

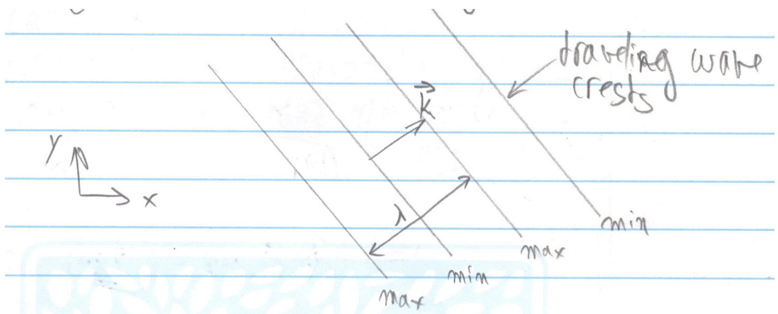
2.1 Definitions

Consider a place wave solution for surface elevation,

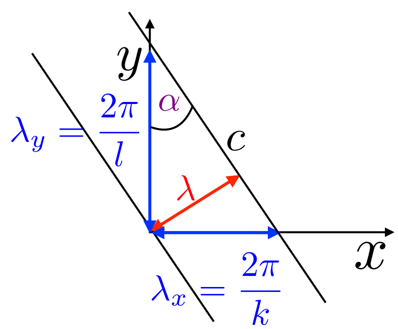
$$\eta(x, y, t) = \eta_0 \cos(kx + ly - \omega t)$$

and we have,

- wave vector: $\vec{k} = (k, l)$
- wavelength: $\lambda = 2\pi/\sqrt{k^2 + l^2}$ (distance between crests)
- period: $T = 2\pi/\omega$ (time between crests)
- amplitude: η_0



To see that the wavelength is given by $\lambda = 2\pi/(k^2 + l^2)^{1/2}$, consider the following figure,



Use $\sin\alpha = \lambda/\lambda_y = \lambda_x/c$, together with $c = \sqrt{\lambda_x^2 + \lambda_y^2}$ to find that

$$\lambda = \lambda_y \sin\alpha = \frac{\lambda_x \lambda_y}{(\lambda_x^2 + \lambda_y^2)^{1/2}}$$

substituting $\lambda_x = 2\pi/k$, $\lambda_x = 2\pi/l$, we get the desired expression,

$$\lambda = \frac{2\pi}{(k^2 + l^2)^{1/2}}.$$

2.2 Phase velocity

Phase speed, is the speed of crests. Consider a fixed y and suppose a given crest is at $x = x_0$ at time t_0 and at $x = x_1$ at time t_1. This implies,

$$\eta(x_0, y, t_0) = \eta(x_1, y, t_1)$$

so that,

$$\eta_0 \cos(kx_0 + ly - \omega t_0) = \eta_0 \cos(kx_1 + ly - \omega t_1)$$

which implies,

$$kx_0 - \omega t_0) = kx_1 - \omega t_1$$

and therefore that

$$\frac{x_1 - x_0}{t_1 - t_0} = \frac{\omega}{k}.$$

let $t_1 \to t_0$ and then the LHS is the speed of propagation of the crust, so we found that

$$c_{ph} = \frac{\omega}{k}.$$

We can express this phase velocity in terms of the wave length λ and period T.

$$c_{ph} = \frac{\omega}{k} = \frac{2\pi/T}{2\pi/\lambda} = \frac{\lambda}{T}.$$

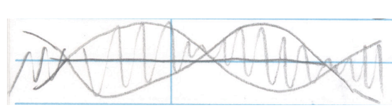
This makes sense, as a wave crest travels a distance equal to the wave length during one period.

In two dimensions, the phase velocity is $(c_{ph}^{(x)}, c_{ph}^{(y)}) = (\omega/k, \omega/l)$; note that this is *not* a vector.

2.3 Group velocity

Consider two waves of similar (k, ω) traveling together,

$$(k - \delta k, \omega - \delta\omega)$$
$$(k + \delta k, \omega + \delta\omega).$$



The surface elevation is then given by

$$\begin{aligned}\eta &= \cos\left[(k-\delta k)x-(\omega-\delta\omega)t\right]+\cos\left[(k+\delta k)x-(\omega+\delta\omega)t\right]\\ &= 2\cos\left[\delta k\, x-\delta\omega\, t\right]\cos\left[kx-\omega t\right]\end{aligned}$$

phase speed of the envelope, which is the velocity of energy propagation and is termed group velocity is,

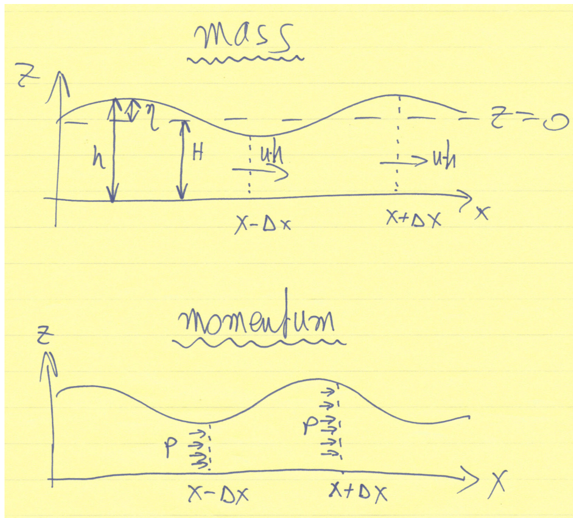
$$\frac{\delta\omega}{\delta k}=\frac{\partial\omega}{\partial k}.$$

In two dimensions, the wave number is a vector, $\vec{k}=(k,l)$, and so is the group velocity,

$$\vec{\mathbf{c}}_g=(c_g^{(x)},c_g^{(y)})=\left(\frac{\partial\omega}{\partial k},\frac{\partial\omega}{\partial l}\right).$$

3 Surface gravity waves – without rotation

Objective: find the dispersion relation $\omega(k)$, the relation between wave length and period, which also tells us about the wave phase and group speeds. We'll derive results for shallow water waves (wave length much larger that ocean depth, this is relevant for beach waves and tsunamis), state the result for deep waves, and discuss scaling relationships for both deep and shallow water waves.



3.1 Shallow water waves scaling argument/ dimensional analysis

The relevant dimensional constants are: H - depth; g - gravity; ω - frequency (2π/period); k - wavenumber (2π/wavelength). Try writing the phase speed $c = \omega/k$ as function of H, g,

$$\begin{aligned}
&[c] = m/s = m^1 s^{-1} = [H]^a [g]^b = m^a (m/s^2)^b = m^{a+b} s^{-2b} \\
&\Rightarrow \quad a + b = 1; \quad 2b = 1 \\
&\Rightarrow \quad a = 1/2; b = 1/2
\end{aligned}$$

so that $c = \sqrt{gH}$ and therefore $\omega = \sqrt{gH}k$.

- Note that these waves are non-dispersive: different wave lengths travel at the same speed.
- The dispersion relation for these shallow water surface gravity waves also explain why such waves arrive parallel to the coast.
- Tsunamis are also shallow water waves: wave length is 1000s of km, and depth of ocean is 4 km. Their propagation speed is $\sqrt{gH} = \sqrt{10 \times 4000} = 200m/s$. Sound velocity in air: 35 m/s.

3.2 Shallow water 1d mass conservation

Consider a channel of width Δy and height $h(x,t)$, and in it a section from $x-\Delta x$ to $x+\Delta x$. The velocity in the x direction is $u(x,t)$, there is no velocity in the y direction. Mass conservation for this small section states,

Rate of change of the total mass between $x-\Delta x$ and $x+\Delta x$ = incoming mass flux $-$ outgoing mass flux.

In equations this is

$$\begin{aligned}\frac{\partial}{\partial t}(h2\Delta x\Delta y\rho_0) &= u(x-\Delta x,t)h(x-\Delta x,t)\rho_0\Delta y - u(x+\Delta x,t)h(x+\Delta x,t)\rho_0\Delta y\\ &= \frac{u(x-\Delta x,t)h(x-\Delta x,t)-u(x+\Delta x,t)h(x+\Delta x,t)}{2\Delta x}2\Delta x\rho_0\Delta y\\ &\approx -\frac{\partial(u(x,t)h(x,t))}{\partial x}2\Delta x\rho_0\Delta y\end{aligned}$$

so that

$$\frac{\partial h}{\partial t}+\frac{\partial(uh)}{\partial x}=0.$$

if $h=H+\eta$, such that H is constant and $\eta \ll H$, $u\ll 1$, we can write $uh=uH+u\eta\approx uH$ and $\partial_t h=\partial_t\eta$ so that

$$\frac{\partial \eta}{\partial t}+H\frac{\partial u}{\partial x}=0.$$

3.3 Shallow water 1d momentum equation

Consider the momentum budget of the same section:

Mass$\times$acceleration=horizontal pressure force at $x-\Delta x$ $-$ pressure force at $x+\Delta x$.

Mass is $\rho_0(2\Delta x)\Delta y h(x,t)$. Acceleration is $\frac{\partial u}{\partial t}$. Integrating the hydrostatic equation

$$p_z=-g\rho_0$$

with $z=0$ being the bottom and with the top being at $z=h$, we have

$$p(x,z)=g\rho_0(h-z).$$

Note that the pressure vanishes at the surface, as it should (we are ignoring atmospheric pressure). The total pressure force at a point x is then given by

$$\Delta y\int_0^{h(x,t)}p(x,z)dz=\Delta y\int_0^{h(x,t)}\rho_0 g(h-z)\,dz=-\Delta y\rho_0 g\frac{1}{2}(h-z)^2\Big|_0^h=\Delta y\rho_0 g\frac{1}{2}h^2$$

We can now write the complete equation,

$$\begin{aligned}
\rho_0(2\Delta x)\Delta y h(x,t)\frac{\partial u}{\partial t} &= \Delta y \rho_0 g \frac{1}{2}(h^2(x-\Delta x) - h^2(x+\Delta x)) \\
&\approx -\Delta y \rho_0 g \frac{1}{2}(2\Delta x)\frac{\partial h^2}{\partial x} \\
&= -\Delta y \rho_0 g (2\Delta x) h \frac{\partial h}{\partial x}
\end{aligned}$$

so that

$$\frac{\partial u}{\partial t} = -g\frac{\partial h}{\partial x}$$

or, in terms of η,

$$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x}$$

3.4 Shallow water 1d wave equation

Combining the results above we have ★★★

$$\begin{aligned}
\frac{\partial \eta}{\partial t} &= -H\frac{\partial u}{\partial x} \\
\frac{\partial u}{\partial t} &= -g\frac{\partial \eta}{\partial x}.
\end{aligned}$$

Take $\frac{\partial}{\partial t}$ of the first equation, take $\frac{\partial}{\partial x}$ of the second equation, multiply the second by H and subtract the second from the first to find,

$$\frac{\partial^2 \eta}{\partial t^2} = gH\frac{\partial^2 \eta}{\partial x^2}$$

which is the wave equation!

Solution, try $\eta = \eta_0 \cos(kx - \omega t)$ to find that this solves the equation only if,

$$\omega^2 = gHk^2 \quad \Rightarrow \quad \omega = \pm\sqrt{gH}k$$

This is the dispersion relation, $\omega(k)$. We can now also calculate the phase velocity (in this case equal to the group velocity) $c = \omega/k = \pm\sqrt{gH}$, just like in the scaling argument. The $\pm$ correspond to two waves traveling in opposite directions.

3.5 Particle trajectories

We can first find the velocity $u(x,t)$ from the above solution for the surface height $\eta(x,t)$. Use the momentum equation

$$u_t = -g\eta_x = gk\eta_0 \sin(kx - \omega t)$$

so that therefore

$$u = (gk/\omega)\eta_0 \cos(kx - \omega t)$$

For a water parcel starting at x_0, we can use

$$dx/dt = u(x,t) \approx u(x_0,t) = (gk/\omega)\eta_0 \cos(kx_0 - \omega t),$$

and we can integrate this to find

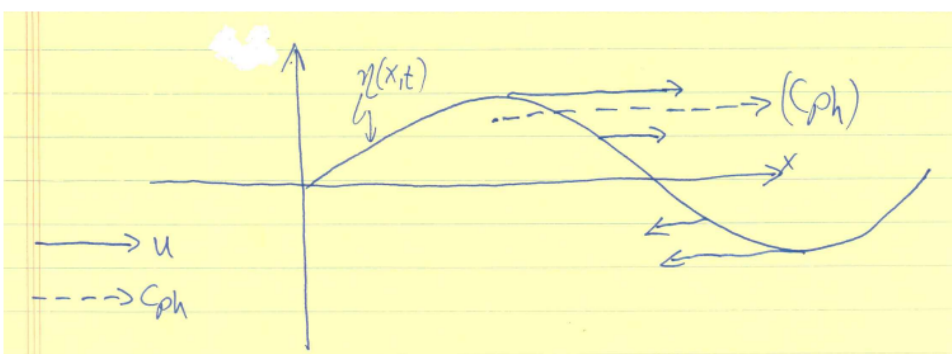
$$x(t) = x_0 - (gk/\omega^2)\eta_0 \sin(kx_0 - \omega t).$$

This tells us that water parcels oscillate back and forth around their initial location x_0 but *do not* propagate with the wave!

Consider the phase between the velocity and the surface elevation,

$$\eta = \eta_0 \cos(kx - \omega t)$$
$$u = \eta_0 \frac{gk}{\omega} \cos(kx - \omega t)$$

so that maximum parcel velocity occurs at the crests and is directed in the same direction as the phase velocity $c = \omega/k$, and the parcel velocity at the troughs is in the opposite direction (because the cosine is negative there). See following schematic figure, this is a good point to discuss breaking waves.



Note that the nondimensional factor appearing in the solution for $x(t)$ may be written as

$$\frac{gk}{\omega^2} = \frac{gk}{gHk^2} = \frac{1}{Hk} = \frac{1}{2\pi}\frac{\lambda}{H} >> 1$$

where the inequality is based on the assumption of shallow water waves that the wave length is larger than the depth, and this equation implies that the amplitude of parcel motion in the horizontal direction is much larger than η_0 which is the amplitude in the vertical direction. Hence the parcel motion is essentially horizontal.

Discuss Tsunami and how it is low-amplitude in open ocean, frequency does not change as it approaches the shallower coastal areas but wave length does, so conservation of energy per wave length dictates increase in amplitude near coast.

3.6 Standing waves

Consider two counter-propagating superimposed waves of the form $\eta(x,t) = \eta_0 \sin(kx - \omega t) = \eta_0 \sin(k(x - ct)$ where $c = \omega/k$, ★★★

$$\eta(x,t) = \eta_0 \sin(k(x - ct)) + \eta_0 \sin(k(x - (-c)t)),$$

The frequencies of the waves are given by $\omega = ck$ and $\omega = -ck$, correspondingly. Thus they propagate in opposite directions, but are identical otherwise. Using the formula for adding cosines, we find,

$$\eta(x,t) = 2\eta_0 \sin(kx)\cos(\omega t).$$

This represents a fixed pattern in space of a wavelength $2\pi/k$ that oscillates with a period $2\pi/\omega$. If we think of this as the oscillations of a string of length L fixed at the edges, the string needs to satisfy $\sin(kx) = 0$ at both ends, which implies $\sin(kL) = 0$, and the possible values of k are therefore $kL = n\pi$, $n = 1, 2, \ldots$. These are standing waves! Note that there are node points in space where the oscillation amplitude vanishes.

3.7 Tidal resonance

Consider an elongated bay of length L, open to the ocean at one end, and closed by a land on the other end. The ocean current u at the ocean side oscillates at the tidal frequency, and this excites shallow-water waves that propagate into the bay. The equation and boundary conditions describing this scenario are then,

$$\begin{aligned} &u_{tt} = (gH)u_{xx} \\ &u(x=0,t) = 0 \\ &u(x=L,t) = u_0 \sin(\omega t). \end{aligned}$$

The waves propagating from the ocean side are expected to be reflected at the other end and lead to opposite-propagating waves and therefore to standing waves, so look for a solution of the form,

$$u(x,t) = A \sin(kx) \sin(\omega t).$$

where ω is the tidal frequency and k is to be determined. Substitute this into the equation to find,

$$-\omega^2 A \sin(kx) \sin(\omega t) = -k^2 (gH) A \sin(kx) \sin(\omega t),$$

which implies that

$$k = \omega/\sqrt{gH}.$$

At $x = 0$, the boundary condition is satisfied,

$$A \sin(k\,0) \sin(\omega t) = 0,$$

while at the other end we have,

$$A \sin(k\,L) \sin(\omega t) = u_0 \sin(\omega t),$$

which implies

$$A = \frac{u_0}{\sin(k\,L)} = \frac{u_0}{\sin\left(\omega\,L/\sqrt{gH}\right)},$$

using the value of k calculated before. We therefore have the final solution for the velocity field due to forced tides in the bay,

$$u(x,t) = \frac{u_0}{\sin\left(\omega\,L/\sqrt{gH}\right)} \sin\left(\omega\,x/\sqrt{gH}\right) \sin(\omega t).$$

The solution implies that when,

$$\omega\,L/\sqrt{gH} = n\pi$$

the sine in the denominator vanishes and the amplitude becomes infinite. This will be modified to a finite large value when we add friction later, but explains the large amplitude tides in some bays. This corresponds to a scenario where the natural seiches of the bay are excited by the tidal frequency. Because η and u are related through the momentum and continuity equations, a large amplitude velocity also implies a large amplitude surface elevation changes, as observed in some locations.

We can also calculate the surface elevation signal, using the mass conservation equation,

$$\eta_x = -u_t/g = -\frac{\omega u_0/g}{\sin\left(\omega\, L/\sqrt{gH}\right)} \sin\left(\omega\, x/\sqrt{gH}\right)\cos\left(\omega t\right)$$

from which we find,

$$\eta = \frac{u_0\sqrt{H/g}}{\sin\left(\omega\, L/\sqrt{gH}\right)} \cos\left(\omega\, x/\sqrt{gH}\right)\cos\left(\omega t\right)$$

and given that for the resonant frequencies we have $\omega/\sqrt{gH} = n\pi/L$, the surface elevation at resonance becomes,

$$\eta = \frac{u_0\sqrt{H/g}}{\sin\left(\omega\, L/\sqrt{gH}\right)} \cos\left(n\pi x/L\right)\cos\left(\omega t\right).$$

The $n = 1$ case shows that the tidal surface elevation variability is maximal at $x = 0$ and $x = L$ with a zero point in between.

As an example, suppose $H = 90$ m, $g = 10$ m/s^2, so that $\sqrt{gH} = 30$ m/s, $\omega = 2\pi/(12\text{ hours})$, and then for the resonance $n = 1$, the length of the basin needs to be,

$$L = \pi\frac{\sqrt{gH}}{\omega} = 648\text{ km}.$$

A wave travels in this example from the open ocean to the other end and back in a time $2L/\sqrt{gH} = 12$ hours. Hence there is a resonance between this propagating wave and the tidal forcing at the bay entrance, leading to the strong response. This simple explanation is meant to only provide a crude insight into tidal resonance, actual details are somewhat more complex.

3.8 Deep 1d water waves scaling argument

Consider an option of infinite depth, so that the depth is no longer a relevant factor. The relevant dimensional constants are: g – gravity; ω – frequency (2π/period); k – wavelength (2π/wavelength). Try writing the frequency ω as function of k, g, ★★★

$$\begin{aligned} &[\omega] = 1/s = [k]^a[g]^b = (1/m)^a(m/s^2)^b \\ \Rightarrow \quad & a = 1/2; b = 1/2 \end{aligned}$$

so that $\omega = \sqrt{gk}$ and therefore $c_{ph} = \omega/k = \sqrt{g/k}$ and $c_g = \partial\omega/\partial k = \sqrt{g/k}/2$. This turns out to be the exact result, if one solves the relevant equations for deep gravity waves. Note that these waves are dispersive: different wave lengths travel at different speeds.

Discuss particle motions in deep gravity waves.

Discuss swell: waves arriving from a remote storm and why swell is long wave/ smooth.

3.9 Finite ocean depth and limits of shallow and deep water

If the depth is not assumed to necessarily be very small nor very large, the dispersion relation is found to be

$$\omega^2 = gk \tanh(kH).$$

Let's see how this general relation behaves for the limits examined above.

First, in the case of shallow water, $\lambda = 2\pi/k \gg H$, we have $kH \ll 1$ and therefore $\tanh(kH) \approx kH$, so that $\omega^2 \approx gHk^2$ as before.

Next, in the case of very deep water, $\lambda = 2\pi/k \ll H$, we have $kH \gg 1$ and therefore $\tanh(kH) \approx 1$, so that $\omega^2 \approx gk$ as before.

4 Buoyancy oscillations

Consider a fluid element in a stratified ocean, displaced by a distance δz. We now allow for acceleration in the vertical dimension, so that the vertical momentum budget $F = ma$ takes the form of acceleration balanced by pressure and gravity forces,

$$\rho_0 \frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z} - g\rho.$$

Consider a density perturbation at a level $z + \delta z$ due to a parcel that was lifted from a level z. The density perturbation is the density at the level at which the parcel originated, minus that at the level to which it arrived, $\delta\rho(\delta z) = \bar{\rho}(z) - \bar{\rho}(z + \delta z)$, or

$$\delta\rho = -\frac{\partial\bar{\rho}}{\partial z}\delta z$$

noting that if $\delta z > 0$, that implies $\delta\rho > 0$, which makes sense as a denser fluid parcel moves up into a lighter fluid, creating a positive density anomaly there. Next, assume the pressure balances the mean density rather than the perturbed density,

$$0 = -\frac{\partial p}{\partial z} - g\bar{\rho}(z).$$

Subtracting this background static momentum balance from the above vertical momentum equation we are left with,

$$\rho_0 \frac{\partial w}{\partial t} = -g\delta\rho.$$

Substitute in this last equation $\delta\rho = -\frac{\partial\bar{\rho}}{\partial z}\delta z$ and $w = \frac{\partial \delta z}{\partial t}$, to find,

$$\rho_0 \frac{\partial^2}{\partial t^2}\delta z = -\left[\frac{-g}{\rho_0}\frac{\partial\bar{\rho}}{\partial z}\right]\delta z$$

and we define the buoyancy frequency to be

$$N^2 \equiv \frac{-g}{\rho_0}\frac{\partial\bar{\rho}}{\partial z}$$

so that the solution is

$$\delta z = A\cos Nt$$

which shows that the displaced parcel oscillates in the vertical direction.

5 Internal shallow water waves

Consider a two layer model, with the lower layer much thicker and thus assumed to be at rest (Fig. 1).

The momentum equations for the two layers are

$$\frac{\partial u_1}{\partial t} = -\frac{1}{\rho_1}\frac{\partial p_1}{\partial x} \quad (1)$$

$$\frac{\partial u_2}{\partial t} = -\frac{1}{\rho_2}\frac{\partial p_2}{\partial x} \quad (2)$$

Assuming the deep velocity vanishes implies that the horizontal pressure gradient is also zero in the lower layer, $\frac{\partial p_2}{\partial x} = 0$. Assuming a hydrostatic vertical momentum balance

$$p_z = -g\rho$$

and integrating this balance in z, we can write the pressure at a depth z in the upper layer as

$$p_1(x, y, z, t) = g(-z + \eta_s(x, y, t))\rho_1$$

so that

$$-\frac{1}{\rho_1}\frac{\partial p_1}{\partial x} = -g\frac{\rho_1}{\rho_1}\frac{\partial \eta_s}{\partial x} = -g\frac{\partial \eta_s}{\partial x}.$$

In the lower layer, the pressure is due to an integral of ρ_1 over the depth of first layer ($h_1 = H_1 + \eta_s - \eta_d$), plus an integral of ρ_2 over the depth range within the second layer, from z to $-H_1 + \eta_d$,

$$p_2(x, y, z, t) = g(H_1 + \eta_s - \eta_d)\rho_1 + g(-H_1 + \eta_d - z)\rho_2$$

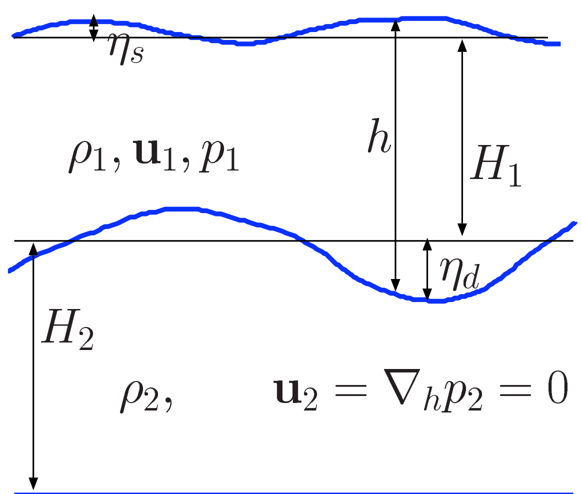


Figure 1: The $1\frac{1}{2}$ layer model

so that

$$\begin{aligned}\frac{1}{\rho_2}\frac{\partial p_2}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{\rho_1}{\rho_2}g\eta_s + \frac{\rho_2-\rho_1}{\rho_2}g\eta_d\right) \\ &\approx \frac{\partial}{\partial x}(g\eta_s + g'\eta_d)\end{aligned}$$

where

$$g' \equiv \frac{\rho_2-\rho_1}{\rho_0}g \approx \frac{\rho_2-\rho_1}{\rho_2}g.$$

The assumption that the deep horizontal pressure gradient vanishes gives

$$g\frac{\partial}{\partial x}\eta_s = -g'\frac{\partial}{\partial x}\eta_d$$

which, together with the observation that $g' \ll g$ so that $\eta_s \ll \eta_d$, implies

$$g\frac{\partial \eta_s}{\partial x} = -g'\frac{\partial \eta_d}{\partial x} \approx g'\frac{\partial h}{\partial x}.$$

The first equality tells us that the upper surface η_s varies in the opposite direction from the deep interface η_d, and at a much smaller amplitude. This means that internal waves have a signal that is seen at the ocean surface and can therefore be observed remotely (e.g., from satellites). The second equality, together with the above relations, finally allows us to write the horizontal pressure gradient in the upper layer as a function of the upper layer thickness

$$-\frac{1}{\rho_0}\frac{\partial p_1}{\partial x} = -g'\frac{\partial h}{\partial x},$$

so our momentum and mass conservation equations may be written as

$$\frac{\partial u_1}{\partial t} = -g'\frac{\partial h}{\partial x} \tag{3}$$

$$\frac{\partial h}{\partial t} + \frac{\partial (u_1 h)}{\partial x} = 0 \tag{4}$$

Approximating h by its mean value H_1 in the second equation as we did for the shallow water waves, we finally have

$$\frac{\partial u_1}{\partial t} = -g'\frac{\partial h}{\partial x} \tag{5}$$

$$\frac{\partial h}{\partial t} = -H_1\frac{\partial u_1}{\partial x} \tag{6}$$

from which we can derive the wave equation

$$\frac{\partial^2 h}{\partial t^2} = (g'H_1)\frac{\partial^2 h}{\partial x^2}$$

This are exactly the same equations we had for a single layer of shallow water, with g replaced by g'. The dispersion relation is

$$\omega = \pm\sqrt{g'H_1}k$$

and the phase and group wave velocities in this case are both equal to $c = \pm\sqrt{g'H_1}$. This also shows that internal waves propagate much slower than surface waves. Note also the above relation between the internal wave displacement of the interface between the layers, and the smaller displacement of the surface, providing a surface signature of internal waves.

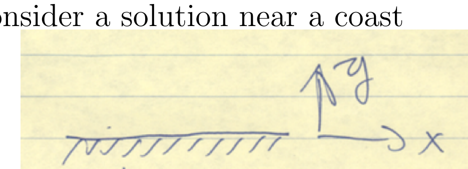
6 Shallow water waves in the presence of rotation

6.1 Coastal Kelvin waves

Start from linearized shallow water momentum and continuity equation in 2d,

$$\begin{aligned}
\frac{\partial u}{\partial t} - fv &= -g\frac{\partial \eta}{\partial x} \\
\frac{\partial v}{\partial t} + fu &= -g\frac{\partial \eta}{\partial y} \\
\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) &= 0
\end{aligned}$$

consider a solution near a coast



and look for a solution with $v = 0$ everywhere, because we know it must vanish at the coast. The equations become

$$\begin{aligned}
\frac{\partial u}{\partial t} &= -g\frac{\partial \eta}{\partial x} \\
fu &= -g\frac{\partial \eta}{\partial y} \\
\frac{\partial \eta}{\partial t} + H\frac{\partial u}{\partial x} &= 0
\end{aligned}$$

Consider a wave solution,

$$\begin{aligned}
u &= \hat{u}(y)\cos(kx - \omega t) \\
\eta &= \hat{\eta}(y)\cos(kx - \omega t)
\end{aligned}$$

the first momentum equation gives

$$-\omega \hat{u} = -gk\hat{\eta}$$

or

$$\hat{u} = \frac{gk}{\omega}\hat{\eta}$$

substituting in the second momentum equation we find

$$\frac{\partial \hat{\eta}}{\partial y} = -\frac{f}{g}\hat{u} = -\frac{fk}{\omega}\hat{\eta}$$

and given that $\omega/k = c$ we can write the solution as

$$\hat{\eta}(y) = \eta_0 \exp\left(-\frac{f}{c}y\right)$$

using the first momentum equation and the continuity equation together we find the dispersion relation,

$$c = \omega/k = \pm\sqrt{gH}$$

but given the above structure in y we see that only the positive root is physical and that the wave must travel with the coast to its right in the northern hemisphere. The final solution may therefore be written as

$$\begin{aligned}
\eta(x, y, t) &= \eta_0 \exp\left(-\frac{f}{c}y\right)\cos(kx - \omega t) \\
u(x, y, t) &= \eta_0 \frac{g}{c} \exp\left(-\frac{f}{c}y\right)\cos(kx - \omega t) \\
c &= \omega/k = +\sqrt{gH}.
\end{aligned}$$

6.2 Poincare waves

Next, away from a boundary, starting again from

$$\begin{aligned}
\frac{\partial u}{\partial t} - fv &= -g\frac{\partial \eta}{\partial x} \\
\frac{\partial v}{\partial t} + fu &= -g\frac{\partial \eta}{\partial y} \\
\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) &= 0
\end{aligned}$$

look for a plane wave solution

$$
\begin{aligned}
u &= u_0 e^{i(kx+ly-\omega t)} \\
v &= v_0 e^{i(kx+ly-\omega t)} \\
\eta &= \eta_0 e^{i(kx+ly-\omega t)}.
\end{aligned}
$$

substituting in the equations we find

$$
\begin{aligned}
-i\omega u_0 - f v_0 &= -igk\eta_0 \\
-i\omega v_0 + f u_0 &= -igl\eta_0 \\
-i\omega\eta_0 + iH\left(ku_0 + lv_0\right) &= 0.
\end{aligned}
$$

Write this in matrix form,

$$
\begin{pmatrix} -i\omega & -f & igk \\ f & -i\omega & igl \\ ikH & ilH & -i\omega \end{pmatrix} \begin{pmatrix} u_0 \\ v_0 \\ \eta_0 \end{pmatrix} = 0
$$

The determinant needs to vanish for a nontrivial solution to exist. Using wolfram alpha with,

```
determinant[{{-i*omega,-f,i*g*k},{f,-i*omega,i*g*l},{i*k*H,i*l*H,-i*omega}}]
```

leads to

$$
-if^2\omega - igHk^2\omega - igHl^2\omega + i\omega^3 = 0
$$

one solution is $\omega = 0$ and then we are left with

$$
\omega^2 = f^2 + gH(k^2 + l^2)
$$

which is the dispersion relation of Poincare waves. When $f = 0$ this reduces to the usual gravity wave dispersion relation in 2d. With $g = 0$ this becomes the inertial motion dispersion relation.