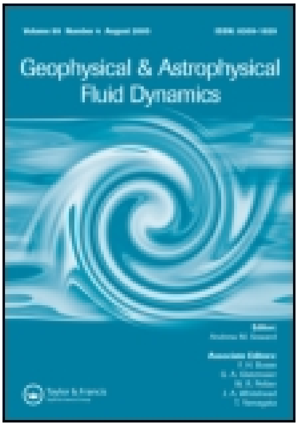

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Generalization of Cowling's theorem

R. Hide [a] & T. N. Palmer† [a]
[a] Meteorological Office, Bracknell, Berkshire, RG12 2SZ, UK

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Generalization of Cowling's Theorem

R. HIDE and T. N. PALMER†
Meteorological Office, Bracknell, Berkshire RG12 2SZ, UK

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An extension of Cowling's neutral point argument shows that no axisymmetric poloidal magnetic field can be maintained by self-exciting dynamo action in an electrically-conducting fluid against the dissipative effects of Ohmic heating. This generalizes previous results to cases where the magnetic field can be non-steady, the fluid compressible and the (scalar) coefficients of magnetic permeability and electrical conductivity dependent on position and time.

1. INTRODUCTION

Cowling (1934) demonstrated in a classic paper that a steady poloidal magnetic field that possesses an axis of symmetry cannot be maintained by motional induction in an electrically-conducting fluid against the effects of Ohmic dissipation. His simple and appealing argument was based on considerations of the behaviour of the magnetic field near an *O*-type neutral point in the meridional plane. Attempts to generalize Cowling's "anti-dynamo" theorem have been the subject of several studies using a variety of techniques [for references see Moffatt (1978), Parker (1979), James, Roberts and Winch (1980), Hide (1981)]. Backus (1957) and Braginskiy (1964), for example, were able to show that non-steady axisymmetric magnetic fields cannot be maintained by motional induction when the fluid is incompressible (a restriction which has led to erroneous speculations in the literature that axisymmetric (non-steady) self-exciting dynamos might be possible in a compressible fluid, see Hide (1981) and Section 4 below). Having first given a definition of self-exciting dynamo action, we show by an extension of Cowling's neutral point technique that

†From September 1981 on secondment at Department of Atmospheric Sciences, University of Washington, Seattle, Washington 98195.

no axisymmetric poloidal magnetic field can be maintained by such action, even when the field is non-steady and the (scalar) coefficients of magnetic permeability and electrical conductivity are dependent on position and time.

2. BASIC EQUATIONS AND DEFINITIONS

Consider a connected body of electrically-conducting fluid V_0 bounded by a surface S_0 with surface element $\mathbf{dS}$. The flux linkage with S_0 of a magnetic field $\mathbf{B}$ that pervades the conducting fluid and the surrounding space is defined as the essentially non-negative quantity

$$N(S_0; t) \equiv \int_{S_0} |\mathbf{B} \cdot \mathbf{dS}|. \tag{2.1}$$

In the absence of permanent magnets, $\mathbf{B}$ would be due entirely to electric currents of density $\mathbf{j}$, and in a so-called "homogeneous self-exciting dynamo" the electromotive forces that produce these electric currents are provided by motional induction, involving fluid motions within V_0 with Eulerian velocity $\mathbf{u}$. By this means some weak adventitious seed field is amplified and maintained against the effects of Ohmic decay. If the fluid motions were suddenly to cease, $N(S_0; t)$ would decay on a time-scale $O(\tau_d)$ where τ_d is the Ohmic decay time based on a characteristic length L of the order of the dimensions of V_0 [see Eq. (2.8)].

In the present paper we define dynamo action as implying that

$$N(S_0; t) \not\to 0 \quad \text{as} \quad t \to \infty. \tag{2.2}$$

For practical purposes this criterion is equivalent to that proposed by Hide (1979). It has advantages over other proposals such as those based on total magnetic energy or equivalent magnetic moment, which can be ambiguous when $\mathbf{B}$ has toroidal as well as poloidal components or when the conducting fluid is not incompressible [see Hide (1981)].

We shall need the so-called "pre-Maxwell" equations of electrodynamics, namely Gauss' law

$$\nabla \cdot \mathbf{B} = 0 \tag{2.3}$$

(which implies of course that

$$\int_{S_0} \mathbf{B} \cdot \mathbf{dS} = 0,$$

cf. Eq. (2.1)), Faraday's law

$$\partial\mathbf{B}/\partial t+\nabla\times\mathbf{E}=0, \tag{2.4}$$

and Ampere's law

$$\nabla\times(\mu^{-1}\mathbf{B})=\mathbf{j}, \tag{2.5}$$

together with Ohm's law in a moving medium

$$\mathbf{j}=\sigma[\mathbf{E}+\mathbf{u}\times\mathbf{B}], \tag{2.6}$$

where $\mathbf{E}$ is the electric field. The magnetic permeability μ and electrical conductivity σ are scalars but they may depend on position and time. When $\mathbf{E}$ and $\mathbf{j}$ are eliminated from these equations it is found that $\mathbf{B}$ satisfies

$$\partial\mathbf{B}/\partial t=-\nabla\times(\sigma^{-1}\nabla\times(\mu^{-1}\mathbf{B}))+\nabla\times(\mathbf{u}\times\mathbf{B}). \tag{2.7}$$

The second term on the right-hand side represents effects due to motional induction. These disappear when $\mathbf{u}=0$; solutions of Eq. (2.7) then correspond to the free decay of $\mathbf{B}$ on a time-scale of the order of

$$\tau_d=\bar{\sigma}\bar{\mu}L^2, \tag{2.8}$$

where $\bar{\sigma}$ and $\bar{\mu}$ are typical values of σ and μ. From Eq. (2.7) there follows the familiar necessary (but not sufficient) condition for dynamo action that the "magnetic Reynolds number"

$$Q\equiv UL\bar{\mu}\bar{\sigma} \tag{2.9}$$

should be sufficiently large, where U is a characteristic speed of fluid flow [see e.g. Moffatt (1978)].

On the basis of Eq. (2.7), we show in what follows that when $\mathbf{B}$ retains an axis of symmetry and remains spatially smooth the condition for dynamo action [Eq. (2.2)] cannot be satisfied for finite $\mathbf{u}$. Motional induction is thus incapable of preventing the Ohmic decay of such (and topologically equivalent, see Section 4 below) magnetic fields. It is of course of interest to investigate the topological nature of departures from axial symmetry of **B** that are associated with dynamo action but this lies beyond the scope of the present paper [see Hide (1979), (1981)]. Here we are concerned with the problem of proving under the most general conditions that axisymmetric magnetic fields will always decay to zero, either monotonically or otherwise.

3. AXISYMMETRIC SYSTEMS

Take a cylindrical co-ordinate system (r, ϕ, z), put $R^2 = r^2 + z^2$, suppose that $\mathbf{B} = (B_r, B_\phi, B_z)$ is axisymmetric, i.e. independent of ϕ everywhere, and that $(B_r^2 + B_z^2)^{1/2} = O(R^{-3})$ as $R \to \infty$. Hence, Eq. (2.3) implies the existence of a twice differentiable scalar field $h(r, z, t)$ satisfying

$$(B_r, B_z) = (r^{-1}\, \partial h/dz,\ -r^{-1}\, \partial h/\partial r), \tag{3.1}$$

and $h(0, z, t) = 0$, $h(r, z, t) = O(R^{-1})$ as $R \to \infty$. In terms of this scalar field, Eq. (2.1) can be written as

$$N(S_0; t) = 2\pi \sum_{m=1}^{M(t)} \left| h(r_m, z_m, t) \right|. \tag{3.2}$$

Here $(r_m(t), z_m(t))$ are the co-ordinates of the so-called "C-lines" on S_0 where $\mathbf{B} \cdot \mathbf{dS} = 0$ [see Hide (1979)], of which at time t there are $M(t)$ in number, where $M(t) \geqq 1$.

There will in general be many local maxima of $|h|$ in the (r, z)-plane, corresponding to the O-type "neutral points" of the meridional components (B_r, B_z) of **B**. Consider in particular the global maximum

$$H(t) = \max_{(r,z)} \left| h(r, z, t) \right|, \tag{3.3}$$

and let $(\hat{r}, \hat{z})$ be the point where this maximum is attained. (The case where this maximum is attained on a more general region of the (r, z)-plane is considered later.) Now as the magnetic field evolves in time the point $(\hat{r}, \hat{z})$ will, in general, move about in the (r, z)-plane. Indeed, there may be times $t_1, t_2, \ldots$ etc. when this movement is discontinuous; however, because h is a continuous field then, by its definition, the non-negative quantity H must be a continuous function of time. Further, if h is differentiable then H must be piecewise differentiable, i.e. it fails to be differentiable only at times $t_1, t_2, \ldots$ etc. (A more rigorous statement about the differentiability of H is given below in the Appendix.) The fact that H is continuous results from its definition in terms of the global maximum of $|h|$; the evolution in time of any other local maximum in $|h|$ (the second largest maximum, for example) cannot be guaranteed to be continuous.

Now consider the disk-shaped surface $\hat{S}$ defined by $r<\hat{r}$, $z=\hat{z}$, $0<\phi \leq 2\pi$ and define the orientation of the surface element $\mathbf{d\hat{S}}$ of $\hat{S}$ by the condition

$$\int_{\hat{S}} \mathbf{B} \cdot \mathbf{d\hat{S}} > 0. \tag{3.4}$$

Using Eqs. (3.1) and (3.3) we have

$$2\pi H(t) = \int_{\hat{S}} \mathbf{B} \cdot \mathbf{d\hat{S}}. \tag{3.5}$$

Except at times $t_1, t_2, \ldots$ etc. Eq. (3.5) can be differentiated, giving

$$2\pi \dot{H}(t) = \int_{\hat{S}} \partial \mathbf{B}/\partial t \cdot \mathbf{d\hat{S}} + \int_{\partial \hat{S}} \mathbf{v} \times \mathbf{B} \cdot \mathbf{dl}, \tag{3.6}$$

where $\mathbf{v}$ is the velocity of the boundary $\partial \hat{S}$ of $\hat{S}$ whose line element (in the ϕ-direction) is $\mathbf{dl}$. By virtue of Stokes' theorem and Eq. (2.7), Eq. (3.6) becomes

$$2\pi \dot{H}(t) = \int_{\partial \hat{S}} [\sigma^{-1} \nabla \times (\mu^{-1} \mathbf{B}) + (\mathbf{v} - \mathbf{u}) \times \mathbf{B}] \cdot \mathbf{dl} \tag{3.7}$$

Now by definition, both $[(\mathbf{v}-\mathbf{u}) \times \mathbf{B}]_\phi = 0$ and $\nabla \mu \cdot \nabla h = 0$ at $(r, z) = (\hat{r}, \hat{z})$; hence Eq. (3.7) can be written as

$$\dot{H}(t) = -(\sigma\mu)^{-1} \nabla^2 h(\hat{r}, \hat{z}, t) \operatorname{sgn}\left(\int_0^{\hat{r}} B_z \, dr\right). \tag{3.8}$$

If h is a maximum at $(\hat{r}, \hat{z})$ then $\nabla^2 h(\hat{r}, \hat{z}, t) \leq 0$, and by the definition of the orientation of $\hat{S}$,

$$\operatorname{sgn}\left(\int_0^{\hat{r}} B_z \, dr\right) = -1.$$

Conversely, if h is a minimum at $(\hat{r}, \hat{z})$ then $\nabla^2 h(\hat{r}, \hat{z}, t) \geq 0$ and

$$\operatorname{sgn}\left(\int_0^{\hat{r}} B_z \, dr\right) = +1.$$

Hence,

$$\dot{H}(t) = -(\sigma\mu)^{-1}\left|\nabla^2 h(\hat{r}, \hat{z}, t)\right| \leqq 0. \tag{3.9}$$

However, since $H(t)$ is continuous for all t *then*

$$\max_{(r,z)} \left|h(r, z, t)\right|$$

is a monotonic continuous non-increasing function of time.

The argument so far has included the "pathological" possibility that $\nabla^2 h(\hat{r}, \hat{z}, t) = 0$ (i.e. that j_ϕ vanishes at the neutral point $(\hat{r}, \hat{z})$), in which case Eq. (3.9) implies that $\dot{H}(t) = 0$. The indefinite maintenance of such a field configuration can, however, be dismissed by the following reasoning. Suppose that $\dot{H}(t) = 0$ indefinitely. Then $\nabla^2 h(\hat{r}, \hat{z}, t) = 0$ and the expansion of $h(r, z, t)$ as a Taylor series about $(\hat{r}, \hat{z})$ gives

$$\left|\nabla h\right| = O(s^2), \qquad \nabla^2 h = O(s), \tag{3.10}$$

where s is the distance from $(\hat{r}, \hat{z})$. Hence, provided that $\sigma\mu$ and $(\mathbf{v} - \mathbf{u})$ are bounded, there will exist a small but finite neighbourhood $\mathcal{N}$ of $(\hat{r}, \hat{z})$ such that at any time the average value of

$$[\sigma^{-1}\nabla \times (\mu^{-1}\mathbf{B}) + (\mathbf{v} - \mathbf{u}) \times \mathbf{B}]_\phi$$

over $\mathcal{N}$ is dominated by $\langle(\sigma\mu)^{-1}\nabla^2 h\rangle$ (where $\langle\ \rangle$ denotes the average value over $\mathcal{N}$). From Eqs. (2.7) and (3.1) we have at (r, z, t)

$$\partial\left|h\right|/\partial t = [\sigma^{-1}\nabla \times (\mu^{-1}\mathbf{B}) + (\mathbf{v} - \mathbf{u}) \times \mathbf{B}]_\phi \operatorname{sgn}\left(\int_0^r B_z\, dr\right), \tag{3.11}$$

and because, in the hypothetical case we are considering, $\dot{H}(t) = 0$, discontinuous jumps in $(\hat{r}, \hat{z})$ do not arise. Therefore, since $\mathcal{N}$ has a fixed volume around $(\hat{r}, \hat{z})$, it follows that

$$\frac{d}{dt}\langle\left|h\right|\rangle \simeq -\langle\left|(\sigma\mu)^{-1}\nabla^2 h\right|\rangle. \tag{3.12}$$

Of course the average $\langle\ \rangle$ over $\mathcal{N}$ applies equally well if the global maximum of $\left|h\right|$ is not an isolated point but on some region $\hat{R}$ of the (r, z)-plane, in which case $\mathcal{N}$ is a fixed neighbourhood of $\hat{R}$.

In this way it is guaranteed that the rate of change of $\langle|h|\rangle$ is decreasing at a finite rate (for finite h). Hence we have reached a contradiction. If $\dot{H}(t)=0$ but $\langle|h|\rangle$ decreases at a finite rate then ultimately h will cease to be differentiable at $(\hat{r},\hat{z})$ or on the boundary of $\hat{R}$, contrary to our earlier assertion that h is differentiable everywhere. Hence we eliminate the possibility that $\dot{H}(t)=0$ indefinitely. By the same reasoning it follows that $\dot{H}(t)$ cannot tend asymptotically to zero for some finite non-zero value of H, for if this were to occur $H(t)$ would become indefinitely small for finite h for an indefinitely long period of time. Hence we can strengthen our statement about the evolution of $H(t)$; not only must H be monotonic and non-increasing, it must ultimately decay to zero. This in turn implies that h itself must ultimately decay to zero everywhere. Hence, since h is differentiable,

$$N(S_0;t)\to 0 \quad \text{as} \quad t\to\infty \tag{3.12}$$

[see Eq. (3.2)], *implying* [by Eq. (2.2)] *that dynamo action cannot occur when* **B** *is axisymmetric everywhere.*

By Eqs. (2.8) and (3.9) the decay of the essentially non-negative quantity $H(t)$ is monotonic and on the Ohmic decay time scale τ_d. The quantity $N(S_0;t)/M(t)$ [see Eqs. (2.1) and (3.2)], which satisfies $0 \leqq N(S_0;t)/M(t) \leqq 2\pi H(t)$, also decays on the same overall time-scale, but not in general monotonically. There may be intervals of time, $O(L/U)$[see Eq. (2.9)], during which $d[N(S_0;t)/M(t)]/dt$ is positive, but the monotonic decay of $H(t)$ in the axisymmetric case ensures that **B** ultimately decays to zero everywhere on the Ohmic time scale.

4. CONCLUDING REMARKS

In the foregoing demonstration that the Ohmic decay of a magnetic field that possesses an axis of symmetry cannot be prevented by motional induction, no restriction is placed on the steadiness of **B**. Nor has attention been confined to the case of an incompressible fluid, which Braginskiy (1964) has discussed on the basis of an energy equation obtained by taking the scalar product of Eq. (2.7) with **B** and integrating over all space, and James, Roberts and Winch (1980) have discussed on the basis of the expansion of Eq. (2.7) in terms of spherical harmonics. Furthermore, the coefficients of magnetic permeability μ and electrical conductivity σ have been allowed to be arbitrary differentiable functions of space and time within the conducting fluid. In extending Cowling's argument we have focussed attention on the overall maximum of $|h|$ in the

(r, z)-plane. Only when $\mathbf{B}$ is steady, the case considered by Cowling (1934), can the argument be applied to any local maximum in $|h|$.

In Section (3) we confined our attention to the case of strictly axi-symmetric magnetic fields, for which the neutral lines are circles concentric with the axis of symmetry, and each meridional field line (B_r, B_z), by rotation around the axis of symmetry, defines a toroidal surface. However the results of Section 3 apply equally well to field configurations where these surfaces are geometrically deformed provided that they remain topologically toroidal, since this implies that the field will be independent of some generalized azimuthal co-ordinate $\tilde{\phi}$ and therefore a scalar potential h of the form of Eq. (3.1) can be introduced to satisfy Eq. (2.3).

In a paper received in preprint form during the preparation of the present paper, Lortz and Meyer-Spasche (1981) arrive at conclusions that are qualitatively similar to ours, using arguments based on certain general extremum theorems for parabolic and elliptical differential equations applied, effectively, to Eq. (2.7). The reader without a background in functional analysis may prefer the more physical but none-the-less rigorous arguments given here.

Contrary to a recent suggestion by Hibberd, thermoelectric effects of the Nernst–Ettinghausen type cannot prevent the decay of an axisymmetric magnetic field [see Hide (1981)]. When such effects are present, wherever the term $\mathbf{u} \times \mathbf{B}$ appears [see Eqs. (2.6), (2.7), (3.7) and (3.11)] it has to be replaced by $(\mathbf{u} - \mathbf{G}) \times \mathbf{B}$, where $\mathbf{G}$ is proportional to the Nernst–Ettinghausen coefficient and the temperature gradient. But $\mathbf{G} \times \mathbf{B}$ has no component in the ϕ direction at a neutral point and consequently Eq. (3.8), which governs the decay of the axi-symmetric magnetic field, is unaffected.

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Appendix

The argument given in Section 3 concerning the differentiability of $H(t)$ [see Eq. (3.3)] can be generalized. Consider the decomposition of an arbitrary time interval $t_A<t<t_B$ into partial intervals $t_{k-1}<t<t_k$, $k=1,2,\ldots,n$; $t_1=t_A$, $t_n=t_B$. Within any partial interval we have

$$|H(t_k)-H(t_{k-1})| \leqq (t_k-t_{k-1})\left[\max_{(t_{k-1}<t<t_k)} \max_{(r,z)} \left|\frac{\partial}{\partial t} h(r,z,t)\right|\right].$$

Since $\partial h/\partial t$ is finite for all (r, z, t) then

$$|H(t_k)-H(t_{k-1})| = O(t_k-t_{k-1}).$$

Hence the variation

$$\sum_{k=1}^{n} |H(t_k)-H(t_{k-1})|$$

of H in the interval $t_A<t<t_B$ is bounded, independent of the particular choice of decomposition. By appealing to Lebesgue's theorem [see, for example, Riesz and Sz-Nagy (1965)], which states that every function of bounded variation possesses a finite derivative almost everywhere, we infer that $H(t)$ is differentiable except at most on a set of measure zero.