Singular perturbation notes

Eli, APM202

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1 ODE reminder from CP §15.1

Consider

$$\varepsilon w'' + w' = 1$$
$$w(0) = w(1) = 0$$

try first $w = w_0 + \varepsilon w_1 + \cdots$ to find $O(1)$ eqn is $w_0' = 1$, so that $w_0 = x + C$ which can satisfy only one b.c. $O(\varepsilon)$ eqn is $w_0'' + w_1' = 0$ or $w_1' = 0$, which cannot satisfy the b.c. hence the failure. Problem is that ε multiples the highest derivative so that this is a singular perturbation problem.

Try instead $w = w_i(x) + w_{bl}(\xi)$ where $\xi = x/\delta$, δ is a small yet unknown parameter, and we expect w_{bl} to be important only near the boundary at $x = 0$. Away from the boundary, $w_i' = 1$, and the solution $w_i = x - 1$ satisfies the b.c at $x = 1$. Near $x = 0$ we have

$$\varepsilon(\partial_{xx} w_i + \frac{1}{\delta^2}\partial_{\xi\xi} w_{bl}) + \partial_x w_i + \frac{1}{\delta}\partial_\xi w_{bl} = 1$$

using the solution for w_i this becomes

$$\varepsilon \frac{1}{\delta^2}\partial_{\xi\xi} w_{bl} + \frac{1}{\delta}\partial_\xi w_{bl} = 0$$

to get both terms to balance we choose the boundary layer thickness to be $\delta = \varepsilon$ so that the equation is

$$\partial_{\xi\xi} w_{bl} + \partial_\xi w_{bl} = 0$$

one integration gives

$$\partial_\xi w_{bl} + w_{bl} = A$$

and the solution is

$$w_{bl} = A + Be^{-\xi}.$$

The combined solution is

$$w = x - 1 + A + Be^{-\xi}.$$

and imposing the two boundary conditions, noting that $e^{-\xi}$ may be neglected at $x = 1$, this becomes

$$w = x - 1 + e^{-\xi} = x - 1 + e^{-x/\varepsilon}$$

What if we solved the equation for w_i such that it satisfies the b.c at $x = 0$ instead of at $x = 1$? In that case, $w_i(x) = x$. The boundary layer is now needed at $x = 1$, so we let

$\xi = (1-x)/\delta$ such that $\xi \to \infty$ as we get away from the boundary, and the equation for $w_{bl}(\xi)$ becomes (note the minus sign difference from above),

$$\partial_{\xi\xi} w_{bl} - \partial_{\xi} w_{bl} = 0.$$

One integration gives

$$\partial_{\xi} w_{bl} - w_{bl} = A,$$

and the solution is

$$w_{bl}(\xi) = Be^{\xi} - A.$$

This solution does not decay away from the boundary layer at $x = 1$ and is therefore not self-consistent. This demonstrates that the boundary layer must be at $x = 0$.

2 PDE example from CP §15.1

Consider

$$\begin{aligned}
&\varepsilon(v_{xx} + v_{yy}) + v_x = y(1 - y^2),\\
&0 < x < 1, \quad 0 < y < 1,\\
&v = 0 \text{ on all boundaries.}
\end{aligned}$$

This happens to be the equation describing the wind-driven circulation in the ocean, e.g., the North Atlantic ocean and the Gulf Stream. In that application v is the stream function...

Write $v = v^i + v^{bl}$ where $v^i(x, y)$ is the interior solution and $v^{bl}(x, y, \varepsilon)$ the boundary layer solution. The interior solution is

$$v^i = y(1 - y^2)x + A(y)$$

which we may write as

$$v^i = y(1 - y^2)(x + B(y)),$$

so that the y b.c. are satisfied for any $B(y)$. We can now satisfy the x b.c. at $x = 0$ by choosing $B = 0$ or at $x = 1$ by choosing $B = -1$. In either of these cases, a boundary layer would be needed at the other end.

Suppose we try $B = -1$ first, and then a boundary layer is needed at $x = 0$. Define $\xi = x/\delta$ and the boundary layer equation becomes

$$\varepsilon(v^{bl}_{xx} + v^{bl}_{yy}) + v^{bl}_x + \varepsilon(v^i_{xx} + v^i_{yy}) + v^i_x = y(1 - y^2)$$

or,

$$\varepsilon(\frac{1}{\delta^2}v^{bl}_{\xi\xi} + v^{bl}_{yy}) + \frac{1}{\delta}v^{bl}_\xi + \varepsilon(v^i_{xx} + v^i_{yy}) + v^i_x = y(1 - y^2)$$

neglecting $O(\varepsilon)$ terms and using the solution for the interior solution,

$$\varepsilon\frac{1}{\delta^2}v^{bl}_{\xi\xi} + \frac{1}{\delta}v^{bl}_\xi \approx 0$$

for both terms to have the same order, choose $\delta = \varepsilon$ to get,

$$v^{bl}_{\xi\xi} + v^{bl}_\xi \approx 0.$$

The solution is, as in the ODE case,

$$v^{bl} = C(y)e^{-\xi} + D(y)$$

The complete solution is now

$$v = y(1 - y^2)(x - 1) + C(y)e^{-x/\varepsilon} + D(y).$$

to satisfy the boundary conditions at $y = 0, 1$ we can write this as

$$v = y(1 - y^2)\left[(x - 1) + E(y)e^{-x/\varepsilon} + F(y)\right].$$

The b.c. at $x = 1$ implies $F = 0$ (note that the exponential term may be neglected there), while the b.c. at $x = 0$ implies $E = 1$, so that our final solution is

$$v = y(1 - y^2)\left[x - 1 + e^{-x/\varepsilon}\right].$$

HW: try $B = 0$ case, show that the right side of the domain cannot support a b.l. so that the above is the only option.