Schrodinger equation:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \cdot \frac{\partial^2}{\partial x^2} \psi + \frac{1}{2} K \cdot x^2 \psi$$

[Schiff (6.16), (13.1) (pp. 24, 67)]

let $\psi = f(t) \cdot u(x)$

$$\Rightarrow \frac{i\hbar f'}{f} = \left(-\frac{\hbar^2}{2m} u'' + \frac{1}{2} K x^2 u\right) \frac{1}{u} = E \quad (p.30)$$

$$\Rightarrow \begin{cases} f(t) = C \cdot e^{-iEt/\hbar} \\ \left(-\frac{\hbar^2}{2m} \partial_{xx} + \frac{1}{2} K x^2\right) u(x) = E \cdot u(x) \end{cases}$$

(eqn 13.1, p. 67)

can rewrite this as

$$u_{xx} + \frac{2mE}{\hbar} u - \frac{Km}{\hbar^2} x^2 u = 0$$

define $\xi = \alpha \cdot x$ with $\alpha^4 = \frac{mK}{\hbar^2}$

& $\lambda = \frac{2E}{\hbar} \left(\frac{m}{K}\right)^2$

$$\Rightarrow \frac{d^2 u}{d\xi^2} + (\lambda - \xi^2) u = 0 \quad (13.2, p 67)$$

[Landau Lifshitz, quantum mechanics, p. 69]

consider behavior for very large ξ.

we expect a solution that decays for both $\xi \to +\infty$ & $\xi \to -\infty$, so try

(†) $u \sim e^{-ax^2}$. at large ξ, eq'n is

$u_{\xi\xi} - \xi^2 u = 0$. subst (†) to get:

$u_\xi \sim -2a\xi e^{-a\xi^2}$, $u_{\xi\xi} \sim -2a \cdot e^{-a\xi^2} + 4a^2\xi^2 e^{-a\xi^2}$

$\approx 4a^2\xi^2 \exp(-a\xi^2)$

to satisfy $u_{\xi\xi} - \xi^2 u \cong 0$ we need

$4a^2 = 1 \Rightarrow a = \frac{1}{2}$. $\Rightarrow u(\xi \to \infty) \sim e^{-\frac{1}{2}\xi^2}$

so now try a solution to the full eq'n (13.2) of the form:

$$u(\xi) = H(\xi) \cdot e^{-\frac{1}{2}\xi^2}$$

[end Landau-Lifshitz]

$$\Rightarrow H'' - 2\xi \cdot H' + (\lambda - 1) \cdot H = 0.$$

[Schiff, pp 68-69]

using the power method around $\xi = 0$ we have

$$H = \sum_{n=0}^{\infty} a_n \xi^n \text{ . subst into eq'n}$$

$$\underbrace{\sum_{n=2}^{\infty} n(n-1) a_n \xi^{n-2}}_{(1)} - 2\underbrace{\sum_{n=1}^{\infty} n a_n \xi^n}_{(2)} + (\lambda-1)\underbrace{\sum_{n=0}^{\infty} a_n \xi^n}_{(3)} = 0$$

define $m = n-2$ for (1), & $n = m$ for (2, 3)

$$\Rightarrow a_{m+2} = a_m \frac{2m+1-\lambda}{(m+2)(m+1)} \quad (\star)$$

$$\text{as } m \to \infty \quad \frac{a_{m+2}}{a_m} \sim \frac{2}{m}$$

For comparison,

$$e^{2x^2} = \sum_{n=0}^{\infty} \frac{(2x^2)^n}{n!} = \cdots \frac{2^n x^{2n}}{n!} + \frac{2^{n+1}}{(n+1)!} x^{2n+2} + \cdots$$

$$\Rightarrow \frac{a_{m+2}}{a_m} = \frac{2^{n+1}/(n+1)!}{2^n/n!} = \frac{2}{n+1} \sim \frac{2}{n}$$

$\Rightarrow$ if series for H is not terminated at a finite n, it will dominate

$u = H(\xi) \cdot e^{-\frac{1}{2}\xi^2}$ & u will diverge at ∞.

$\Rightarrow$ must make sure that $(\star)$ leads to vanishing of a_{m+2} at some point.

this happens if $2m+1-\lambda = 0$

$$\Rightarrow \lambda = 2m+1$$

The finite polynomial solution is therefore calculated from (A), with $\lambda - 1 = 2m$

Need two basis solutions, try

$[a_0 = 1, \ a_1 = 0]$ (a)

$[a_0 = 0, \ a_1 = 1]$ (b)

→ let $\lambda = 2m+1$, $m = 0 \Rightarrow \lambda = 1$

(a): $a_0 = 1$, $a_2 = (2\cdot 0 + 1 - 1)/(0+2)(0+1) = 0$

$\Rightarrow H_0 = 1$

(b) $a_1 = 1$, $a_3 = (2\cdot 1 + 1 - 1)/(1+2)(1+1) \neq 0$ } diverges.

$a_5 \neq 0 \ldots$

→ let $\lambda = 2m+1$, $m = 1 \Rightarrow \lambda = 3$

(a) $a_0 = 1$, $a_2 =$ (diverges...)

(b) $a_0 = 0; a_1 = 1; \ a_3 = (2\cdot 1 + 1 - 3)/(\ )(\ ) = 0$

$\Rightarrow H_1 = \xi$.

→ let $\lambda = 2m+1$, $m = 2$, $\lambda = 5$ ... etc.

so for each value of m (or, equivalently, λ), one series is finite & one diverges.

$\Rightarrow \ H_0 = 1, \ H_1 = \xi, \ H_2 = 4\xi^2 - 2, \ \ldots$

≡ "Hermite polynomials"...

to summarize:

$$\psi(x,t) = \sum_m C_m e^{iE_m t/\hbar} \cdot e^{-\frac{1}{2}\xi^2} \cdot H_n(\xi)$$

where $\xi = \left(\frac{mk}{\hbar^2}\right)^{\frac{1}{4}} \cdot x$

and because λ is discretized as

$\lambda_m = 2m+1 \quad , \quad m = 0, 1, 2, \ldots$

so is the energy E (the separation variable!):

$$E_m = (2m+1)\frac{\hbar}{2} \cdot \left(\frac{k}{m}\right)^2$$

the constants C_m are calculated from the initial conditions if needed.