wind driven circulation and vorticity dynamics, including western bndry currents

consider an ocean with a flat bottom, uniform density, and horizontal velocities that are independent of depth. The momentum equations are

[acceleration + coriolis = pressure + friction + wind stress]

$$\frac{\partial u}{\partial t} - fv = \frac{-1}{\rho}\frac{\partial p}{\partial x} - ru + \quad (1)$$

$$\frac{\partial v}{\partial t} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} - rv + \quad (2)$$

f, the coriolis force may be approximated using a taylor expansion:

$$f = 2\Omega \sin\theta \cong 2\Omega \sin\theta_0 + 2\Omega \cos\theta_0 \cdot (\theta - \theta_0)$$

$$= [2\Omega \sin\theta_0] + \left[\frac{2\Omega}{R}\cos\theta_0\right]\left[r\cdot(\theta-\theta_0)\right]$$

(3) $$= f_0 + \beta \cdot y$$

where θ_0 is a latitude in the middle of the ocean, R earth's radius, & y the northward distance in meters. Note: $\beta = \frac{df}{dy}$.

$(\tau^{(x)}, \tau^{(y)})$ are the wind stress components. H is the ocean depth.

[wind driven circulation]

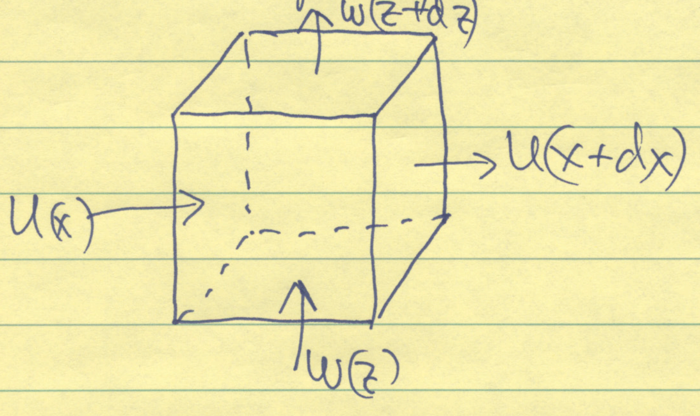
now take the curl of the momentum equations, which means $\frac{\partial(2)}{\partial x} - \frac{\partial(1)}{\partial y}$ to find

$$(4) \quad \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \beta \cdot v$$
$$= -r\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \ldots$$

now, we need to derive a few related results in order to continue.

(A) mass conservation equation:

consider a small volume of water, fixed in space & calculate its mass budget:



rate of change of mass in this box is equal to inflow − outflow. ($dx = \Delta x \ldots$)

$$\Rightarrow \frac{\partial}{\partial t}(\rho \cdot \Delta x \Delta y \Delta z) = [u(x) - u(x+dx)]\,dy\,dz$$
$$+ [v(y) - v(y+dy)]\,dx\,dz$$
$$+ [w(z) - w(z+dz)]\,dx\,dy$$

$\parallel$ 0 $(\rho = \text{const})$

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[wind driven circulation]

$$(5) \Rightarrow \boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0}$$

"continuity equation" or "mass conservation for an incompressible fluid"

B Ekman pumping:

Integrate the continuity equation over the upper 50m of the ocean:

$$\int_{-50}^{0}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)dz + \int_{-50}^{0}\frac{\partial w}{\partial z}dz = 0$$

$$(6) \Rightarrow \frac{\partial}{\partial x}\int u\,dz + \frac{\partial}{\partial y}\int v\,dz = -\cancel{w(\text{surface})} + w(-50).$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad = 0$$

we found before that the horizontal transport at the surface due to the wind stress is equal to:

$$(7) \int_{-50}^{0} u\,dz = M_{(x)} = \frac{\tau^{(y)}}{\rho \cdot f}$$

"Ekman transport"

$$(8) \int_{-50}^{0} v\,dz = M_{(y)} = -\frac{\tau^{(x)}}{\rho f}$$

[wind-driven circulation]

now subst (7,8) into (6) to find

(9) $$\text{"Ekman pumping"} = w(-50_m) = \frac{\partial}{\partial x}\left(\frac{\tau^{(y)}}{\rho f}\right) - \frac{\partial}{\partial y}\left(\frac{\tau^{(x)}}{\rho f}\right).$$
$$= \text{curl}\,\frac{\vec{\tau}}{\rho f}.$$

(c) vorticity:

vorticity is defined as the curl of the velocity field.

$$\vec{\zeta} = \vec{\nabla}\times\vec{u} = \text{curl}\,\vec{u} = \begin{bmatrix} \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \end{bmatrix}$$

we denote the vertical component by ζ, so that $\zeta \equiv \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$.

*what does ζ mean physically?:

consider a "solid body rotation" of, say, water in a rotating bucket:
in cylindrical coordinates, the velocity is

$$v^{(r)} = 0, \quad v^{(\theta)} = \omega \cdot r.$$

[wind driven circulation]

the vorticity is therefore

$$\zeta = \text{curl}\left(v^{(r)}, v^{(\theta)}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(r \cdot v^{(\theta)}\right) - \frac{\partial}{\partial \theta}\left(v^{(r)}\right)$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left\{r \cdot (\omega \cdot r)\right\} = 2 \cdot \omega$$

$\Rightarrow$ vorticity = 2 · rotation rate

① "planetary" vorticity:

If the ocean is at rest, it still has some vorticity due to the earth rotation. this would be solid body rotation then.

* at the north pole, $\omega = \Omega$ so that the vorticity is $2\omega = 2\Omega$.
* at the south pole, $\omega = -\Omega$ (local vertical direction is opposite, hence the minus), $\Rightarrow$ vorticity $= -2\Omega$.
* in between, at some latitude θ:

planetary vorticity $= 2\Omega \sin\theta = f$!

Back to the vorticity equation (4) on p. 2:

first note that $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial w}{\partial z}.$ $\boxed{\#}$ ~~now w~~ is

Next, integrate (4), use $\boxed{\#}$ & (9), and re-define $\int dz\, u \to u$, $\int dz\, v \to v$.
integral is from bottom to base of Ekman layer ~50m

Second, use the notation $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \equiv \zeta$,

$$\underbrace{\frac{\partial \zeta}{\partial t}}_{\boxed{1}} + \underbrace{\beta v}_{\boxed{2}} = \underbrace{-r\zeta}_{\boxed{3}} + \underbrace{fw(z=-3km)}_{\boxed{4}}$$

"vorticity equation"

Interpretation:

$\boxed{1}$: rate of change of the vorticity.

$\boxed{2}$: $\beta v = v \cdot \frac{df}{dy}$: as a fluid parcel moves in the north-south direction, its planetary vorticity changes due to the change of f as function of y.

$\boxed{3}$ $-r\zeta$: friction damps the velocity & therefore also the vorticity.

[wind-driven circulation]

[4]: This is the effect of Ekman pumping!
to understand this, consider a cylinder of ocean water extending from top to bottom. This fluid column rotates either because of relative vorticity: $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$

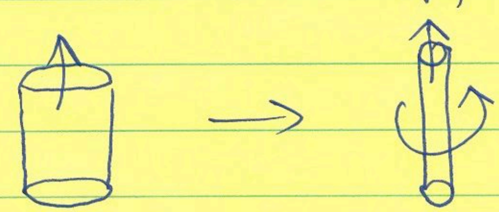
or planetary vorticity $f = 2\Omega \sin\theta$.

→ Now if the w(z=-3km) is up, the fluid column is stretched & because of angular momentum conservation spins faster ⇒ increased vorticity!

→ similarly if the w(z=-3km) is down, column is compressed & spins slower.

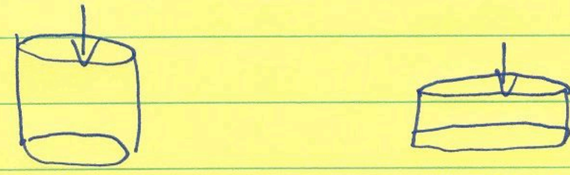
* w(z=-3km) g effect of vorticity:

w(z=-3km) >0 :



t_1 $t_2 > t_1$

w(z=-3km)<0 <0 :



t_1 $t_2 > t_1$

[ ]

we can now consider the vorticity budget in the ocean interior, far from the east or west boundaries, & then near the bndries:

* <u>interior vorticity budget:</u>

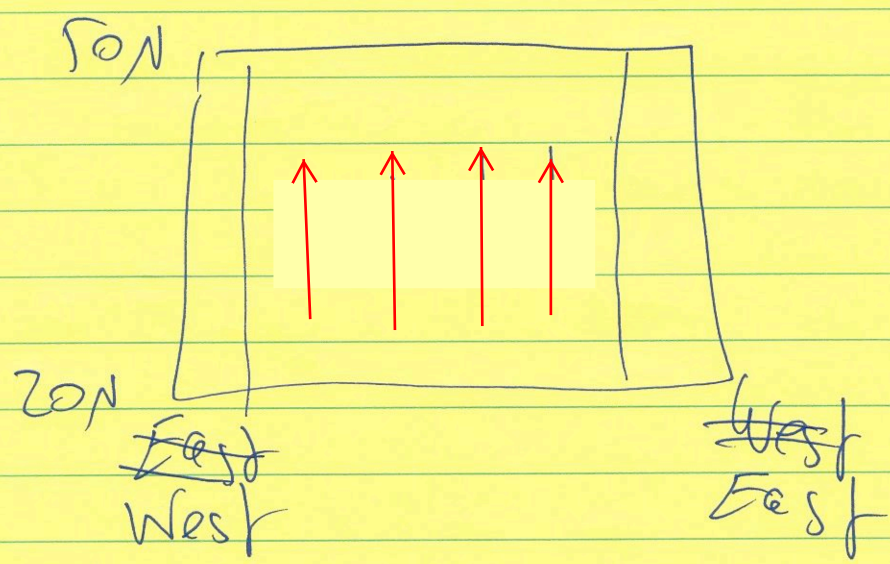
the friction is generally small
assume a steady state

$$\Rightarrow \boxed{\beta v = fw(z=-3km)}$$ { ~~Sverdrup balance~~

~~Ekman pumping is downward in subtropical gyre (20°-50°N)~~

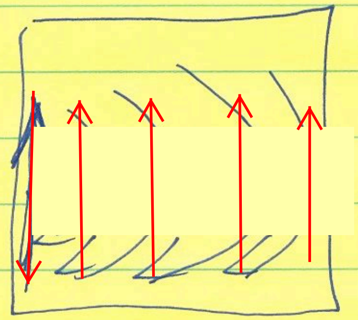
$\Leftrightarrow fw(z=-3km) < 0$

$\Rightarrow$ northward velocity

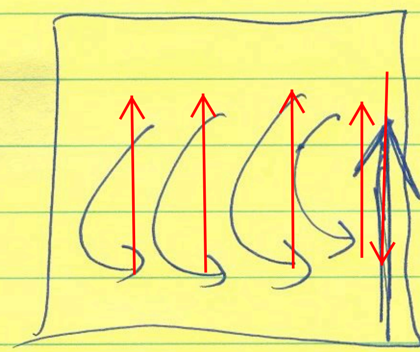


[wind-driven circulation]

where should the northward return flow be?



OR



* vorticity balance in bndry current:

v is large in bndry current, so both βv & the $r \cdot \frac{\partial v}{\partial x}$ part of $r \cdot \zeta$ are larger than curl τ.

$$\Rightarrow \quad \boxed{\beta v = -r \frac{\partial v}{\partial x}}$$

vorticity balance in bndry.

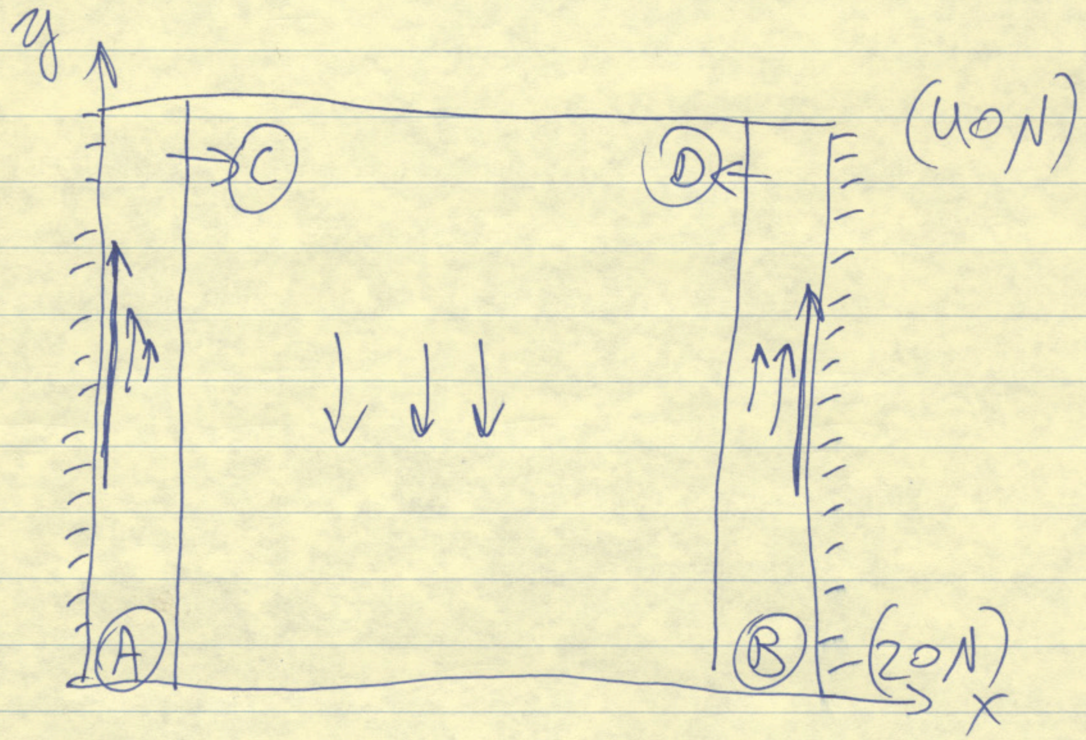
now $v > 0$ in bndry current.

$-r\frac{\partial v}{\partial x}$ is positive only if bndry current is on the west

$\Rightarrow$ Western bndry current

(Stommel 1949)

heuristic explanation of WBC



in WBC : $-r\zeta \cong -r\frac{\partial v}{\partial x}$

in (A) : $-r\frac{\partial v}{\partial x} > 0$: fluid obtains vorticity as it travels north.

in (B) : $-r\frac{\partial v}{\partial x} < 0$: fluid looses vorticity as it travels north.

In interior: vorticity $= f = f_0 + \beta y$,

$\Rightarrow$ fluid looses vorticity as it travels south.

$\Rightarrow$ (A) compensates for interior loss, allows fluid to remerge with interior at (C);

WBC heuristics continued

if bndry current is in B, on the other hand, vorticity is lost at both interior & bndry current, so flow cannot be in a steady state.