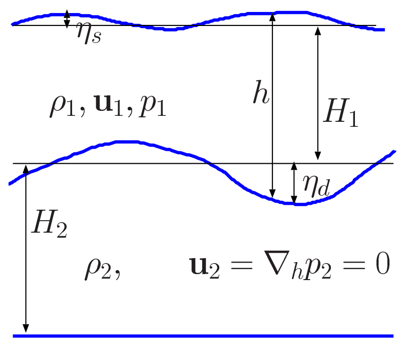

Figure 18: The $1\frac{1}{2}$ layer model

1.2 A brief equatorial dynamics background

1.2.1 Importance of thermocline dynamics and reduced gravity models

The phenomenology above indicates that motions of the equatorial thermocline are critical to ENSO's dynamics. We therefore start by deriving the simplest equations that describe the thermocline dynamics. Consider a two layer model, with the lower layer much thicker and thus assumed to be at rest (Fig. 18).

The momentum equations (Boussinesq approximation)

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} + 2\Omega\times\mathbf{u} = -\frac{1}{\rho_0}\nabla p + \mathbf{g}\rho/\rho_0 + \nu\nabla^2\mathbf{u}$$

then imply that because the horizontal velocity in the lower layer is zero, $\mathbf{u}_{2H} = (u_2, v_2) = 0$, the horizontal pressure gradients are also zero in the lower layer, $\nabla_H p_2 = 0$, with $\nabla_H = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ being the two dimensional horizontal gradient, and where ∇ above stands for the three dimensional gradient operator. Assuming a hydrostatic vertical momentum balance (because $H \ll L$)

$$p_z = -g\rho$$

and integrating this balance in z, we can write the pressure at a depth z in the upper layer as

$$p_1(x, y, z, t) = g(-z + \eta_s(x, y, t))\rho_1$$

so that

$$-\frac{1}{\rho_0}\nabla p_1 \approx -g\nabla\eta_s.$$

In the lower layer, the pressure is

$$p_2(x, y, z, t) = g(H_1 + \eta_s - \eta_d)\rho_1 + g(H_1 + \eta_d - z)\rho_2$$

so that

$$\begin{aligned}\frac{1}{\rho_0}\nabla_H p_2 &= \nabla_H\left(\frac{\rho_1}{\rho_0}g\eta_s + \frac{\rho_2-\rho_1}{\rho_0}g\eta_d\right)\\ &\approx \nabla_H(g\eta_s + g'\eta_d)\end{aligned}$$

where $g' \equiv \frac{\rho_2-\rho_1}{\rho_0}g \approx \frac{\rho_2-\rho_1}{\rho_2}g$. That this deep horizontal pressure gradient vanishes gives

$$g\nabla_H\eta_s = -g'\nabla_H\eta_d$$

which, together with the observation that $g' \ll g$ so that $\eta_s \ll \eta_d$, implies

$$g\nabla_H \eta_s = -g'\nabla_H \eta_d \approx g'\nabla_H h.$$

Together with the above relations this finally allows us to write the horizontal pressure gradient in the upper layer as a function of the upper layer thickness

$$-\frac{1}{\rho_0}\nabla p_1 = -g'\nabla h.$$

1.2.2 The equatorial β plane

The objective now is to find a convenient representation of the effect of the earth rotation near the equator [43, 20]. We start with horizontal momentum equations for a $1\frac{1}{2}$ layer fluid as above, on a sphere, where we use

$$2\vec{\Omega} \times \mathbf{u} = 2\Omega \begin{pmatrix} w\cos\theta - v\sin\theta \\ u\sin\theta \\ -u\cos\theta \end{pmatrix}.$$

to find

$$\begin{aligned} \frac{du}{dt} + \frac{uw}{r} - \frac{uv}{r}\tan\theta + 2\Omega(w\cos\theta - v\sin\theta) &= -\frac{g'}{r\cos\theta}\frac{\partial h}{\partial \phi} + \mathcal{F}_\phi \\ \frac{dv}{dt} + \frac{wv}{r} - \frac{u^2}{r}\tan\theta + 2\Omega u\sin\theta &= -\frac{g'}{r}\frac{\partial h}{\partial \theta} + \mathcal{F}_\theta \end{aligned}$$

where

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \frac{u}{r\cos\theta}\frac{\partial}{\partial \phi} + \frac{v}{r}\frac{\partial}{\partial \theta} + w\frac{\partial}{\partial r},$$

and where $(\mathcal{F}_\phi, \mathcal{F}_\theta)$ represent the forcing and dissipation terms, and θ, ϕ the latitude and longitude. Next, assume linear momentum dynamics, and use the fact that $w \ll (u, v)$. Also, write the vertical coordinate as $r = r_0 + z$ where r_0 is the earth radius, so that within a thin layer of fluid (ocean thickness $\ll$ earth radius) we have $1/r = 1/(r_0 + z) \approx 1/r_0$, and therefore,

$$\begin{aligned} \frac{\partial u}{\partial t} - 2\Omega\sin\theta v &= -\frac{g'}{r_0\cos\theta}\frac{\partial h}{\partial \phi} + \mathcal{F}_\phi \\ \frac{\partial v}{\partial t} + 2\Omega\sin\theta u &= -\frac{g'}{r_0}\frac{\partial h}{\partial \theta} + \mathcal{F}_\theta. \end{aligned}$$

Next, we restrict our attention to near-equatorial regions, where we can define local Cartesian coordinates around some central location (θ_0, ϕ_0)

$$\begin{aligned} x &\equiv r_0\cos\theta_0(\phi - \phi_0) \\ y &\equiv r_0(\theta - \theta_0), \end{aligned}$$

as well as expand the Coriolis force as

$$\begin{aligned} 2\Omega\sin\theta &\approx 2\Omega\sin\theta_0 + 2\Omega\cos\theta_0(\theta - \theta_0) \\ &= f_0 + \beta y \end{aligned}$$

with $\beta \equiv 2\Omega \cos\theta_0 / r_0$. An expansion around the equator $\theta_0 = 0$ leads to $f_0 = 0$. Using a simple linear friction law and incorporating the wind stress forcing

$$\begin{aligned}\mathcal{F}_\phi &= \mathcal{F}_x = -\varepsilon u + \tau^x/(\rho_0 H) \\ \mathcal{F}_\theta &= \mathcal{F}_y = -\varepsilon v + \tau^y/(\rho_0 H)\end{aligned}$$

we obtain the final set of β-plane momentum equations for a $1\frac{1}{2}$ layer model

$$\frac{\partial u}{\partial t} - \beta y v = -g' \frac{\partial h}{\partial x} - \varepsilon u + \frac{\tau^x}{\rho_0 H} \quad (1)$$

$$\frac{\partial v}{\partial t} + \beta y u = -g' \frac{\partial h}{\partial y} - \varepsilon v + \frac{\tau^y}{\rho_0 H}. \quad (2)$$

and with three unknowns (u, v, h) we need a third equation which is provided by the linearized mass conservation equation (which also includes on the rhs a rough linear parameterization of entrainment (mixing) at the base of the water layer above the thermocline)

$$\frac{\partial h}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = -\varepsilon h. \quad (3)$$

1.2.3 Equatorial waves

The derivation here follows Gill [20]. Consider first the case of an equatorial Kelvin wave, which is a special solution of (1,2,3) for the case of zero meridional velocity ($v = 0$), no forcing and no dissipation. In this case, these equations reduce to

$$\begin{aligned}\frac{\partial u}{\partial t} &= -g' \frac{\partial h}{\partial x} \\ \beta y u &= -g' \frac{\partial h}{\partial y} \\ \frac{\partial h}{\partial t} + H\frac{\partial u}{\partial x} &= 0\end{aligned}$$

Note the geostrophic balance in the y-momentum equation. Substituting $e^{i(kx-\omega t)}$ dependence for all three variables, we get from the first that $u = (kg'/\omega)h$, so that the third one gives the dispersion relation

$$\omega^2 = (g'H)k^2$$

which is the dispersion relation of a simple shallow water gravity wave. The second equation then gives $\beta y \frac{kg'}{\omega} h = -g' \frac{\partial h}{\partial y}$, or

$$\frac{\partial h}{\partial y} = -\frac{\beta k}{\omega} y h.$$

We are searching for equatorial-trapped solutions, and we note that the solution for the y-structure decays away from the equator only when $k > 0$. This implies that the wave solution we have found must be eastward propagating! Using the dispersion relation, with

$$c \equiv \sqrt{g'H} \approx (9.8 \times 10^2 * cm\, sec^{-2} \times 5 * 10^{-3} \times 100 \times 10^2 cm)^{1/2} \approx 2.2 m/sec$$

we finally have

$$h_{Kelvin}(x, y, t) \propto e^{-\frac{1}{2}(\beta/c)y^2} e^{i(kx-\omega t)}.$$

Note that the decay scale away from the equator is the equatorial Rossby radius of deformation defined as

$$L^R_{eq} \equiv \sqrt{c/2\beta} \approx (c/(2 \times 2.3 \times 10^{-11} m^{-1} sec^{-1}))^{1/2} \approx 220km$$

Next is the derivation of the full set of equatorial waves, where we now do not assume that the meridional velocity v vanishes. Substitute $h(x, y, t) = h(y)e^{i(kx-\omega t)}$ dependence, and similarly for (u, v), and derive a single equation for h to find the parabolic cylinder equation (Gill, [20], section 11.6.1)

$$\frac{d^2v}{dy^2} + \left(\frac{\omega^2}{c^2} - k^2 - \frac{\beta k}{\omega} - \frac{\beta^2}{c^2}y^2\right) v = 0.$$

The solutions that vanish at $y \to \pm\infty$ occur only for certain relations between the coefficients, and these relations serve as the dispersion relation

$$\frac{\omega^2}{c^2} - k^2 - \frac{\beta k}{\omega} = (2n + 1)\frac{\beta}{c}. \tag{4}$$

Note that the Kelvin wave dispersion relation is formally a solution of this dispersion relation for $n = -1$ (simply check that $\omega = ck$ satisfies (4) for $n = -1$). The meridional structure of the waves in this case of equatorially trapped solutions is expressed in terms of the Hermit polynomials

$$v = 2^{-n/2}H_n((\beta/c)^{1/2}y) \exp(-\beta y^2/2c) \cos(kx - \omega t)$$

and is shown in Fig. 19, where

$$H_0 = 1; \qquad H_1 = 2x; \qquad H_2 = 4x^2 - 2; \qquad H_3 = 8x^3 - 12x; \qquad H_4 = 16x^4 - 48x^2 + 12$$

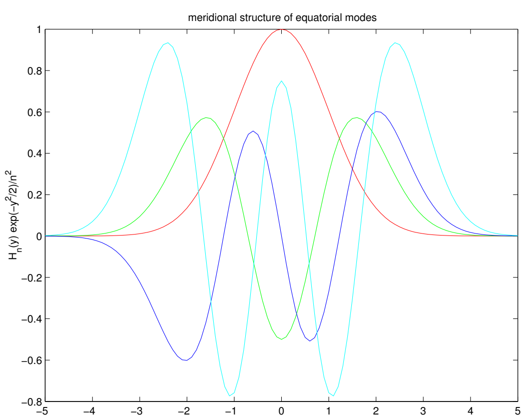

Figure 19: The latitudinal structure of the first few equatorial modes: $H_n(y) \exp(-y^2/2)/n^2$.

The dispersion relation is plotted in Fig. 20.

So, we have a complete set of waves, the Kelvin ($n = -1$), Yanai ($n = 0$), Rossby and Poincare ($n > 0$) waves. As seen in the plot, the dispersion relation includes two main sets of waves for $n > 0$. For high frequency, we can neglect the term $\frac{\beta k}{\omega}$, to find the Poincare gravity-inertial waves

$$\omega^2 \approx (2n + 1)\beta c + k^2c^2,$$

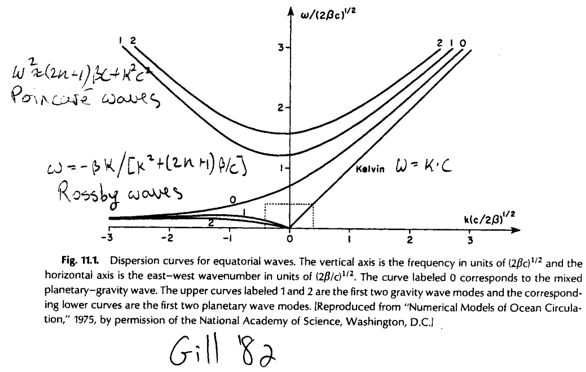

Figure 20: The Equatorial wave dispersion relation, (Gill [20], p 438, Fig. 11.1)/

while for low frequency, we can neglect the term ω^2/c^2 in the dispersion relation to find the westward propagating Rossby wave dispersion relation

$$\omega = \frac{-\beta k}{k^2 + (2n+1)\beta/c}.$$

Typical speeds of long Rossby waves would therefore be

$$\omega/k = \frac{-c}{2n+1}$$

so that the first Rossby mode ($n = 1$) travels at a 1/3 of the Kelvin wave speed, implying a roughly 2.5 months crossing time for Kelvin and 8 months for Rossby waves (based on 15,000 km basin width).

Note (Fig. 19) that the first Rossby mode has a zero at the equator and two maxima away from the equator, while the Kelvin wave has a maximum at the equator. This tells us something about how a random initial perturbation will project on the different modes. That is, a forcing pattern or an initial perturbation that is centered at the equator may be expected to excite Kelvin waves, while a forcing or initial perturbation that has components off the equator will tend to excite Rossby waves. The above discussion centers on the first baroclinic mode, but may be generalized to higher vertical baroclinic modes, although for our purposes this is not essential.

1.2.4 Ocean response to wind perturbation

Consider first the mean state of the thermocline. The steady state ($\partial u/\partial t = 0$) momentum equation (1) in a reduced gravity model, at $y = 0$ ($\beta y v = 0$) in the presence of easterly wind forcing and neglecting frictional effects ($-\varepsilon u = 0$) is

$$0 \approx -g' \frac{\partial h}{\partial x} + \frac{\tau^x}{\rho_0 H}$$

so that an easterly wind stress is balanced by a pressure gradient due to a thermocline tilt, with the thermocline closer to the surface in the East Pacific. This mean state of the thermocline results in the cold tongue there, as observed, via the mixing of cold sub-thermocline water with the surface water, as will be discussed more quantitatively below.

Regarding the interannual equatorial thermocline variability, at this stage we just note that a wind perturbation that corresponds to a weakening of the mean easterlies in the central Pacific affects the thermocline depth in the central Pacific. It creates downwelling Kelvin waves (that is, waves that propagate a downwelling signal,