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Hydromagnetism. I. A Review

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This paper gives a survey of the theories (Part I) and phenomena (Part II) of cosmic magnetism, extending from geomagnetism over solar and sunspot magnetism to stellar and interstellar magnetic fields. The theoretical treatment is purely classical and Maxwelliam. Most cosmic fluids, being highly ionized, are excellent conductors; this implies that the displacement current can always be neglected. The fundamental equations of hydromagnetism are the electromagnetic field equations together with the hydrodynamic equation, both containing coupling terms between magnetic field and motion. Among other theoretical developments, the "dynamo" theory which ascribes the cosmic magnetic fields to amplifying processes in the moving fluid is described and it is shown that it is well suited to represent the observed phenomena of the generation and maintenance of cosmic magnetic fields.

CONTENTS

BASIC DYNAMICAL NOTIONS

IMAGINE a race of intelligent beings who lived on a planet covered with a highly viscous atmosphere. If their astronomers discovered the phenomena of turbulence, so important in astrophysics, there would at once arise an argument of the learned as to the fundamental nature of these effects. Can they be described in terms of classical, continuum physics? Do they, instead, involve molecular actions that are not readily set in evidence in experiments performed in the laboratory? Or do they perhaps involve modifications of the basic physical laws through terms that are significant only in large dimensions and become negligible on the scale of the laboratory?

In our terrestrial environment we are altogether familiar with the phenomena presented by turbulence. We are convinced that they are fully describable in terms of the continuum equations of classical hydrodynamics, in spite of the fact that an actual mathematical analysis of even the simplest forms of turbulence is of formidable complexity. The trend in the explanation of hydromagnetic phenomena has been similar. The magnetic fields of the earth, sunspots, numerous stars, and the interstellar gaseous medium are becoming amenable to an understanding and to mathematical treatment along the lines of conventional physics, specifically classical continuum physics. As we proceed it will appear more clearly why it is so difficult, often next to impossible, to duplicate these large-scale phenomena in the laboratory. The reasons lie in certain dimensional conditions, not in any fundamental deficiency of the classical field equations. One cannot produce visible light by a magnetron,—but this does not indicate any shortcomings of Maxwell's equations.

With the exception of highly rarefied gases where classical continuum theory sometimes fails, we can base our interpretation on the classical laws that hold for a moving, electrically conducting fluid in which there exist electromagnetic fields. The equations of motion of the fluid are the conventional Navier-Stokes equations,

$$\frac{d\mathbf{v}}{dt}=-\frac{1}{\rho}\nabla p-\nabla U+\nu\nabla^2\mathbf{v}+\frac{1}{\rho}\mathbf{J}\times\mathbf{B}. \qquad (1)$$

Here ρ, p, U, ν have usual meaning of density,

* Supported by the Office of Naval Research.

pressure, gravitational potential, and kinematic viscosity; the last term on the right is the *ponderomotive force* which the electromagnetic field exerts on the fluid; it will be discussed presently. The behavior of the electromagnetic field is given by the usual Maxwell equations; in this review we shall, as a rule, use rationalized mks units. A detailed dimensional analysis[1] shows that in all geophysical and astrophysical forms of hydromagnetism the displacement current is altogether negligible; also negligible are all purely electrostatic effects. By the same token all relativistic effects of order higher than v/c are negligible. Under these conditions we can use the electromagnetic field equations in the simple form

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t},\quad \nabla\times\mathbf{B}=\mu\mathbf{J}, \tag{2}$$

where of course the magnetic field obeys the subsidiary condition $\nabla\cdot\mathbf{B}=0$. Now the fluid is assumed electrically conducting; on moving relative to the electromagnetic field it produces a motional induction. We can therefore write the current as

$$\mathbf{J}=\sigma\mathbf{E}+\sigma\mathbf{v}\times\mathbf{B}, \tag{3}$$

where the second term on the right is the motional induction term. The last terms in Eqs. (1) and (3), respectively, represent the coupling between the field and fluid motion; if these terms are not small the fluid and field are no longer separate dynamical entities; we must think of the combination fluid-field as a single dynamical system. The ponderomotive force term in Eq. (1) can be written, by Eq. (2),

$$\mathbf{F}=\frac{1}{\rho}\mathbf{J}\times\mathbf{B}=\frac{1}{\mu\rho}(\nabla\times\mathbf{B})\times\mathbf{B}. \tag{4}$$

We can furthermore eliminate **B**: Taking the curl of the second of Eqs. (2), and using Eq. (3) and the first of Eqs. (2), we obtain

$$\frac{\partial\mathbf{B}}{\partial t}=\nabla\times(\mathbf{v}\times\mathbf{B})+\nu_m\nabla^2\mathbf{B}. \tag{5}$$

Here use has been made of the vector identity,

$$\nabla\times\nabla\times\mathbf{B}=-\nabla^2\mathbf{B},$$

[1] W. M. Elsasser, Phys. Rev. 95, 1 (1954).

valid since $\nabla\cdot\mathbf{B}=0$; the quantity

$$\nu_m=1/\mu\sigma \tag{6}$$

will be designated as the *magnetic viscosity.* On using the last expression (4) for the ponderomotive force in Eq. (1), we are left with a set of equations containing the vectors **v** and **B** only; the basic equations of field-motion are now given by the combination of Eqs. (1) and (5). We have tacitly assumed that μ and ρ are constant throughout; this simplification will be maintained in most of the subsequent discussion.

Rewriting Eq. (3) by means of Eqs. (2) and (6) as

$$\nu_m\nabla\times\mathbf{B}=\mathbf{E}+\mathbf{v}\times\mathbf{B}, \tag{7}$$

we next investigate the order of magnitude of each of the three terms in Eq. (7). Let L, T, V stand for the order of magnitude of a length, time, and velocity, respectively. We see from the first Eq. (2) that **E** may, in order of magnitude, be replaced by $V\mathbf{B}$. Thus the two terms on the right of Eq. (7) are both of the order $V\mathbf{B}$. But the left side is of order $(\nu_m/L)\mathbf{B}$. If we keep V and ν_m constant we can make the left-hand side of Eq. (7) arbitrarily small by going to sufficiently large linear dimensions. We introduce the two dimensionless constants,

$$R=LV/\nu,\quad R_m=LV/\nu_m, \tag{8}$$

where ν is the ordinary kinematic viscosity of the fluid, ν_m is its magnetic viscosity defined by Eq. (6). The quantity R is the well-known Reynolds number. The frictional term $\nu\nabla^2\mathbf{v}$ in Eq. (1) is of order $1/R$ compared to the dynamical acceleration terms. It is well known that the condition for the appearance of turbulence in a fluid is that R be numerically large. We designate R_m as the *magnetic Reynolds number.* We find that the left-hand side of Eq. (7) is of order $1/R_m$ compared to the right-hand side. Thus if R_m is numerically large, as it is in fluids of cosmic dimensions, the two terms on the right of Eq. (7) cancel very nearly and the actual electric currents corresponding to a given magnetic field can become very small. This result is purely dimensional, characteristic of large linear extensions; it does require that conductors be in motion, but has nothing to do with their being fluid. If we apply the same considerations to

Eq. (5) we find that the dissipative term $\nu_m\nabla^2\mathbf{B}$ is of the order $1/R_m$ compared to the two other terms. This is in complete analogy to the behavior of the frictional term in the hydrodynamic Eqs. (1) and thus affords a justification for the names given ν_m and R_m. But, it is useful to emphasize that the analogy between these quantities and the corresponding mechanical ones is rather incomplete. R_m for instance is not a measure of the transition between laminar and turbulent flow in hydromagnetic motion. If a magnetic field exists in a conducting fluid, the ponderomotive forces Eq. (4) are such that they tend to keep the lines of force from being contorted; they therefore give a certain stiffness to the fluid which some authors have chosen to designate as "magnetic viscosity," a use of this term radically different from the one adopted here. We shall later on deal briefly with such mechanical effects of the field and their influence on the stability of fluid motion. The aforementioned complications of terminology do not appear so long as we are dealing with purely kinematical problems, that is with the differential Eq. (5) in which the velocity field is assumed to be given.

We can give a simple physical interpretation to the magnetic Reynolds number, R_m: Consider an electric conductor in which a system of currents is flowing. In the absence of electromotive forces such currents will decay exponentially. We can find the decay time by familiar methods; we can estimate it, for instance, from Eq. (5) on setting $\mathbf{v}=0$, which shows that the decay time is of the order $\tau=L^2/\nu_m$. Hence we may write, from Eq. (8),

$$R_m=\tau V/L=\tau/T, \tag{9}$$

where T refers to the mechanical motions of the fluid; it is the time required by a fluid particle to travel a distance equivalent to the extension L of the conductor. The fact that R_m is numerically large for bodies of cosmic dimensions thus corresponds to very large spontaneous decay times of electric currents and magnetic fields compared to the mechanical periods. The fluid can undergo *extensive internal deformation* during a time in which the spontaneous decay of the electromagnetic fields is quite small. This is the essential feature of cosmic hydromagnetism; it can clearly not be duplicated in the laboratory where electromagnetic decay times rarely exceed the order of milliseconds.

To visualize conditions of cosmic hydromagnetism, let us obtain some numerical estimates. For the earth's fluid metallic core we may estimate[2] $\sigma=10^6$ mks, which is one-tenth of the conductivity of ordinary iron, giving $\nu_m\sim 1$. If we take $V\sim 0.01$ cm/sec and $L\sim 10^4$ km we get $R_m\sim 10^3$. For the sun, Cowling[3] has shown that the free decay time of a current system covering the entire sun is of the order of 10^{10} years which, by Eq. (9), clearly corresponds to extremely large values of R_m. Sunspots are often of the same size as the earth; we may further estimate $\nu_m\sim 10$ from Cowling's data. Velocities of the order of 1 km/sec occur, giving $R_m\sim 10^9$. Generally speaking, the conductivity of the highly ionized gases making up cosmic matter is good; it does not fall below that of metals by many powers of ten. Hence, with velocities of a few km/sec and the large linear dimensions of cosmic bodies we obtain extremely large values of R_m.

What now is the consequence of the fluid's being able to move a great deal while the magnetic field decays but very little? The effects may be visualized by means of a fundamental integral formula, due to Cowling. To derive it we apply Stokes' theorem to the first equation (2) and on using Eq. (7) obtain

$$\int\frac{\partial B_n}{\partial t}d\sigma=-\int\mathbf{E}\cdot d\mathbf{s}$$

$$=\int(\mathbf{v}\times\mathbf{B})\cdot d\mathbf{s}-\nu_m\int(\nabla\times\mathbf{B})\cdot d\mathbf{s}. \tag{10}$$

In the first integral on the left we can put $\partial/\partial t$ in front of the integral sign if the contour is considered fixed in space. On the other hand, let a surface and its contour move bodily with the fluid, sweeping out a disk-shaped volume during time dt (Fig. 1). Consider the flux passing through the small strip which forms the edge of this disk. If we write $\mathbf{v}=d\mathbf{r}/dt$ we can express

[2] W. M. Elsasser, Revs. Modern Phys. **22**, 1 (1950).
[3] T. G. Cowling, Monthly Notices Roy. Astron. Soc. **105**, 166 (1945).

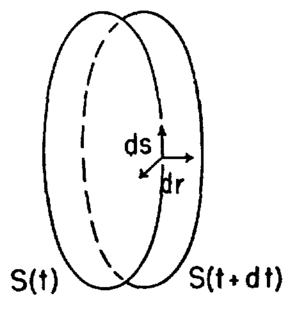

FIG. 1. Contour.

this flux as

$$\int (d\mathbf{r}\times d\mathbf{s})\cdot\mathbf{B} = -dt\int (\mathbf{v}\times\mathbf{B})\cdot d\mathbf{s}.$$

From this we may conclude that

$$\frac{\partial}{\partial t}\int B_n d\sigma - \int (\mathbf{v}\times\mathbf{B})\cdot d\mathbf{s} = \frac{d}{dt}\int B_n d\sigma, \quad (11)$$

where on the right there appears the *substantial* derivative, referring to the flux crossing a surface that moves bodily with the fluid. To justify Eq. (11) we notice that the first integral represents the change of flux due to $\partial\mathbf{B}/\partial t$ for a surface fixed in space. The second integral must then represent the change in flux for fixed **B** due to the displacement of the surface. That this is so follows at once if we apply to Fig. 1 the formula $\int B_n d\sigma = 0$, whose validity for any closed surface fixed in space results from $\nabla\cdot\mathbf{B}=0$. Finally, substituting Eq. (11) into Eq. (10) we have

$$\frac{d}{dt}\int B_n d\sigma = \nu_m \int (\nabla\times\mathbf{B})\cdot d\mathbf{s}, \quad (12)$$

where the right hand is of order $1/R_m$. For an ideal conductor ($R_m=\infty$) we have

$$\frac{d}{dt}\int B_n d\sigma = 0. \quad (13)$$

Equations (12) and (13) are, by the way, also valid for compressible fluids, no assumptions about incompressibility having been made in the derivation. The result may be stated in more physical terms by saying that in an ideal fluid conductor the magnetic lines of force are carried along bodily with the fluid; they are "frozen" into the fluid. One can show that Cowling's integral (13) is equivalent to the differential equation (5) if in the latter we drop the dissipative term. It is possible to integrate these equations for the ideal conductor with respect to time and to obtain expressions for the field at time t in terms of the field at time $t=0$ and of the finite deformation which the fluid has undergone.[4]

For a fluid of finite conductivity, the right-hand side of Eq. (12) represents a "slippage" of the lines of force through the fluid; this is expressed alternatively in the last term of Eq. (5) as a "diffusion" of the field. It follows from well-known principles that this operates in such a way as to smooth out contortions of the field lines; generally speaking the effect of the diffusive terms is to decrease the field energy. In deriving Eqs. (12) and (13) we have made no use of the hydrodynamic equations of motion; these results pertain essentially to electrodynamics as applied to deformable large-scale conductors.

Return now to the basic hydromagnetic equations (1) and (5). In these there appear coupling terms which produce an interaction between the velocity field **v** and the magnetic field **B**. The energy-conservation law can be derived in the conventional fashion, but energy is not conserved for each of the separate fields; energy can indeed be transferred from the fluid motion to the magnetic field and conversely. This can conveniently be demonstrated by means of Cowling's flux-conservation theorem (13) which tells us that the lines of force are deformed by the motion of the fluid. The simplest kind of deformation that leads to *amplification* of the

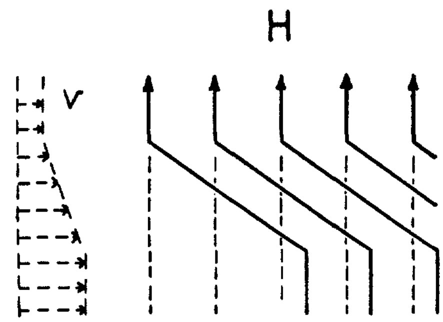

FIG. 2. Amplification of magnetic field by a linear velocity shear normal to field.

[4] S. Lundquist, Phys. Rev. **83**, 307 (1951); Arkiv Fysik **5**, No. 15 (1952).

magnetic field consists in a velocity-shear field normal to the local magnetic field. This is illustrated in Fig. 2 where we assume the velocity in the x-direction, the velocity profile being shown on the left. Assume that the fluid is set in motion at $t=0$ and that at that time the magnetic field is homogeneous and in the y-direction, as shown by the dashed lines. After some lapse of time the field-lines have been deformed into the shape indicated by the heavy lines. An x-component of **B** has been created; its energy is superadded to the energy of the original field. The total energy content of the magnetic field has increased at the expense, of course, of work done by the kinetic energy of the fluid. It is apparent that the process of Fig. 2 is capable of a great many variations, depending on the complexity of the geometry assumed for the fluid motion or the original field.

This brings us to the central problem of hydromagnetism, namely, the generation and maintenance of the observed cosmic magnetic fields by the motion of fluid conductors. The simple amplifying device of Fig. 2 provides no more than a hint that the observed fields can be so explained. The outlook for this type of explanation is quite favorable, as we shall show in these pages. Let us for the moment confine ourselves to the remark that any alternative sources of cosmic magnetic fields are unlikely to be effective. Barring extravagant, speculative assumptions there are two types of causes for such magnetic fields which may readily be conceived; one is ferromagnetism and the other consists in an emf producing electric currents in the cosmic conductors. Ferromagnetism is hardly of interest as a cause of solar and stellar magnetic fields, but it has in former times been advanced as an explanation for the earth's field. It is known at present that the earth's field originates essentially in the fluid, metallic core, and the existence of ferromagnetic materials under the physical conditions prevailing in the core is altogether improbable.

The idea of an impressed emf causing currents in a cosmic conductor seems at first more attractive. In the earth, thermoelectric forces caused by a temperature difference between polar and equatorial regions at the same depth, would give rise to current systems that generate a toroidal magnetic field (the term will be explained in the following). It has recently been proposed that the observed field is a combination of such effects with hydromagnetic induction,[5] though notions about thermoelectricity in the deep interior of the earth are of necessity rather qualitative.

When we come to rarefied, ionized gases one is inclined to have recourse to any emf that can be produced by the difference in inertia of the positive and negative ions (differential diffusion of plasma in gravitational and other fields). The theory of effects of this type has been developed[6] and there exist laboratory experiments verifying their existence in ionized gases and even in mercury.[7] These effects are rather minute, in the laboratory at least, and require careful shielding of the earth's field and other external magnetic fields.

We come now to a general criticism of the hypotheses involving an emf in cosmic fluids. The efficiency of any such mechanism must be compared with the efficiency of the purely inductive, hydromagnetic amplification, and in this comparison the emf loses out: First, the hydromagnetic amplifying processes are purely kinematical; they are indeed independent of the material constants of the medium. The equivalent of Eq. (13) in differential form is obtained by dropping the last term in Eq. (5); thus

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B}), \tag{14}$$

and this approximation is the better, the larger the linear dimensions involved. The dimensional character of this relation is simple enough. If we write the integral of Eq. (14) symbolically $B(t)=FB(o)$, where F is a linear operator, then F depends only on the nondimensional combination VT/L which shows that for a system of given dimensions L, the time required for amplification can be made as short as one pleases by increasing the velocities sufficiently.

[5] S. K. Runcorn, Trans. Am. Geophys. Union **35**, 49 (1954).
[6] L. Biermann and A. Schlüter, Z. Naturforsch. **5a**, 65 (1950); Phys. Rev. **82**, 863 (1951).
[7] Burhorn, Griem, and Lochte-Holtgreven, Nature **172**, 1054 (1953); Physik. **137**, 175 (1954).

This is not of course an argument indicating that systematic amplification from very small beginnings does actually occur. Leaving the discussion of this problem for later and assuming that systematic processes of amplification do occur, we may at once conclude that the time required to reach some sort of saturation field is of order V/L, where V is representative of the prevailing fluid velocities.

Consider next the magnetic fields produced by an impressed emf. Let us represent this effect by means of a term $\sigma\mathbf{E}^i$ on the right of Eq. (3). This gives rise to a term $\nabla\times\mathbf{E}^i$ on the right of Eq. (5). In order to have a very simple model useful for dimensional arguments, let us now assume that the mean effect of the induction term $\nabla\times(\mathbf{v}\times\mathbf{B})$ for a sufficiently irregular velocity field can be assimilated to a turbulent mixing process. Following a mode of thought common in hydrodynamics we introduce a turbulent magnetic diffusivity ν_m' which will be of the order of $R_m\nu_m$, thus large compared to ν_m. We then write in place of Eq. (5)

$$\frac{\partial\mathbf{B}}{\partial t}=\nu_m'\nabla^2\mathbf{B}+\nabla\times\mathbf{E}^i. \qquad (15)$$

We do not of course imply that Eq. (15) represents faithfully the dynamical processes, but it will be adequate for order-of-magnitude estimates. Now in a solid body the growth of a magnetic field under an impressed emf would also be described by Eq. (15) except that for ν_m' we would have to write $\nu_m=1/\mu\sigma$, as given by Eq. (6). The time of growth of the field to saturation is inversely proportional to ν_m. In the turbulent case where $\nu_m'=R_m\nu_m$ we see from Eqs. (6) and (8) that $1/\nu_m'\sim 1/LV$. The rise-time is reduced by a factor $1/R_m$ as compared to the case of a solid conductor; it is, on the simple model, of the same order as the rise-time of inductive hydromagnetic amplification discussed in connection with Eq. (14). On the other hand, the stationary value of the field produced by an emf, as obtained by setting $\partial B/\partial t=0$ in Eq. (15), is also inversely proportional to ν_m', thus the saturation field corresponding to the impressed currents *is reduced by a factor of order* $1/R_m$, again compared to a solid conductor. This justifies our previous contention that the physical preconditions of hydromagnetism, namely, mechanical motions in large fluid conductors, favor intrinsic inductive amplification of magnetic fields over the effects of impressed electromotive forces.

We are thus led to concentrate our attention upon mechanisms of inductive amplification, as expressed by Eq. (14). Two types of processes have been considered as conducive to cosmic magnetic fields by amplification. One of them is turbulence. If an interaction between fluid motion and the magnetic field exists and if it were possible to ignore dissipation altogether, then one could apply the principles of statistical mechanics to such a frictionless hydromagnetic system. If one argued in this way, he would readily be led to the conclusion that the statistical equilibrium of mutual energy transfer between motion and field corresponds to *equipartition* of the energy, expressed by

$$[\rho\mathbf{v}^2]_{\mathrm{Av}}=[\mathbf{B}^2/\mu]_{\mathrm{Av}}. \qquad (16)$$

Now, however, turbulence is a dissipative process and certainly far from thermodynamical equilibrium. Thus we must not take the equipartition condition (16) too seriously, but it is found valuable as a guide in unravelling the complex observed phenomena. There is one thing we may infer from this argument, namely that, assuming statistical disorder of the fluid motion, the field energy, if much below equipartition to begin with, is likely to increase on the average. It would suffice for this to start with some very small stray fields which would then progressively increase as time goes on. Any field thus produced would have essentially random characteristics. There is good evidence for a very strong random component of the earth's field, for instance, and so the model of statistical amplification has its range of validity. On the other hand, some of the features of cosmic magnetic fields seem to be rather stable. Among these is the earth's main dipole, the sunspot cycle, and the characteristics of magnetic stars, stationary or periodic. It is hard to conceive of these as produced by purely random amplification without some ordering principle of the motion. In a technical generator, a dynamo, the current is ordered by virtue of the mechanical design embodying insulators, commutators, etc.

In a *hydromagnetic dynamo* the conductor is simply connected, and thus an equivalent ordering agency can be based *only on purely dynamical properties* of the fluid motion. It would be impossible, of course, to have a fluid dynamo for $R_m \ll 1$, since then the field would have decayed before amplification could take place. The dynamo problem presents itself naturally in two stages: First, we may consider the kinematical problem that arises when $\mathbf{v}$ is assumed given and fields satisfying Eq. (5) are sought. In this way we may demonstrate the fundamental possibility of fluid dynamos, but hardly their occurrence in nature. When the magnitude of the induced field approaches equipartition, the ponderomotive forces which the field exerts upon the fluid become comparable to the purely mechanical forces. We are then confronted with a problem of genuine hydromagnetic dynamics. Clearly, the joint solution of Eqs. (1) and (5) in any but the simplest cases will prove exceedingly difficult.

We now turn to a survey of the results so far obtained in applying the basic equations of hydromagnetism to specific problems. A few rather special theoretical developments that are closely related to astrophysical or geophysical observations will be mentioned in Part II.

HYDROMAGNETIC WAVES

Conventional sound waves in a fluid are longitudinal and are based on the existence of a finite compressibility. In an incompressible fluid of classical hydrodynamics no genuine wave motion exists. This is altered when a strong magnetic field prevails in the fluid. It was discovered by Alfvén in 1942[8] that purely transverse waves are then possible. We shall here assume incompressibility, $\nabla\cdot\mathbf{v}=0$, and postulate the fact that the waves are transverse.

To derive the equations for Alfvén waves we make use of some well-known vector identities. We have for the induction term in Eq. (5), since

$$\nabla\cdot\mathbf{B}=\nabla\cdot\mathbf{v}=0,$$
$$\nabla\times(\mathbf{v}\times\mathbf{B})=(\mathbf{B}\cdot\nabla)\mathbf{v}-(\mathbf{v}\cdot\nabla)\mathbf{B}. \quad (17)$$

For the ponderomotive force [Eq. (4)] we can write

$$(\nabla\times\mathbf{B})\times\mathbf{B}=(\mathbf{B}\cdot\nabla)\mathbf{B}-\tfrac{1}{2}\nabla(\mathbf{B}^2). \quad (18)$$

We now linearize the hydromagnetic equations by setting

$$\mathbf{B}=\mathbf{B}_0+\mathbf{b},\ \mathbf{b},\ \mathbf{v}\ \text{small},$$

meaning that we neglect squares and products of $\mathbf{b}$ and $\mathbf{v}$. We assume the large, static field $\mathbf{B}_0$ homogeneous, that is independent of x,y,z,t. With these assumptions the expressions (17) and (18) reduce to

$$\nabla\times(\mathbf{v}\times\mathbf{B})=(\mathbf{B}_0\cdot\nabla)\mathbf{v},$$
$$(\nabla\times\mathbf{B})\times\mathbf{B}=(\mathbf{B}_0\cdot\nabla)\mathbf{b}-\nabla(\mathbf{B}_0\cdot\mathbf{b}), \quad (19)$$

but the last gradient term vanishes because of the assumed transversality of $\mathbf{b}$. Furthermore, $d\mathbf{v}/dt=\partial\mathbf{v}/\partial t$, since we neglect second-order terms in $\mathbf{v}$. We assume absence of dissipation, $\nu=\nu_m=0$ and, on substituting Eq. (19) into the hydromagnetic Eqs. (1) and (5), obtain

$$\frac{\partial\mathbf{v}}{\partial t}=\frac{1}{\mu\rho}(\mathbf{B}_0\cdot\nabla)\mathbf{b}-\frac{1}{\rho}\nabla p,\quad \frac{\partial\mathbf{b}}{\partial t}=(\mathbf{B}_0\cdot\nabla)\mathbf{v}.$$

Let now $\mathbf{B}_0$ be in the z-direction. Then these equations reduce to the simple form

$$\frac{\partial\mathbf{v}}{\partial t}=\frac{B_0}{\mu\rho}\frac{\partial\mathbf{b}}{\partial z},\quad \frac{\partial\mathbf{b}}{\partial t}=B_0\frac{\partial\mathbf{v}}{\partial z}, \quad (20)$$

where we have omitted the ∇p term for a readily apparent reason: Equations (20) do not contain $\partial/\partial x$, $\partial/\partial y$; they correspond to waves travelling in the z-direction independent of x and y, that is, plane waves. Clearly, for any such waves ∇p must also be in the z-direction. But the vectors $\mathbf{v}$ and $\mathbf{b}$ are always perpendicular to z; hence we must have $\nabla p=0$. (A closer analysis shows that ∇p is of the second order in the small quantities $\mathbf{v}$, $\mathbf{b}$.) By cross elimination Eq. (20) yields the wave equations,

$$\frac{\partial^2\mathbf{v}}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2\mathbf{v}}{\partial t^2}=0,\quad \frac{\partial^2\mathbf{b}}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2\mathbf{b}}{\partial t^2}=0,$$
$$c^2=B_0^2/\sqrt{\mu\rho}. \quad (21)$$

The velocity of propagation of the transverse Alfvén waves is thus, $c=B_0/(\mu\rho)$. There is no dispersion.

[8] H. Alfvén, *Cosmical Electrodynamics* (Oxford University Press, London, 1950).

We further notice that by Eq. (20) the vectors **v** and **b** are parallel: if we choose a plane of polarization, then the motion as well as the perturbation magnetic field lie in this plane. As remarked by Alfvén, this type of wave motion is closely analogous to the familiar waves traveling along a taut string. The magnetic field B_0 generates the longitudinal tension which must exist in the string to make transverse waves possible. If the fluid is displaced normally to the direction of B_0 the magnetic field lines follow this displacement, by virtue of the flux-conservation law, Eq. (13). The ponderomotive forces thus set up tend to return the field lines to their straight configuration, but they overshoot, and oscillate. Assuming a harmonic wave in which both **v** and **b** have a factor $\exp i(kz-\omega t)$, where $\omega/k=c$, we readily verify from Eq. (20) that $b^2=\mu\rho v^2$, in other words we have at any instant equipartition of the perturbation energy between the motion and the perturbation field. This must also hold for any linear superposition (Fourier integral) of such waves since the individual harmonic components are orthogonal to each other.

The physical importance of waves in a fluid arises from the fact that they permit transport of energy at a rate in excess of the mean velocity of the fluid itself. In an incompressible, nonmagnetic fluid there exists no genuine wave motion and the energy of a local disturbance remains captive: it can only be convected along with the fluid. In a highly compressible fluid the energy of a local disturbance can be radiated away in the form of sound waves, but the efficiency of this process is usually small. If a strong magnetic field exists in the fluid we have for Alfvén waves, $v=b/\sqrt{(\mu\rho)}$ and $c=B_0/\sqrt{(\mu\rho)}$, hence $v/c=b/B_0$, and from $b\ll B_0$ there follows $v\ll c$. Any perturbation whose energy density is small compared to the energy density of the over-all field is rapidly propagated along the field lines; this propagation can be in either direction, since the wave equations (21) admit solutions of the form

$$v=v(z-ct),\quad b=b(z-ct),$$

or else of the form

$$v=v(z+ct),\quad b=b(z+ct).$$

Parker[9] has shown that when the energy of B_0 is large compared to any perturbation energy, all perturbations can be represented as linear superpositions of Alfvén waves, the terms omitted in this representation being small of higher order in b/B_0. It appears then that the presence of a strong magnetic field is a necessary and sufficient condition for any local perturbation to migrate faster than convectively. A "strong" field here means one in which the energy of the homogeneous component of the field is well above the equipartition value expressed in Eq. (16). So far as empirical conditions are concerned, there is evidence that the field in the earth's core is above the equipartition value, but the consequences of this fact in terms of wave motion have not yet been evaluated. In a sunspot with a field of, say 0.2 mks (2000 gauss) and a velocity of, say 1 km/sec, we have equipartition for a density, $\rho_0=3\times10^{-5}$ g/cm^3. This corresponds to a level well below the photosphere. For higher levels, the velocity of Alfvén waves is, under the assumptions made, given numerically by $c=\sqrt{(\rho_0/\rho)}$ km/sec. It has been indicated by a number of authors that Alfvén waves should be of importance in the observed regions of sunspots.

We mention briefly some other theoretical work on hydromagnetic waves. As Walén[10] has pointed out, the hydromagnetic Eqs. (1) and (5) admit also of wave solutions of finite amplitude, that is for an arbitrary value of b/B_0. The necessary condition for this is found to be equipartition of the wave energy, that is, $b^2=\mu\rho v^2$. These waves are particular integrals of the hydromagnetic equations; they are no longer arbitrarily superposable as is the case for waves of infinitesimal amplitudes; waves traveling in the same direction can be superposed, but a linear superposition of waves of finite amplitude traveling in opposite directions is in general no longer a solution of the hydromagnetic equations. Since for $b\sim B_0$ the waves travel with a velocity comparable to v, the velocity of turbulent elements, the observation of waves of finite amplitude would be a difficult matter.

[9] E. N. Parker, Phys. Rev. **99**, 241 (1955).

[10] C. Walén, Ark. Mat. Astron. Fysik **30A**, No. 15 (1944); **31B**, No. 3 (1944); **33A**, No. 18 (1946).

Ferraro[11] and Roberts[11a] have derived the formulas that govern the reflection and refraction of Alfvén waves at the boundary of two media of different densities. Lehnert[12] has investigated the propagation of these waves in the presence of a Coriolis force and finds that there exist two circularly polarized modes which travel with different velocities. Dungey[12a] and Roberts[12b] have investigated the attenuation of hydromagnetic waves. de Hoffman and Teller[13] have analyzed hydromagnetic shock waves; further study of such waves originates from Helfer[14] and Lüst.[15] Baños[16] has given a general classification of hydromagnetic waves, while retaining all terms compatible with linearization of the basic equations.

The papers quoted above deal with wave motion mainly under astrophysical conditions. The relatively small hydromagnetic effects that can be produced in the laboratory have also been investigated. Lundquist[17] has studied solutions of the linearized hydromagnetic equations with special regard to the feasibility of laboratory experiments on waves; he performed one experiment with torsional hydromagnetic waves in a tank filled with mercury in the presence of a strong magnetic field. Liquid sodium is better suited than mercury for hydromagnetic laboratory experiments owing to its high electrical conductivity and low density. Lehnert[18] has done extensive experimentation with torsional waves in a cylindrical vessel filled with liquid sodium. He has also carried through the theoretical analysis in detail, obtaining on the whole a satisfactory agreement between experiment and theory. Even with a favorable substance such as sodium, damping effects play an important role under laboratory conditions. In conclusion we might mention that Anderson[19] has computed the small perturbation effects owing to the influence of the earth's magnetic field upon compressional sound waves traveling in a conductor such as sea water.

TURBULENCE AND INSTABILITY

We now turn to the problems of hydromagnetic turbulence. Their importance in connection with any satisfactory theory of hydromagnetism will be obvious from what has preceded. In a turbulent medium we may expect amplifying processes based on a velocity shear, as in Fig. 2, to occur with a certain frequency, on statistical grounds. This question has been investigated by Batchelor.[20] He bases his ideas on statistical relationships derived from the magnetic induction Eq. (5). The formal structure of this equation is not new; it may be shown[21] to be completely analogous to the Helmholz equation valid in ordinary hydrodynamics for the vorticity, say $\boldsymbol{\omega}$:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times (\mathbf{v} \times \omega) + \nu \nabla^2 \boldsymbol{\omega}. \tag{22}$$

Now a good deal is known about the behavior of $\boldsymbol{\omega}$ from experiments in wind tunnels. Batchelor uses the similarity of Eqs. (5) and (22) to infer the behavior of magnetic fields in a turbulent, conducting medium; to quote him: "It is well known in the subject of turbulence that on the average particles of the fluid tend to diffuse apart and that this process lengthens lines which move with the fluid. Lines of vorticity do not move with the fluid exactly when $\nu \neq 0$, but inasmuch as they do so approximately, they tend to lengthen." If the same argument is applied to the magnetic field lines one may conclude that on starting with a small rms value of the field this value will increase as a result of the stretching of the field lines. The analogy with the vortex lines seems trustworthy enough when the magnetic field is so small that the ponderomotive forces exerted by it are negligible (it is to be remembered that these are quadratic in the field strength). One may also infer that owing to the linear character of the differential equations involved the rate of increase with time

[11] V. C. A. Ferraro, Astrophys. J. **119**, 393 (1954).
[11a] P. H. Roberts, Astrophys. J. **121**, 720 (1955).
[12] B. Lehnert, Astrophys. J. **119**, 647 (1954); **121**, 481 (1955).
[12a] J. W. Dungey, J. Geophys. Research **59**, 323 (1956).
[12b] P. H. Roberts, Astrophys. J. **122**, 315 (1955).
[13] F. de Hoffman and E. Teller, Phys. Rev. **80**, 629 (1950).
[14] H. L. Helfer, Astrophys. J. **117**, 177 (1953).
[15] R. Lüst, Z. Naturforsch. **8a**, 277 (1953).
[16] A. Baños, Jr., Phys. Rev. **97**, 1435 (1955).
[17] S. Lundquist, Phys. Rev. **76**, 1805 (1949).
[18] B. Lehnert, Phys. Rev. **94**, 815 (1954).
[19] N. S. Anderson, J. Acoust. Soc. Am. **25**, 529 (1953).
[20] G. K. Batchelor, Proc. Roy. Soc. (London) **201**, 405 (1950).
[21] W. M. Elsasser, Phys. Rev. **69**, 106, **70**, 202 (1946); **72**, 821 (1947).

of the field is exponential, a conclusion first indicated by Schlüter and Biermann.[22] Another highly plausible assumption is that, given this rapid amplification for small fields, the equilibrium of hydromagnetic turbulence corresponds to the equipartition law, Eq. (16). Batchelor's contention, however, that for the largest-size eddies the associated magnetic fields must remain below equipartition, seems open to question. There are, moreover, limitations to the dynamical analogy between $\boldsymbol{\omega}$ and $\mathbf{B}$: For a given velocity field in ordinary turbulence the dynamical forces on the fluid are

$$(\mathbf{v}\cdot\nabla)\mathbf{v}=\boldsymbol{\omega}\times\mathbf{v}-\tfrac{1}{2}\nabla(\mathbf{v}^2),$$

whereas in hydromagnetic turbulence the forces are given by the sum of this expression and the ponderomotive forces, Eqs. (4) or (18). These formulas are by no means symmetrical in $\boldsymbol{\omega}$ and $\mathbf{B}$.

The further, detailed structure of the turbulent spectrum depends no doubt on the relative magnitude of the "viscosities" ν and ν_m. In ordinary turbulence the smallest eddy size (short-wave cutoff of the spectrum) is that for which the viscous forces become comparable to the inertial forces. It is plausible that in hydromagnetic turbulence, if equipartition at least of the smaller eddies is assumed, the short-wave cutoff of the turbulent spectrum is given by the larger one of the two quantities, ν and ν_m (Batchelor's much more radical conclusion that there can be no magnetic amplification at all for $\nu_m>\nu$ does not seem acceptable without further justification). Elsasser[1] has estimated the ratio ν/ν_m for ionized cosmic gases based on simple gas-kinetic considerations and finds that if the gas is assumed to be hydrogen,

$$\nu/\nu_m=2\cdot 10^{-4}\alpha/\rho, \tag{23}$$

where α is the degree of ionization, ρ the density in mks units. In interstellar gases with their low density, the value of ν/ν_m is very large; that is, the effect of magnetic diffusivity (e.g., the Joule's heat generated) is entirely negligible compared to the effects of mechanical viscosity. In the far interior of stars, on the other hand, if hydromagnetic fields are present, energy dissipation is almost exclusively by electromagnetic means.

The kinematical properties of a field of turbulence can be expressed in terms of correlations between the velocities of fluid particles some distance apart. This method has been developed in ordinary nonmagnetic turbulence, expecially by von Karman and Howarth; it has been systematically extended to the hydromagnetic case by Chandrasekhar.[23] Further work is due to Lundquist[24] and an extension of Chandrasekhar's theory to compressible fluids to Krzywoblocki.[25] Finally, Lehnert[26] has studied the free decay of hydromagnetic turbulence subject to the influence of an external, homogeneous magnetic field. He finds that in the presence of such a field the turbulent regime can remain neither homogeneous nor isotropic. It develops strongly axisymmetrical properties in that the Fourier components along the direction of the field are damped out preferentially. On the other hand, this influence of the magnetic field is counteracted, in a rotating system, by a Coriolis force, provided the axis of rotation is inclined relative to the magnetic field. On the basis of his theoretical developments Lehnert is able to give an explanation of the inhibition of turbulence in mercury by a magnetic field observed in the experiments to be reported presently.

Such experiments were first carried out by Hartmann[27] who let mercury flow down a channel of rectangular cross section. He found that under conditions of a turbulent regime the pressure difference required between the ends of the channel, in order to maintain a constant rate of total flow, decreases when the magnetic field is increased. It is readily understandable that the presence of a magnetic field should reduce the amount of turbulence observed; the forces produced by the field give the fluid an added stiffness, which is equivalent to an increase in the effective eddy viscosity. Some-

[22] A. Schlüter and L. Biermann, Z. Naturforsch. **5a**, 237 (1950).

[23] S. Chandrasekhar, Proc. Roy. Soc. (London) **204**, 435; **207**, 306 (1951); see also a forthcoming paper by the same author in Proc. Roy. Soc. (London) (to be published).

[24] S. Lundquist, Arkiv Fysik **5**, No. 15 (1952).

[25] M. Z. E. Krzywoblocki, Acta Phys. Austriaca **6**, 157, 250 (1952/3).

[26] B. Lehnert, Quart. Appl. Math. **12**, 321 (1955).

[27] J. Hartmann, Kgl. Danske Videnskab. Selskab. Mat.-fys. Medd. **15**, No. 6 (1937); F. Lazarux, *ibid.*, **15**, No. 7 (1937).

what similar experiments in a rectangular channel, shaped as a horizontal ribbon, were carried out more recently by Murgatroyd.[28] Again, the onset of turbulence depends on the strength of the magnetic field and can be expressed empirically in terms of certain nondimensional parameters, in general agreement with the theoretical work available. Even at the highest Reynolds numbers that could be obtained ($R=10^5$) turbulence could be successfully suppressed by a sufficiently strong magnetic field. Finally, Lehnert[29] observed a regime of eddying motions developed in mercury between a stationary and a rotating cylinder. The torque between the two cylinders, for a given angular velocity, increases with increasing field strength. This is in the opposite direction to what one would expect on the assumption of a mere suppression of turbulence. In fact, in one set of experimental conditions an appreciable decrease of the torque was observed when the field was introduced. The magnetic fields used by the authors quoted above were of the general order of 10 000 gauss.

Let us return now to the theory. The difficulties in the treatment of any hydromagnetic problem are the same as those of more conventional hydrodynamics: by and large only linear problems are open to exhaustive mathematical analysis. There are few of these in hydrodynamics; one such class is represented by highly viscous, laminar flow, slow enough so that the $(\mathbf{v}\cdot\nabla)\mathbf{v}$ terms in the equations of motion may be neglected. The Poiseuille and Stokes formulas of viscous flow are familiar examples. The technique can be extended to the slow, laminar flow of a viscous, conducting fluid in a homogeneous external magnetic field. The case of viscous flow between two parallel infinite plates has been treated by Stuart[30] and the case of laminar flow in a rectangular pipe by Shercliff.[31]

As a rule, of course, the equations of hydromagnetism are essentially nonlinear. Linearization of such a system of differential equations means the application of perturbation theory. The solutions of any perturbation problem are of two types, either oscillatory about an equilibrium or stationary state, indicating that the state is stable, or else increasing in amplitude as time goes on, indicating that the original state was unstable. The use of this method in classical hydrodynamics is well known, for instance for the determination of the transition between the laminar and the turbulent regime which occurs when certain wave modes of the laminar flow become unstable. For a periodically damped viscous flow Rayleigh developed a perturbation technique that permits one to compute the first appearance of the well-known Bénard cells of convection in a viscous fluid under the influence of a temperature gradient. Chandrasekhar extended these techniques to the hydromagnetic case in a series of remarkable papers (reviewed by Chandrasekhar[32]). They discuss stability problems of a conducting fluid in the presence of magnetic fields as well as of Coriolis forces, corresponding to the actual situation in astrophysics. The calculations bear out in a quantitative form the notion that a magnetic field has an inhibiting effect on the onset of convective motion; moreover, they lead to other quantitative predictions which could not easily have been foreseen intuitively. It is rather difficult to give an adequate account of these beautiful investigations within the confines of a brief review such as this where lack of space prevents us from exhibiting the analytical procedures; we can therefore only give a rough enumeration of the results.

The first group of these papers[33] deals with the onset of cellular convection of the Bénard type. In a nonconducting fluid such convection sets in when the so-called Rayleigh number (a nondimensional combination containing the coefficients of heat conduction and viscosity, the temperature gradient, and the fourth power of the depth of the layer) exceeds a certain critical value, in the neighborhood of 1000. If the fluid is an electrical conductor and a magnetic field is applied the critical Rayleigh number, R_c becomes a function of another nondimensional

[28] W. Murgatroyd, Phil. Mag. **44**, 1348 (1953).
[29] B. Lehnert, Arkiv Fysik. **5**, 69 (1952); Tellus **4**, 63 (1952).
[30] J. T. Stuart, Proc. Roy. Soc. (London) **221**, 189 (1954).
[31] J. A. Shercliff, Proc. Cambridge Phil. Soc. **49**, 136 (1953).
[32] S. Chandrasekhar, Monthly Notices Roy. Astron. Soc. **113**, 667 (1953).
[33] S. Chandrasekhar, Phil. Mag. **43**, 501, 1317 (1952); **45**, 1177 (1954).

parameter, $Q=B^2\sigma d^2/\rho\nu$ (where d is the depth of the layer); asymptotically, for large numerical values of Q and R_c they are proportional to each other. The theoretical predictions have just received a splendid quantitative experimental verification by Nakagawa.[34]

Next, Chandrasekhar[35] studied the stability of a viscous conducting fluid enclosed between rotating cylinders and subjected to a magnetic field. Again, the onset of rotational instability (leading eventually to turbulence) depends on the magnitude of the parameter Q; the inhibition of instability increases, of course, with increasing Q, the effect being more pronounced here than in the case of thermal instability. The same mathematical technique as is used to analyze the effect of a magnetic field may be employed to determine the effect on convective stability of a Coriolis force.[36] The effect of rotation is a stabilizing one, the critical Rayleigh number increasing as function of a nondimensional parameter $T=4\omega^2d^4/\nu^2$. Asymptotically, for large numerical values, R_c becomes proportional to $T^{\frac{2}{3}}$. Still more complicated is the case where a magnetic field and a Coriolis force act simultaneously upon the convective layer; Chandrasekhar[32] presents extensive numerical results of such calculations. Some rather curious phenomena appear as a result of the competition of these two effects. Perhaps it is best to let the author himself speak: "For a value of Q slightly less than 1000 the wave number of the cells which appear at marginal stability will suddenly decrease from $a=18.2$ to $a=3.4$. In other words, if we start with an initial situation in which T has the value 10^6 and no magnetic field is present and gradually increase the strength of the magnetic field, then at first the cells which appear at marginal stability will be elongated; but when the magnetic field has increased to a value corresponding to $Q=1000$, cells of two very different sizes will appear simultaneously: one set which will be highly elongated and another set which will be much less elongated. As the magnetic field increases beyond this value, the critical Rayleigh number will actually begin to decrease. However, for sufficiently large Q the inhibition due to the magnetic field will predominate and will take control of the situation. This is an unexpected sequence of events and I do not know if one could have predicted it."

In the problems of stability just discussed the fluid is considered as incompressible, apart from those variations of density (usually small) required to produce convective motion. We encounter a different type of stability problems in astrophysics when dealing with highly rarefied gases, as in the envelopes of stars or in interstellar gas clouds. Here the fluid can change its shape rather freely; instability then indicates that an equilibrium configuration will go over into another one which usually is rather radically different from the first. Thus Jeans has shown that a very large homogeneous gas cloud is unstable under the mutual gravitational attraction of its parts and will collapse into lumps, a fact fundamental for cosmogonic speculation. A given perturbation in such a gas is unstable when its wavelength exceeds the value $c(\pi/G\rho)^{\frac{1}{2}}$, where c is the velocity of sound in the gas and G the gravitational constant. Chandrasekhar[37] has shown that this stability criterion remains unaffected by the presence of either a Coriolis force or a magnetic field.

Consider a rarefied gas in which there prevails a very strong magnetic field (e.g., in the envelope of a magnetic star or in the atmosphere above a sunspot). Now all mechanical forces acting on the fluid other than the ponderomotive force, Eq. (4), derive from a scalar potential; hence hydrostatic equilibrium can prevail only when Eq. (4) is also the gradient of a scalar. The differential equation,

$$(\nabla\times\mathbf{B})\times\mathbf{B}=\nabla\varphi, \qquad (24)$$

represents a necessary condition for hydrostatic equilibrium in the presence of a magnetic field. Following Lundquist[4,38] who first studied such hydromagnetic equilibrium fields we may transform Eq. (24) by means of the identity (18) which gives

$$(\mathbf{B}\cdot\nabla)\mathbf{B}=\nabla\psi, \qquad (25)$$

with $\psi=\varphi+\frac{1}{2}B^2$. It may readily be proved, by

[34] Y. Nakagawa, Nature **175**, 417 (1955).
[35] S. Chandrasekhar, Proc. Roy. Soc. (London) **216**, 293 (1953).
[36] S. Chandrasekhar, Proc. Roy. Soc. (London) **217**, 306 (1953).
[37] S. Chandrasekhar, Astrophys. J. **119**, 7 (1954).
[38] S. Lundquist, Arkiv Fysik. **2**, No. 35 (1950).

the way, that the only solution of Eq. (25) for $\nabla\psi=0$ is a homogeneous field, $\mathbf{B}=$const. On the other hand, there do exist nontrivial solutions of Eq. (24) for $\nabla\varphi=0$. These were extensively investigated by Lüst and Schlüter[39] who satisfy Eq. (24) by setting $\nabla\times\mathbf{B}=\alpha\mathbf{B}$ with α some scalar function. They assume spherical symmetry and find that solutions exist for which the magnetic field is entirely confined to a shell between two concentric spheres. A general solution consists of a set of such spherical shells. The fluid within any single shell can rotate as a whole without affecting the other shells. A somewhat simpler case has been integrated by Dungey[40] referring to an atmosphere, the closed lines of force being confined to a slab bounded by parallel planes, gravity being normal to these planes.

A problem which has received considerable attention is that of the stability of a cosmic magnetic field having the shape of a cylinder of infinite length. Lundquist[4] found that a cylinder having a helical twist of the field lines will, under certain conditions, become unstable relative to a bending motion normal to the cylinder axis. Chandrasekhar and Fermi[41] consider such a cylinder as a model of a spiral arm of a galaxy. There are three forces acting in a radial direction, normal to the cylinder axis, namely, the gas pressure, the magnetic force, and gravity. Using numerical values for the spiral arm in which we are located one finds that the gas pressure is a relatively small fraction of the two other forces, so that the equilibrium of the interstellar gas in our spiral arm depends essentially on a balance of the magnetic forces which tend to expand the gas laterally, and gravity. If there were no magnetic field the gas would have collapsed to the center of the arm and would no doubt have long since been converted into stars. Thus the presence of interstellar gas is in itself a strong theoretical argument in favor of a galactic field (about 6×10^{-6} gauss to balance gravity) whose existence has been deduced from astronomical observations on the polarization of star light (see Part II). Chandrasekhar and Fermi proceed to investigate the stability properties of a long magnetic cylinder in detail. Such a cylinder is unstable for all periodic transverse deformations of the boundary whose wavelength exceeds a certain critical value. This value depends on the field strength, the magnetic field having again a stabilizing influence. Now the spiral arms of the galaxy show every sign of dynamical instability, but in the absence of a magnetic field the calculated lifetime of such a system could not exceed some 10^8 years, which is obviously too short. The assumption of a magnetic field of the aforementioned magnitude can be shown to raise this lifetime to some 10^9 years which is satisfactory.

We have omitted from our review a number of theoretical investigations dealing with the modifications of shape, and of the mechanical oscillations of stars brought about by an internal magnetic field. This work seems primarily of purely astronomical interest. As Chandrasekhar[32] has remarked, a magnetic field decreases the oscillatory stability of a star and hence lengthens the fundamental period of oscillation. The periods of magnetic variables (see Part II) are as a rule very much longer than one would calculate for a nonmagnetic star. If this lengthening was to be attributed entirely to the internal field, this field would in some cases have to be very strong indeed (of the order of hundreds of thousands of gauss).

DYNAMO MODELS

We have yet to come to grips with the one problem which, while rather arduous and frustrating from the purely mathematical viewpoint, is clearly basic for hydromagnetism, namely, the question how the observed cosmic magnetic fields are actually created by amplificatory processes. This is the hydromagnetic dynamo problem. Expressed in mathematical form it is essentially nonlinear; no amount of effort will exhibit the governing features in a linearized approximation. This being so one cannot but rely on intuitive, semiquantitative arguments and one must probably continue to do so for some time to come. We have already encountered the process of statistical amplification of magnetic fields in a turbulent regime,

[39] R. Lüst and A. Schlüter, Z. Astrophys. **34**, 263 (1954).
[40] J. W. Dungey, Monthly Notices Roy. Astron. Soc. **113**, 180 (1953).
[41] S. Chandrasekhar and E. Fermi, Astrophys. J. **118**, 113, 116 (1953).

but many of the observed fields are too regular by far to be explained by random motions.

Some writers have suggested that cosmic magnetic fields are the remnants of a primeval field created by some remote cosmogonic process. This idea is, of course, based on the very long calculated lifetimes, far exceeding the age of the universe, in conductors of sufficient size (a star, an interstellar gas cloud, etc.). It is true that the origin of the galactic magnetic fields is still quite obscure. Apart from this case the assumption of a survival of primeval fields must be viewed with very great caution since, as we have mentioned before, the lifetime of magnetic fields in a fluid is determined, not by molecular constants but by the rate of *turbulent* mixing, and this might reduce the computed lifetimes by many powers of ten. The interior regions of the stars are highly quiescent and the question as to whether a magnetic field could remain there for a time comparable to the age of the star is a rather intriguing one. It does, however, have little direct application to the case of stellar magnetism most immediately observed, namely, the magnetic fields on the sun, particularly in sunspots. While the interior of the sun is in all likelihood highly quiescent, it is surrounded by a convective shell. This is known as the hydrogen convection zone, so named because as one descends in the layer the temperature rises to the point where hydrogen is ultimately completely ionized. The progressive ionization of hydrogen with depth leads to an increase in the specific heat which can be shown to engender convective instability. The exact depth of this layer is not known, but it may amount to about 15–20% of the solar radius. There can be no doubt that mixing in this layer is intense enough to prevent any magnetic field from surviving for more than a short time unless maintained by a suitable mechanism.

Objections have been raised against hydromagnetic dynamos on general, as it were, philosophical grounds. They are usually based on a too rigid interpretation of the conservation of magnetic flux, as expressed, for instance, by Cowling's integral theorem, Eq. (13). One must be careful in reasoning about the concept of magnetic "lines of force." As McDonald[42] has shown in detail, the condition $\nabla\cdot\mathbf{B}=0$ does not entail that the lines return upon themselves, or else go to infinity. They can be "ergodic," that is, of infinite length in a finite volume; they can begin or terminate in any "singular" point or line of the field defined by $\mathbf{B}=0$. Again, Cowling's conservation theorem for the component of the flux normal to any surface does not imply that the amount of flux in a given volume is constant; it can be increased indefinitely by the type of shear motion shown in Fig. 2, or its generalizations to be discussed presently. This was emphasized by Alfvén[43]; on the other hand, as Bondi and Gold[44] point out, the internal deformation of an ideally conducting fluid, while leading to amplification, does not give rise to a stationary mechanism because in the absence of dissipation the lines of force get "tangled up" without limit. A dynamo theory must therefore take account of the diffusive smoothing of the field as an essential part of the model.

The first quantitative study of a dynamo mechanism was made by Cowling.[45] He uses a greatly simplified geometrical model, assuming that the lines of force of the magnetic field as well as the trajectories of the fluid particles are confined to the meridional planes. The field created then always remains in the meridional planes. Cowling was able to prove rigorously that under these conditions a stationary dynamo cannot exist. It is likely that this result is a special case of a more general one which says that no dynamo is possible when the fluid motion is essentially two-dimensional, that is when the particles lying in a certain surface always remain on that surface. While this latter statement cannot as yet be rigorously proved it can at least be made highly plausible.[46] It is analogous to certain arguments in turbulence theory where it is highly plausible, although apparently it has not been rigorously proved, that a fluid with a similar constraint to two-dimensional motion cannot become turbulent.

It appears, therefore, that highly symmetrical patterns of fluid motion are not favorable for

[42] K. L. McDonald, Am. J. Phys. 22, 586 (1954).

[43] H. Alfvén, Tellus 2, 74 (1950).

[44] H. Bondi and T. Gold, Monthly Notices Roy. Astron. Soc. 110, 607 (1950).

[45] T. G. Cowling, Monthly Notices Roy. Astron. Soc. 94, 39 (1934).

[46] See a forthcoming review of the author to appear in Revs. Modern Phys.

dynamos since they will tend to result in essentially two-dimensional flows. One can state in physical terms the conditions that tend to invalidate the symmetry restrictions and that correspond to the observed hydromagnetic fields. The basic principles seem to be the simultaneous presence of *convection* and *rotation* in a cosmic fluid. Convection is probably nonspecific; it is merely the usual agency whereby intense and rapid motions in the interior of a cosmic fluid are set up. Now, if convection alone was active the hydrodynamic regime would be that of Bénard cells; the fluid particles would describe closed curves confined to planes, and no dynamo would result. If the system rotates, the paths of the fluid particles are twisted into three-dimensional shapes by the action of the Coriolis force; we shall presently study this mechanism in some detail. Observation indicates that most magnetic stars are rapidly rotating objects (see Part II).

Consider now a spherical shell filled with fluid that is in radial convection (e.g., by heat developed on the inside) while at the same time rotating about an axis through the center of the figure. We shall think of the convection as an irregular, eddying motion where some fluid particles move radially outward, others inward. Now under the action of the Coriolis force, which expresses nothing but the conservation of angular momentum, a particle moving away from the axis of rotation decreases its angular velocity about this axis, a particle moving toward the axis increases its angular velocity. We may conclude, without going here through the mathematical analysis, that in a stationary convective regime there exists *a mean gradient of angular velocity*, in such a way that the angular velocity decreases as we go away from the axis of rotation.

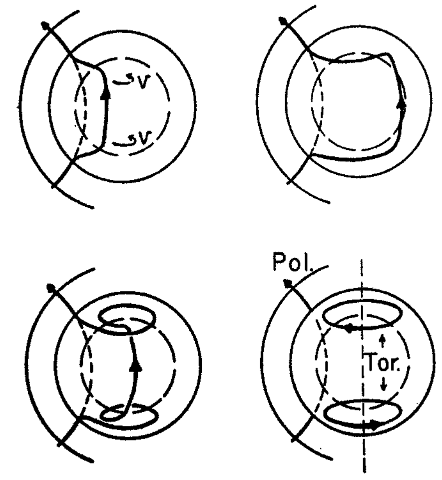

FIG. 3. Generation of toroidal field in a nonuniformly rotating fluid sphere.

Assume now that a magnetic field, whose lines of force are originally in the meridional planes, exists within this nonuniformly rotating fluid. The lines of force, being attached to the fluid particles, will be deformed in the manner shown in Fig. 3. It appears from the last of these diagrams that the final result may be represented as the superposition of two fields, the "primary" whose lines of force are in the meridional planes, designated as *poloidal*, and the "secondary" or induced field whose lines of force are circles about the axis, designated as *toroidal*.[47] This amplificatory mechanism[21] can be expressed in terms of our basic equations (5). In polar coordinates we have, since $\mathbf{v}$ has only one component, v_φ,

$$\frac{\partial B_r}{\partial t}=\nu_m[\nabla^2\mathbf{B}]_r,\quad \frac{\partial B_\theta}{\partial t}=\nu_m[\nabla^2\mathbf{B}]_\theta$$

$$\frac{\partial B_\varphi}{\partial t}=\frac{\mathbf{i}_\varphi}{r}\left[\frac{\partial}{\partial r}(rv_\varphi B_r)+\frac{\partial}{\partial \vartheta}(v_\varphi B_\theta)\right]+\nu_m[\nabla^2\mathbf{B}]_\varphi, \quad (26)$$

where $\mathbf{i}_\varphi$ is a unit vector in the φ direction and where, as is well known, the components of $\nabla^2\mathbf{B}$ in curvilinear coordinates differ from the expression $\nabla^2 B_r$, etc. (They are given, for instance, on p. 116 of Morse and Feshbach.[48]) If we set $\nu_m=0$ the last equation can be integrated at once, providing B_r, B_θ are *fixed*; the toroidal field B_φ then increases linearly with time. This amplifying mechanism is clearly an application of the basic velocity-shear model of Fig. 2. It constitutes the first step toward a dynamo, but in itself is not a dynamo as we may see from the fact that the primary, poloidal field B_r, B_θ itself decays exponentially. If the ex-

[47] Poloidal and toroidal fields, explained here only for rotationally symmetrical fields, are defined more generally in terms of spherical harmonics by Eqs. (30), (31) below. They are the transverse (divergence-free) solutions of the vector wave equation, $\nabla^2\mathbf{A}+k^2\mathbf{A}=0$; they correspond to the **M** and **N** vectors in the terminology of Stratton [*Electromagnetic Theory* (McGraw-Hill Book Company, Inc., New York, 1941)].

[48] Morse and Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company Inc., New York, 1953).

ponential solutions of the first two Eqs. (26) are inserted into the last equation, we find that the toroidal field B_φ does also ultimately decay, after an initial amplification. The whole system might be compared to a radioactive family consisting of a long-lived mother substance and a daughter substance of high-energy output. If we start with the pure mother substance the energy output of the system will increase until the equilibrium amount of the daughter substance has accumulated, but thereafter the output will diminish gradually as the amount of mother substance becomes exhausted. The formation of the toroidal field in the earth as described by Eq. (26) leads to a number of theoretical problems, both electrodynamical and geophysical which have been investigated in considerable detail by Bullard.[49] One significant feature of the toroidal field is that it vanishes in an insulator or vacuum surrounding the conducting sphere; it will therefore not show up in measurements at the earth's surface.

We have not yet, however, arrived at a dynamo model; to obtain one we must find a process which maintains the primary, poloidal field whose existence, up to now, has been postulated. Cowling's result quoted above may readily be extended to say that no rotationally symmetrical fluid motion can possibly amplify the poloidal field if the existing field is any rotationally symmetrical linear combination of a poloidal and a toroidal field. We notice here a rather remarkable topological asymmetry: the amplification of the toroidal field from the poloidal primary is a rotationally symmetrical process, but there exists no reverse to it. It therefore becomes necessary to consider motions that no longer are symmetrical about the axis of rotation: *a hydromagnetic dynamo must be essentially three-dimensional.* We therefore search for a pattern of fluid motion with suitable asymmetry that is to be superposed upon the nonuniform rotation generating the toroidal field. Clearly, this desired pattern cannot essentially depend on the magnetic viscosity ν_m, since the latter merely tends to smooth out existing features of the magnetic field but does not create new ones. We should therefore be able to describe the process in a first approximation by Eq. (14). This equation admits of an arbitrary scale transformation for the length L, provided only we change the scales of V and T correspondingly. Thus, if motions of this type can be found at all, it should be possible to construct them on an arbitrarily small scale. This suggests at once that we look to the local convective eddies as providing the required mechanism. Parker[50] has shown that not only can the local eddies do this, but that there are dynamical reasons for the effective pattern. The essential dynamical feature is again rotation, the action of the Coriolis force upon the local motions. Let us conceive of the convective regime as consisting of a series of streams of fluid rising radially outward while the remainder of the fluid sinks correspondingly. Now consider one such rising stream. At its lower end there must be lateral convergence and at its upper end divergence of the fluid. Assume for example's sake an eddy at the pole where the stream would be in the direction of the earth's axis. As the fluid converges it will be deflected (to the right in the northern hemisphere) at the same time it rises and the net result will be a spiralling motion (Fig. 4). At the top of the stream the fluid diverges and the Coriolis force, acting now in the opposite sense will "uncoil" the spiral again. If the convective stream is not along the earth's axis the geometry of the spiralling motion will be somewhat more complicated.

Consider the deformation of the toroidal magnetic field by such a spiral. This is shown in Fig. 5 where the axis of the convective stream is assumed normal to the field lines. We see how the field lines are lifted and at the same time undergo a circular twist. If the twist is of the order

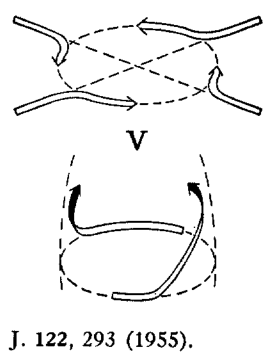

FIG. 4. Coriolis effect on locally converging and rising eddy of fluid.

[49] E. C. Bullard, Proc. Roy. Soc. (London) **197**, 433; **199**, 413 (1949).

[50] E. N. Parker, Astrophys. J. **122**, 293 (1955).

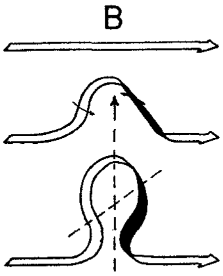

FIG. 5. Creation of a magnetic loop normal to initial toroid field by means of the type of local motion depicted in Fig. 4.

of 90° the result is apparent from Fig. 5: a closed loop of magnetic force in a plane perpendicular to the original lines has been created. The detailed mathematical analysis has been carried out by Parker. It shows that the actual configuration of the field lines is slightly more complicated, but the complications will vanish if averaged over a number of adjacent eddies. The individual loops will coalesce in a manner to be described presently. The right sign for amplification of the poloidal field (regenerative feedback) is achieved when a rising current is coupled with the circulation resulting from influx (cyclonic circulation—in a projection upon the equatorial plane the sense of rotation is the same as that of the earth). Clearly, then, the reverse sign (resulting in degenerative feedback) will apply when a sinking motion is coupled with an influx. In order to apply this model to the formation of a hydromagnetic dynamo it is necessary, therefore, to introduce a special postulate, namely, that the convective motions are asymmetrical with respect to rising and sinking eddies. For anyone acquainted with the remarkable asymmetries of motion found in geophysical hydrodynamics (e.g., in the atmosphere) this will hardly appear a startling assumption. A number of possibilities offer themselves for producing such an asymmetry, but in view of their more or less speculative character it is hardly interesting to enumerate them. Now we notice that the formation of loops suitable for feedback takes place only when the angle of rotation of the eddy is less than 180° and is, in fact, not too far from 90°. To understand that this can be so we must take into account the mechanical forces that counteract eddy formation. The most important among these is the ponderomotive force set up by the deformation of the toroidal magnetic field. We have reason to assume that this field is quite strong; indeed, the theory shows that the toroidal field will in general be appreciably stronger than the poloidal field inside the fluid; this is due to the facility with which the toroidal field is amplified by non-uniform rotation.[49] If we assume that there is a geometrical or dynamical difference, or both, between rising and sinking convective eddies, then it is fairly easy to account for a differential effect upon the twisting of the magnetic lines of force that would make one type of feedback loop preponderate. In a familiar, crude picture, the lines of force act like so many rubber strings that oppose deformation, the force being proportional to the square of the field strength **B** by Eq. (4); thus it is easily conceivable that the effect is sensitive to relatively small differences in the dynamics of the upgoing and downgoing eddies. A detailed analysis of such a differential effect would be rather difficult as the result depends on a great many parameters.

There arises of course the question of the *stability* of such a dynamo. Now when the toroidal field becomes too strong its mechanical forces will stop the required twist of the eddies; they will in fact ultimately stop any convection. The much more difficult question is what happens when the field becomes weak. If we assume that the average twist of the eddies is less than 90°, then a decrease of the field should lead to an intensification of the feedback loops and hence ultimately of the field itself. The conclusion seems inevitable that a stable dynamo regime is

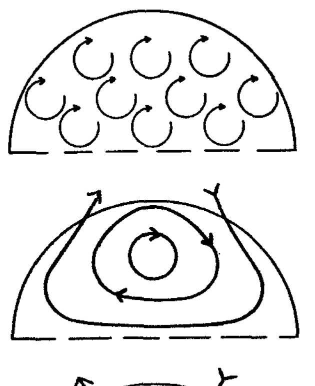

FIG. 6. Coalescence of loops to regenerate poloidal dipole field.

possible only when the toroidal field is strong enough to reduce the mean twisting of the convective eddies induced by the Coriolis force to an angle of less than 90°.

So far we have only discussed individual feedback loops. The final step of the feedback process consists in allowing these loops to coalesce (Fig. 6) so as to form an over-all poloidal field. Parker[50] has carried out the pertinent analysis. To make the problem tractable he assumed first that the magnetic loops are numerous so that they can be represented by a smoothed-out rate of appearance which will be a continuous function of the coordinates. It is convenient to represent the poloidal field by its vector potential where, in the usual way,

$$\mathbf{B}_{\text{pol}} = \nabla \times \mathbf{A}. \qquad (27)$$

Clearly, in case of rotational symmetry of the poloidal field, **A** has only one component, A_φ; we shall write A for short. In terms of A a local feedback loop appears as a "hill" of A. (This model is in contradiction with the usual assumption $\nabla \cdot \mathbf{A} = 0$, but the difficulty disappears as soon as we average over circles of latitude.) Hence the assumptions regarding the feedback mechanism can be expressed by stating that there are sources of A proportional to the strength of the toroidal field, say B (short for B_φ) with a proportionality factor Γ that measures the rate of creation of loops. Taking into account dissipation of the poloidal field, we arrive at the equation,

$$\frac{\partial A}{\partial t} - \nu_m \nabla^2 A = \Gamma(r,\vartheta) B. \qquad (28)$$

Rewriting the least of Eqs. (26) in terms of A and writing v for v_φ, it becomes

$$\frac{\partial B}{\partial t} - \nu_m \nabla^2 B = (\nabla v) \times (\nabla A). \qquad (29)$$

Equations (28), (29) constitute the *dynamo equations*. Note that the gradients in Eq. (29) refer to differentiation in meridional planes; also, the symbol ∇^2 has a meaning slightly different from the conventional one [designating here the operation—*curl curl* applied to a vector component as discussed in connection with the foregoing Eq. (26)]. The dynamo equations embody our physical assumptions regarding the amplificatory feedback cycle; they are linear and their degree of complexity is not such that their integration would be entirely out of reach of conventional techniques.

In order to obtain a simple poloidal field from Eq. (28), Parker assumes $\Gamma B = \text{const}$ inside a sphere of radius $R \leq a$, where a is the radius of the earth's core, and $\Gamma B = 0$ for $R < r < a$. This permits a rigorous solution of Eq. (28) by the conventional method of development into normal modes comprising spherical harmonics and Bessel functions. For reasons of symmetry only dipole, octupole, and higher terms of odd order will be present. The result of the calculation is that for $R = a$ the ratio of the rms values of octupole and dipole fields at the surface of the core is near 0.15 (corresponding to about 0.045 at the surface of the earth). If R becomes even slightly smaller than a, this ratio decreases very rapidly; there is hence no serious difficulty of explaining by this model the preponderance of the dipole in the earth's poloidal field. We note finally that the dynamos here described are not stationary generators, strictly speaking; they are only *stationary in the mean* since the individual eddies must appear and then die out after having twisted the toroidal field lines by a suitable amount.

A different approach to the terrestrial dynamo problem was suggested by Bullard,[49] elaborated by Takeuchi and Shimazu[51] and most extensively by Bullard and Gellman.[52] The approach is kinematical; that is, a definite velocity field is assumed, but not justified by dynamical arguments. If we set $\partial \mathbf{B}/\partial t = 0$ and take $\mathbf{v}$ as given, our basic Eq. (5) may be considered as an eigenvalue problem in **B** with suitable boundary conditions at the surface of the conducting sphere. The method used for integrating Eq. (5) is that of expressing **B** as well as **v** in terms of a series of orthogonal vector modes of the sphere.[21,47] Designating any toroidal mode by a vector **T**, any poloidal mode by **S**, these modes are defined as follows: Let $Y_n^m = P_n^m(\cos\vartheta)\cos m\varphi$

[51] H. Takeuchi and Y. Shimazu, J. Phys. Earth **1**, 1, 57 (1952); J. Geophys. Research **58**, 47 (1953).
[52] E. Bullard and H. Gellman, Trans. Roy. Soc. (London) **247**, 213 (1954).

(or else $\sin m\varphi$) be a spherical surface harmonic. The component of a toroidal mode are

$$(\mathbf{T}_n{}^m)_r=0,\quad (\mathbf{T}_n{}^m)_\vartheta=\frac{T_n{}^m(r)}{r\sin\vartheta}\frac{\partial Y_n{}^m}{\partial\varphi},$$

$$(\mathbf{T}_n{}^m)_\varphi=-\frac{T_n{}^m(r)}{r}\frac{\partial Y_n{}^m}{\partial\vartheta},\quad (30)$$

and the components of the poloidal modes are

$$(\mathbf{S}_n{}^m)_r=\frac{n(n+1)}{r^2}S_n{}^m(r)\,Y_n{}^m,$$

$$(\mathbf{S}_n{}^m)_\vartheta=\frac{1}{r}\frac{\partial S_n{}^m(r)}{\partial r}\frac{\partial Y_n{}^m}{\partial\vartheta},\quad (31)$$

$$(\mathbf{S}_n{}^m)_\varphi=\frac{1}{r\sin\vartheta}\frac{\partial S_n{}^m(r)}{\partial r}\frac{\partial Y_n{}^m}{\partial_\varphi},$$

where $T_n{}^m(r)$ and $S_n{}^m(r)$ are as yet undetermined functions of r.

A fluid motion must now be assumed which should be as simple as possible but must be such as to lead to a feedback process. Bullard found that the simplest such fluid motion consists of a linear combination of two normal modes, one of them being of the type $\mathbf{T}_1{}^0$, representing a nonuniform rotation about the earth's axis; the radial function of this was conveniently taken as a simple algebraic function $T(r)=r^3(1-r)^2$, which rises to a maximum and vanishes at the surface of the sphere. The second velocity mode must clearly be one that lacks rotational symmetry, since otherwise no dynamo could result. Bullard finds that a suitable choice is $\mathbf{S}_2{}^2$, where again he assumes a convenient algebraic expression for the radial dependence. Physically speaking, $\mathbf{S}_2{}^2$ represents a system of four radial streams, alternately ascending and descending, centered about four equidistant points on the equator. This mode does not in itself produce feedback from the toroidal into the poloidal field. The necessary asymmetrical twist of the field lines is brought about by the combination of this mode with the nonuniform circulation $\mathbf{T}_1$. The total fluid motion is then $\mathbf{T}_1+\epsilon\mathbf{S}_2{}^2$, where ϵ is an adjustable parameter.

Next, the magnetic field is developed into a series of normal modes of the above type. If this development is substituted into the differential equation (5), assuming stationarity, we obtain an infinite system of coupled differential equations for the radial functions $S_n{}^m(r)$ and $T_n{}^m(r)$. In order to solve this system approximately one breaks off the spherical harmonic series after a finite number of terms; this leaves one with a finite system of differential equations. The aforementioned authors solved this system numerically—Bullard and Gellman[52] in particular—by using the large electronic computing facilities of the British National Physical Laboratory. The lowest magnetic mode appearing in the solution is of course the poloidal dipole $\mathbf{S}_1$. There are three magnetic modes with $n=2$, one of them is the toroidal field of rotational symmetry about the earth's axis. (The lowest harmonic component of the toroidal field is a quadrupole as may be seen by inspection of Fig. 3: the field has opposite sign in the two hemispheres.) There are three magnetic modes with $n=3$ and five modes with $n=4$. We cannot describe here the very elaborate calculations carried out in order to determine the relative magnitudes and the shape of the radial eigenfunctions in this scheme. There is one quite serious drawback to this particular approach to a dynamo theory: the solution of the induction equation (5) for the stationary case by the method outlined would prove the existence of stationary dynamos *only* if it was ascertained that the spherical-harmonic series, which formally solves the differential equations, converges. The numerical evidence is that convergence if any is very slow; efforts to prove convergence have so far failed.

We now return to the more physically motivated feedback process enbodied in the dynamo equations (28), (29). Parker[50] showed that these equations possess wave-type solutions, the *dynamo waves*. Consider a spherical shell thin enough so that we may neglect curvature and

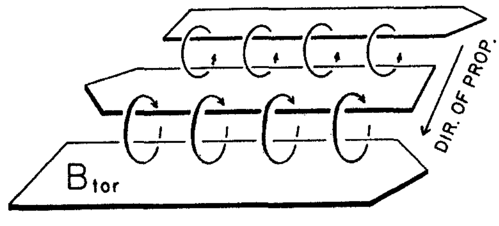

FIG. 7. Schematic of dynamo waves.

approximate it by a slab bounded by two parallel planes. Introduce a Cartesian system of coordinates whose x-axis points east, the y-axis north, and the z-axis upwards, normal to the bounding planes. Let now

$$B=B_0e^{i\omega t-iky},\quad A=A_0e^{i\omega t-iky}. \tag{32}$$

In introducing these into Eqs. (28) and (29) we shall assume $\Gamma=$const and $\nabla v\times\nabla A=\gamma A_0$, where γ is another constant. The last equality corresponds, say, to a uniform velocity shear, $\partial v_x/\partial z=$const., combined with a linear variation of A_0, say $\partial A_{0x}/\partial y=$const., which latter represents a poloidal field in the z-direction. On substitution into Eqs. (28) and (29) we obtain the characteristic equation (condition that the determinant of the system vanish) in the usual way, giving

$$i\omega+\nu_mk^2=(\Gamma\gamma)^{\frac{1}{2}}.$$

There are no solutions such that ω and k are both real. Thus the waves are exponentially increasing, or exponentially damped, with respect to both t and y; if we let $\omega=\omega_1+i\omega_2$ we find

$$k=\nu_m^{-\frac{1}{2}}[(\Gamma\gamma)^{\frac{1}{2}}+\omega_2-i\omega_1]^{\frac{1}{2}} \tag{33}$$

representing a wave exponentially increasing in amplitude while it travels to the south. Figure 7, while not corresponding faithfully to the wave solution (32), (33), exhibits the physical essentials of the dynamo waves in terms of alternating strands of toroidal fields together with alternating sets of poloidal feedback loops. In the dynamo waves the toroidal and poloidal field have a phase shift of 90° relative to each other. Parker has pointed out that such waves are rather closely akin to what we observe in the solar convective zone during a sunspot cycle where the magnetic field migrates systematically from higher to lower heliographic latitudes. Thus the dynamo waves seem at least the beginning of an ulterior theory of the sunspot cycle; we shall revert to them when we discuss the observations on solar magnetic fields in Part II.

Practical Aids for Physics Teachers

My Most Successful Experiment in Teaching Physics

All old teachers have "tricks of the trade" which seem to make for success. In the hope of spurring others to recount their experiences, I shall describe my own most successful experiment in teaching physics.

About a month before the end of the academic year the students in my course in general physics would be briefed on the "Information Please" climax of the course to be held at the last lecture period. They were asked to hand in, a week before the end of the course, *five written questions on the physics of daily life*. These should not be mathematical, "catch questions," nor impossibly difficult (e.g., "explain gravitation"), and not directly answered in their text. The ideal question would be one about which the student had thought and in whose answer he was really interested. It is obvious that the search for good questions required reviewing and thinking over the entire course and might well be the most worthwhile exercise of the entire year. A good proportion of the questions submitted over the years were interesting—some of them exceedingly interesting.

These 500 or more questions were read—preferably by the assistants in the course—and the best 25 placed in a sealed envelope.

On coming to the last lecture the students would find the stage all set, with a Master of Ceremonies, a Registrar, and four comfortable chairs for the "Guest Artists." These were filled by the four best students of the year as their names were announced.

The Master of Ceremonies would then open the sealed envelope and read the first question: John Smith wants to know "why are clouds white and the sky blue?" The Registrar at once noted that John Smith's question was one of those accepted, and this fact gave him an additional point (or similar small credit) on his semester's grade.

The question was first passed to our "guest artists," and if they were unable to answer it I would take a try at it myself. In the, possibly, one question in four, which none of us could answer satisfactorily in my opinion, John Smith *got an additional point*—amid echoing cheers! This put a premium on ingenious and carefully thought-out questions.

The interest of the students was always greatly aroused. In case this was put on a few days before the end of the course they would talk of nothing else in the remaining class sessions, and these practically had to be devoted to rehashing the whole matter. In view of the remarkable and continuing interest of the public in radio and T.V. quiz programs it would seem quite natural to take a leaf from the books of the program arrangers, and this is, of course, just what has been done here.

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