Shallow water quasi-geostrophy as a PDE perturbation methods example

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1 Basic equations

The development here closely follows Pedlosky (1987) and section 5 is based on the original work of Stommel (1948). Consider an ocean of depth H, uniform density ρ, coordinates (x, y) =(east,north), with a Coriolis force which is a function of latitude $f = f_0 + \beta y$, wind forcing $\tau^{(x)}, \tau^{(y)}$, surface height deviation from rest state $\eta(x, y, t)$ gravity g, bottom friction coefficient r. The (linearized) momentum equations ($F = ma$) correspond to the balance

$$\text{acceleration} = \text{Coriolis force} + \text{pressure force} + \text{friction} + \text{surface wind stress.} \tag{1}$$

The (linearized) mass conservation equation states that the velocity divergence leads to local sea level rise. Together these are,

$$\begin{aligned} u_t &= fv - g\eta_x - ru + \tau^{(x)}/H \\ v_t &= -fu - g\eta_y - rv + \tau^{(y)}/H \\ \eta_t + Hu_x + Hv_y &= 0. \end{aligned} \tag{2}$$

Boundary conditions are that the normal velocities vanish at the boundaries,

$$\begin{aligned} u(x = 0, y) &= u(x = L, y) = 0 \\ v(x, y = 0) &= v(x, y = L) = 0 \end{aligned} \tag{3}$$

and the initial conditions are of some specified initial velocities and sea surface height.

2 Scaling, non-dimensionalization, small parameters

Define scales for each variable, and corresponding non dimensional variables denoted by primes such that

$$x = x'L,\ y = y'L,\ t = t'T,\ u = u'U,\ v = v'U,\ \eta = \eta'\eta_0,\ \tau = \tau_0\tau', \tag{4}$$

so that the equations now take the form

$$\frac{U}{T}u'_{t'} = (f_0U)(1+\beta Ly'/f_0)v' - \frac{g\eta_0}{L}\eta'_{x'} - rUu' + \frac{\tau_0}{H}\tau'^{(x)} \tag{5}$$
$$\frac{U}{T}v'_{t'} = -(f_0U)(1+\beta Ly'/f_0)v' - \frac{g\eta_0}{L}\eta'_{y'} - rUv' + \frac{\tau_0}{H}\tau'^{(y)}$$
$$\frac{\eta_0}{T}\eta'_{t'} + \frac{HU}{L}(u'_{x'} + v'_{y'}) = 0.$$

We now drop the primes, so non-primed variables are non-dimensional in the followings (a confusing but common practice). Next, let $T = L/U$ and rearrange the equations a bit,

$$\frac{U}{f_0L}u_t = (1+\beta Ly/f_0)v - \frac{g\eta_0}{fUL}\eta_x - \frac{r}{f_0}u + \frac{\tau_0}{f_0UH}\tau^{(x)}$$
$$\frac{U}{f_0L}v_t = -(1+\beta Ly/f_0)u - \frac{g\eta_0}{fUL}\eta_y - \frac{r}{f_0}v + \frac{\tau_0}{f_0UH}\tau^{(y)}$$
$$\frac{L\eta_0}{HUT}\eta'_{t'} + (u'_{x'} + v'_{y'}) = 0.$$

Typical scales in, say, the north Atlantic are

$$L = 10^6 m,\ H = 10^3 m,\ U = 0.1 m/s,\ f_0 = 10^{-4}s^{-1},\ \beta = 10^{-11}m^{-1}s^{-1}.$$

We expect the large-scale balance to be between the Coriolis force and the pressure gradient, so we **choose** the scale for the sea surface height accordingly to be

$$\eta_0 = \frac{f_0UL}{g},$$

and also define the "Rossby number" as

$$\epsilon = \frac{U}{f_0L}.$$

The friction coefficient r is also small, so we define a non dimensional friction coefficient, treat it as being order one $E = O(1)$ *although it is still small as we will see below*, such that,

$$\frac{r}{f_0} = \epsilon E.$$

Similarly, the wind stress term is also small, so we assume it to also be order epsilon and define a nondimensional wind stress amplitude as $\mathcal{T} = O(1)$ such that

$$\frac{\tau_0}{f_0UH} = \epsilon\mathcal{T}.$$

Next, define a Froud number $F = O(1)$ such that

$$\frac{L\eta_0}{HUT} = \frac{f_0LU}{Hg} = \frac{U}{f_0L}\frac{f_0^2L^2}{Hg} \equiv \epsilon F.$$

Finally, define a nondimensional scale for the variations of the Coriolis force in latitude, $\hat{\beta} = O(1)$,

$$\beta L/f_0 = \epsilon\hat{\beta}$$

With these definitions, the final nondimensional equations become

$$\begin{aligned}
\epsilon u_t &= (1+\hat{\beta}y)v - \eta_x - \epsilon E u + \epsilon\mathcal{T}\tau^{(x)} \\
\epsilon v_t &= -(1+\hat{\beta}y)u - \eta_y - \epsilon E v + \epsilon\mathcal{T}\tau^{(y)} \\
\epsilon F\eta_t + u_x + v_y &= 0.
\end{aligned} \tag{6}$$

3 Zeroth order dynamics: geostrophy

Expand all nondimensional variables in a perturbation series,

$$\begin{aligned}
u &= u^0 + \epsilon u^1 + \epsilon^2 u^2 + \dots \\
v &= v^0 + \epsilon v^1 + \epsilon^2 v^2 + \dots \\
\eta &= \eta^0 + \epsilon\eta^1 + \epsilon^2\eta^2 + \dots,
\end{aligned}$$

substitute into the equations and keep first only the order one terms, to find

$$\begin{aligned}
0 &= v^0 - \eta^0_x \\
0 &= -u^0 - \eta^0_y \\
u^0_x + v^0_y &= 0.
\end{aligned} \tag{7}$$

The momentum equations form a balance between the Coriolis force and the pressure gradient, known as "geostrophy". Note that the two momentum equations are consistent with and, in fact, imply the third mass conservation equation: the zeroth order, geostrophic, velocities are non-divergent. Consequently, we can define a stream function for these velocities $\psi \equiv \eta^0$, such that $v^0 = \psi_x$ and $u^0 = -\psi_y$.

This zeroth order balance does not include any time derivatives and therefore cannot be used to calculate the time-evolution of the flow, nor to satisfy any initial conditions. We thus need to proceed to the next order.

4 Perturbation analysis and quasi-geostrophic vorticity equation

Proceed to order ϵ to find,

$$\begin{aligned}
u^0_t &= v^1 + \hat{\beta}yv^0 - \eta^1_x - Eu^0 + \mathcal{T}\tau^{(x)} \\
v^0_t &= -u^1 - \hat{\beta}yu^0 - \eta^1_y - Ev^0 + \mathcal{T}\tau^{(y)} \\
F\eta^0_t + u^1_x + v^1_y &= 0.
\end{aligned} \tag{8}$$

Take ∂_x of the second equation minus ∂_y of the first, defining the vorticity $\zeta^0 = v^0_x - u^0_y$, and the curl of the wind, $\mathrm{curl}\tau = \tau^{(y)}_x - \tau^{(x)}_y$,

$$\zeta^0_t = -(u^1_x + v^1_y) - \hat{\beta} y(u^0_x + v^0_y) - \hat{\beta} v^0 - E\zeta^0 + \mathcal{T}\mathrm{curl}\tau$$

Use the $O(1)$ and $O(\epsilon)$ mass conservation equations (7c, 8c) to write this as

$$\zeta^0_t = F\eta^0_t - \hat{\beta} v^0 - E\zeta^0 + \mathcal{T}\mathrm{curl}\tau$$

Use the $O(1)$ momentum equations (7a,b) to write

$$v^0 = \eta^0_x$$
$$\zeta^0 = \eta^0_{xx} + \eta^0_{yy} = \nabla^2\eta^0,$$

to find out final nondimensional "quasi-geostrophic potential vorticity" equation

$$\partial_t(\nabla^2\eta^0 - F\eta^0) + \hat{\beta}\eta^0_x = -E\nabla^2\eta^0 + \mathcal{T}\mathrm{curl}\tau. \tag{9}$$

This gives us a time-dependent equation that can satisfy both initial conditions (for η) and boundary conditions. We therefore found that in order to find how the zeroth order variables change in time, we must go to the order ϵ equations.

Consequence: Rossby waves. Setting the wind forcing and friction to zero and looking for a wave solution, $\psi \equiv \eta^0 = e^{i(kx+ly-\omega t)}$, we find $\omega = -\beta k/(k^2 + l^2 + F)$.

5 Singular perturbation: the Gulf Stream as a boundary layer

Consider the steady state circulation, where the steady vorticity equation takes the form

$$\hat{\beta}\eta^0_x = -E\nabla^2\eta^0 + \mathcal{T}\mathrm{curl}\tau. \tag{10}$$

remembering that the nondimensional friction coefficient E is in fact small even though we kept it in the order ϵ equations, the dominant balance in this equation is therefore,

$$\hat{\beta}\eta^0_x = \mathcal{T}\mathrm{curl}\tau. \tag{11}$$

Suppose the nondimensional wind stress forcing is given by

$$(\tau^{(x)}, \tau^{(y)}) = (-\cos\pi y, 0), \quad 0 < y < 1.$$

This allows us to calculate the v^0 velocity,

$$v^0 = \eta^0_x = \mathcal{T}\mathrm{curl}\tau/\hat{\beta} = (\mathcal{T}/\hat{\beta})\pi\sin\pi y,$$

and the u velocity is found from that using the $O(1)$ mass conservation equation (7c),

$$u^0_x = -v^0_y = -(\mathcal{T}/\hat{\beta})\pi^2 \cos \pi y$$

so that

$$u^0 = -(x-1)(\mathcal{T}/\hat{\beta})\pi^2 \cos \pi y$$

where $x = 1$ is the (nondimensional) eastern boundary location, and this solution guarantees that the normal velocity vanishes there, $u(x = 1) = 0$, as it should. However, the u velocity does not vanish at $x = 0$!

To resolve this, we note that the transition from (10) to (11) involved a singular perturbation, as we neglected the highest derivative in x. We therefore need a boundary layer near $x = 0$ to satisfy the boundary condition there.

Define a local stretched coordinate near $x = 0$, $\xi = x/\delta$ with a yet unspecified nondimensional boundary layer width $\delta \ll 1$. In the boundary layer, write the solution as a sum of the above variables and the boundary layer components, such that the surface elevation is $\eta(x, y) + \tilde{\eta}(\xi, y)$, and the velocities are $u(x, y) + \tilde{u}(\xi, y)$ and $v(x, y) + \tilde{v}(\xi, y)$. Substituting this into (10) and subtracting the equation for the non-tilde variables, we have

$$\delta^{-1}\hat{\beta}\tilde{\eta}^0_\xi = -E(\delta^{-2}\tilde{\eta}^0_{\xi\xi} + \tilde{\eta}^0_{yy}).$$

which may be approximated by

$$\hat{\beta}\tilde{\eta}^0_\xi = -E\delta^{-1}\tilde{\eta}^0_{\xi\xi}$$

in order for the balance to make sense, the boundary layer width must be,

$$\delta = E/\hat{\beta},$$

and our boundary layer equation becomes

$$\tilde{\eta}^0_\xi = -\tilde{\eta}^0_{\xi\xi}$$

which is equivalent to

$$\tilde{v}^0(\xi, y) = -\tilde{v}^0_\xi.$$

The boundary conditions for the tilde quantities are

$$\begin{aligned} &u(x = 0, y) + \tilde{u}(\xi = 0, y) = 0 \\ &(\tilde{\eta}, \tilde{u}, \tilde{v}) \to 0 \text{ for } \xi \to \infty. \end{aligned} \tag{12}$$

The boundary layer solution is therefore,

$$\tilde{v}^0(\xi, y) = A(y)e^{-\xi}$$

and using continuity again (7c), which takes the following form within the boundary layer,

$$\delta^{-1}\tilde{u}^0_\xi + \tilde{v}^0_y = 0,$$

we find the eastward velocity $\tilde{u}$ in the boundary layer,

$$\tilde{u}^0(\xi, y) = \delta A'(y) e^{-\xi}$$

This solution already satisfies $\tilde{u}(\xi \to \infty) = 0$ at the eastern boundary. The other boundary condition (12) gives,

$$A'(y) = \frac{\pi^2 L \mathcal{T}}{\delta \hat{\beta}} \cos \pi y$$

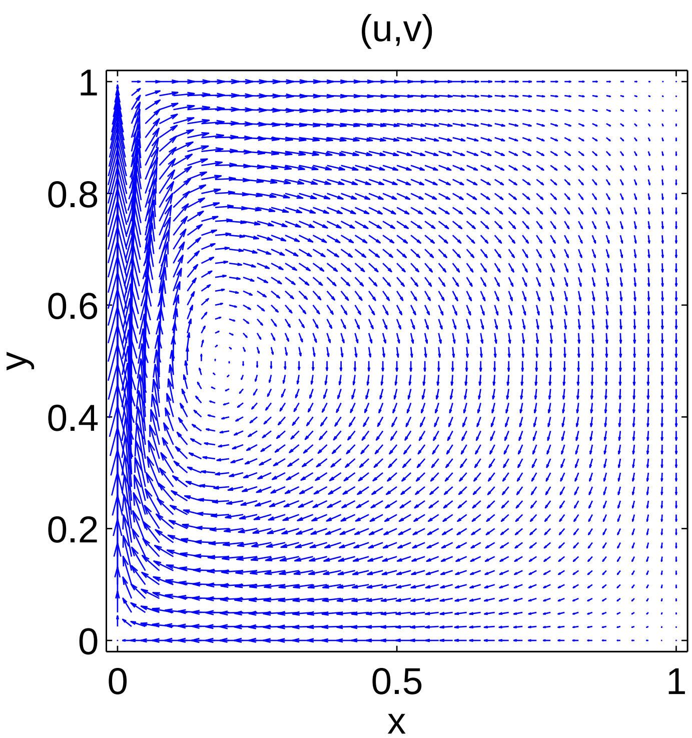
so that

$$A(y) = \frac{\pi L \mathcal{T}}{\delta \hat{\beta}} \sin \pi y.$$

Writing the interior and boundary layer solutions together, we have

$$u^0 = \frac{\mathcal{T}\pi^2}{\hat{\beta}} \left((x-1)\cos \pi y + \cos \pi y \, e^{-x/\delta} \right)$$
$$v^0 = -\frac{\mathcal{T}\pi}{\hat{\beta}} \left(\sin \pi y - \delta^{-1} \sin \pi y \, e^{-x/\delta} \right)$$

and this solution is shown in the figure below.



References

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Stommel, H. (1948). The westward intensification of wind-driven ocean currents. *Transactions, American Geophysical Union*, 29(2):202–206.