Potentially large equilibrium climate sensitivity tail uncertainty

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Arising from P. M. Cox, C. Huntingford & M. S. Williamson Nature **553**, 319322 (18 January 2018) 10.1038/nature25450

Cox *et al.*[1] use an 'emergent constraint' approach to characterize the probability distribution of equilibrium climate sensitivity (ECS) as having a central or best estimate of 2.8 degrees Celsius with a 66-percent confidence interval of 2.2-3.4 degrees Celsius. This implies, by their calculations, that the probability of ECS exceeding 4.5 degrees Celsius is less than 1 percent. They characterize such kind of result as "renewing hope that we may yet be able to avoid global warming exceeding 2 K". We share the desire for less uncertainty around ECS.[2,3] However, we are afraid that the emergent constraint on ECS is largely a function of the underlying near-normal probability density function (PDF) chosen by them (via underlying Gaussian, white-noise, and least-squares linear-regression assumptions). We do not attempt to evaluate Cox *et al.*'s physical modeling (aside from the normality assumption), leaving that task to other physical scientists. We take Cox *et al.*'s 66-percent confidence interval as given,

and explore the implications of applying alternative probability distributions. For example, moving from a normal to a log-normal distribution, while giving identical probabilities for being in the range 2.2-3.4K, increases the probability of exceeding 4.5K around five-fold.

Our methodology is straightforward. We simply wish to show that the aforementioned Cox *et al.* hopeful upper-tail result is, at least in part, a consequence of the PDF chosen to represent ECS as a random variable. Cox *et al.* in effect choose an underlying near-normal PDF. This partially accounts for their optimistic conclusions. Had they, for example, chosen to run a quantile regression that would have allowed for skewed confidence limits, the results might have given very different, and sometimes much less hopeful, assessments of upper tail probabilities. We proxy for such different formulations by examining here the consequences of alternative probability distributions.

Let x stand for ECS and let $f_\theta(x)$ represent a PDF of family θ. We consider three families of two-parameter PDFs: *Normal* ($\theta = N$), *Pareto* ($\theta = P$), *Lognormal* ($\theta = L$). For each such family, we fix the two free parameters by an appropriate condition characterizing the central estimate and by simply imposing, as if given, Cox *et al.*'s condition that 66 percent of the probability lies within the interval [2.2, 3.4]. Mathematically, this Cox *et al.* 66-percent condition is represented for each θ by the equation

$$\int_{2.2}^{3.4} f_\theta(x)\, dx = 0.66. \tag{1}$$

After calibration, we calculate for each θ the probability that ECS exceeds 4.5 degrees Celsius, denoted as $Prob(S_\theta > 4.5)$. This upper-tail behavior is our object of greatest interest here, as it is in much of climate science. Mathematically,

$$Prob(S_\theta > 4.5) \equiv \int_{4.5}^{\infty} f_\theta(x)dx. \tag{2}$$

A *thin-tailed* PDF $f(x)$approaches zero exponentially ($f(x) \propto \exp(-\lambda x)$ for some $\lambda > 0$) or faster as $x \to \infty$. A *fat-tailed* PDF $f(x)$ approaches zero polynomially ($f(x) \propto x^{-k}$ for some $k > 0$) or slower as $x \to \infty$. (The *ratio* of a fat-tailed PDF divided by a thin-tailed PDF therefore approaches infinity as $x \to \infty$.) An *intermediate-tailed* PDF has a tail which goes to zero slower than exponentially but faster than polynomially.

The prototype thin-tailed PDF is the Normal:

$$f_N(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right). \tag{3}$$

Cox *et al.*, whose underlying PDF is effectively Normal, characterize the central or best estimate of climate sensitivity to be 2.8 degrees Celsius. We interpret this as signifying here that $\mu = 2.8$ in (3). The standard deviation σ in (3) is then determined by condition (1) for $\theta = N$, and turns out to be $\sigma = 0.629$. For these two parameter values, we calculate $Prob(S_N > 4.5) = 0.34\%$, confirming Cox *et al.*'s calculation of "the probability of ECS exceeding 4.5 degrees Celsius to less than 1 per cent".

Because the normal PDF is symmetric, mean, mode, and median are identical. When the PDF is right-skewed, mode $<$ median $<$ mean, and we have to choose which of these three measures of central tendency should represent a 'best estimate of 2.8 degrees Celsius'. For the purpose of this set of numerical exercises we choose the median, which is in between the mode and the mean. This particular measure of central tendency has the intuitively appealing and readily visualizable characterization that half the probability is above the median while the other half is below the median.

A candidate for the prototype two-parameter fat-tailed PDF is the Pareto:

$$f_P(x) = \frac{a\, b^a}{x^{a+1}}, \tag{4}$$

for $x \geq b$, while $f_P(x) = 0$ elsewhere. The positive parameter b represents the minimum possible value of x, while the positive parameter a is known as the so-called tail index (smaller values of a correspond to fatter tails). Parameters b and a can be simultaneously fixed or calibrated by setting the median of $f_P(x)$ equal to 2.8 and by imposing the condition (1) for $\theta = P$. The result of this particular curve-fitting exercise is $b = 2.164$ and $a = 2.69$. With these two parameter values, and for what it is worth without thinking deeply, we mechanically calculate $Prob(S_P > 4.5) = 13.95\%$.

In our opinion, the Normal and Pareto distributions represent two extreme poles in upper-tail behavior. To use the Normal is to choose an *extremely* thin upper-tailed PDF. To use the Pareto is to choose an *extremely* fat upper-tailed PDF. This leads us directly to consider fitting an intermediate-tailed PDF. The Lognormal distribution is of the form

$$f_L(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right), \tag{5}$$

where x is constrained to be non-negative. A convenient property of the Lognormal PDF is that its tail goes to zero slower than exponentially but faster than polynomially, making it intermediate between a thin-tailed and a fat-tailed PDF. It also makes the Lognormal a good compromise candidate for consideration in the present context.

The median of the Lognormal PDF (5) is μ, set here to $\mu = 2.8$. The appropriate value of σ that appears in the Lognormal PDF (5) is then fixed or calibrated by condition (1) for $\theta = L$, and turns out to be $\sigma = 3.04$. For these two parameter values, we calculate $Prob(S_L > 4.5) = 1.82\%$. (Note that, perhaps by coincidence, this is very close to the geometric mean of the comparable thin-tailed Normal and the fat-tailed Pareto probabilities: $\sqrt{0.34 \cdot 14.95}\% = 2.18\%$.)

This is the end of our story. What is its moral? Tail behavior is often postulated rather than empirically derived, because typically it is statistically very difficult, and sometimes even impossible, to estimate the probabilities of extreme values from data, since there are so few extreme values in the existing time-series data. This is overwhelmingly true for estimates of ECS tail-probability distributions. Cox *et al.* may have found a useful new way of measuring the 'best estimate' of ECS. In doing so, however, they have effectively assumed a near-Normal distribution around the best estimate. While this near-Normal analysis may be used to justify statements around the 'best estimate' of ECS, it does not justify statements concerning its tail behavior.

Here we demonstrate that $Prob(S > 4.5)$ can vary enormously, depending on what tail behavior the underlying PDF is representing. Our leading candidate for compromise moderation, the intermediate-tailed Lognormal PDF, has $Prob(S_L > 4.5) = 1.82\%$. This is a probability over five times greater than what we impute to be the Cox *et al.* estimate of $Prob(S_N > 4.5) = 0.34\%$ and, we think, does not give much sustenance to the hope that extremely high values of ECS are exceedingly rare.

References

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Competing Interests The authors declare that they have no competing financial interests.