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Hydromagnetic Dynamo Theory*

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1. INTRODUCTORY SURVEY

We use the term hydromagnetism synonymously with magnetohydrodynamics which is preferred by some authors. We think that hydromagnetism recommends itself by its brevity; but above all we hope that a clearcut terminology will soon be established by usage, whether it be the one used here or another.

This article is more specific than a simple review of the field of hydromagnetism (for which see Elsasser, 1955 and 1956).‡ Sections 2 to 5 do give a fairly comprehensive definition of hydromagnetism in terms of the approximations used and the basic equations of field-motion that follow from these approximations. The interaction of electromagnetic fields with electrically conducting fluids can, in principle, give rise to a boundless variety of problems of mathematical physics. In practically all astrophysical and geophysical problems one can, in an excellent approximation, neglect the displacement current in Maxwell's equations. One can, furthermore, in a very good approximation, neglect all relativistic terms of quadratic and higher order in v/c where v designates the velocity of the fluid. What remains in this approximation is a combination of the electromagnetic field equations with the Euler (or Stokes) equations of fluid motion with suitable coupling terms between motion and field. This system of equations will be designated as the hydromagnetic equations.

The problem on which we report here is that of using these equations to study the mechanism whereby the most conspicuous cosmic magnetic fields, those of the earth, of sunspots and the sun, and of magnetic stars are generated and maintained. This is only a segment of the broader dynamics of hydromagnetic fields, but perhaps the most intriguing of its aspects. A system which can maintain magnetic fields (either stationary fields or at least average fields) owing to motions in electrically conducting fluids, will be designated as a *hydromagnetic dynamo.*

The most ancient and the most important of these problems is of course that of the magnetic field of the earth. Some of our mathematics will be specifically adapted to this problem. The geomagnetic field is the result of convective motions in the earth's fluid metallic core. The author has elsewhere (1950) reviewed the geophysical setting and the extensive array of observational data substantiating the model of the geophysical dynamo. The investigation of geomagnetism has for a very long time suffered from a fatal weakness, namely, the isolation and apparent uniqueness of the phenomenon. This deadlock was broken in 1908 when Hale discovered the existence of sunspot magnetic fields of the order of several thousand gauss. A great deal is now known about sunspot magnetism (Kuiper, 1953) and recently Babcock (1955) has described a systematic investigation of weak solar fields, of the order of a few gauss, with a new instrument. Only in recent years has it become known through Babcock's work (Babcock and Cowling, 1953) that numerous stars have intense magnetic fields which amount to several thousand gauss on the average over the star's surface. Most, if not all, of these stellar fields are time dependent, they are approximately periodic, though far from sinusoidal. The theory of these phenomena is still almost nonexistent. In the present review we are concerned mostly with the geomagnetic dynamo, with an occasional digression toward the solar dynamo. Stellar dynamos will not be treated as such, but there is good evidence to the effect that the principles which underlie the terrestrial and solar dynamos can be applied to the stars as well.

Hydromagnetic theory is probably closer to fluid dynamics than to any other branch of theoretical physics. The equations of hydrodynamics are quadratic in the fluid velocity $\mathbf{v}$, and the hydromagnetic equations are similarly of second order in the pair of field variables $\mathbf{v}$ and $\mathbf{B}$ (which enter the equations in a comparable manner as we shall see). Clearly, in the dynamo problem, that is in the problem of generating and maintaining magnetic fields which draw their energy from the

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† Some readers might wish to become acquainted with the dynamo theory, including its fundamental formulas, but without the desire to enter into too many detailed mathematical derivations. While we have avoided the awkward device of appendices we have tried to keep the presentation so that such a reader should be able to pass fairly rapidly over the more intricate formalism without losing the continuity of the argument. This applies particularly to Secs. 4 to 8.

‡ See bibliography at end of article.

mechanical energy of the fluid, the nonlinear character of the equations is altogether essential. One could no more describe a dynamo by a set of linearized equations than one could analyze the self-excited oscillations of a radio transmitter in terms of linear mechanics.

The dynamics of nonlinear systems is still in its infancy. As any glance at a book on nonlinear mechanics will show, the analysis is essentially confined to systems of one degree of freedom, with an occasional sally into the theory of coupled circuits, systems of two degrees of freedom. The hydromagnetic equations on the other hand represent the nonlinear dynamics of a continuum, a system with an infinity of degrees of freedom. Under the circumstances the dynamo theory can be no more than an extremely crude approximation to an integration of the hydromagnetic equations. The point of view taken here is that the existence of hydromagnetic dynamos is based in the first place upon *empirical* arguments pertaining to astrophysics and geophysics. Starting from this idea one can try to disentangle the main features of the observed dynamos, using as many dynamical, formal arguments as is possible at each step. The result, as we show below, is a reasonably clearcut scheme, hypothetical it must be admitted, but plausible in view of its close correspondence to experience.

The existence of dynamos has not been proved rigorously in the sense of having been derived from the hydromagnetic equations without recourse to data of experience. Progress in this more abstract direction has been made by Bullard and, independently, by Takeuchi and Shimazu (1952 and 1953). Their approach has been extensively presented by Bullard and Gellman (1954) so that we can be brief about it. Essentially, one starts with a given type of stationary fluid motion which one has good reason to consider as conducive to dynamo action. The assumption of fixed flow does away with the problems of mechanics and leaves one with the electromagnetic (induction) problem alone. One then seeks stationary eigenvalue solutions of the induction equation, representing steady-state dynamos. The major difficulty lies in the proof of convergence of the series which formally solve the induction equation. Convergence, if any, appears to be very slow and has not so far been established.

There is one rather fundamental difference between these approaches to the dynamo problem and the methods presented here. The authors quoted try to establish the existence of stationary dynamos, whereas in the models described below one is concerned with the much less restrictive condition that dynamos exist which are *stationary in the mean.* To elucidate this distinction, consider turbulence, the most conspicuous of the nonlinear phenomena of fluid dynamics. Clearly, a steady-state turbulent regime is stationary in the mean, and in general in the mean only. This does not create a presumption as to whether or not the mean physical effects (e.g., eddy friction) could be duplicated in a system of rigorously stationary flow. Such might be the case, but the problem is complicated beyond the requirements of the physical data by the added postulate of rigorous stationariness. The example of turbulence is appropriate because, as we shall see, the feedback cycle of our hydromagnetic dynamo models is completed by the inductive action of a set of eddies; the theory would be greatly complicated if one was not permitted to carry out averages over this particular type of eddy motion without inquiring into rigorous stationarity.

Let us now, by way of summary, explain the qualitative conditions, three in number, requisite for the operation of the dynamo models described below. The first of these is that the system in question have *large linear dimensions.* The requirement will be expressed quantitatively later on. Rendering it in simple physical language, the requirement is that the electromagnetic free-decay time of a current in the conducting fluid be large, specifically, larger than the Fourier periods of the fluid motion. If this condition is not fulfilled the magnetic field will decay so fast that the feedback couplings required for the dynamo fail to be effective.

The other two conditions are dynamical. It may be shown that types of fluid motion which are effectively two-dimensional, that is, where the trajectory of a particle is restricted to some surface, a two-dimensional manifold, cannot give rise to dynamo action. A consequence of this requirement is that hydromagnetic dynamos have a *low degree of geometrical symmetry.* Thus if the motions have rotational symmetry about an axis, no dynamo is possible. The low degree of symmetry may be achieved by the action of a strong Coriolis force, and there are observational indications to the effect that efficient hydromagnetic dynamos are correlated with fairly rapid rotation of the celestial bodies in which they occur. Thus our second requirement is *rotation,* and we shall show how the Coriolis force enters essentially into each of the two processes which together make up the complete feedback cycle of our dynamos.

Finally, there must be a source of energy that will generate and sustain the three-dimensional motions which in turn provide dynamo action. In the earth and other celestial bodies that exhibit magnetic fields, *convection* is found to be the driving agency producing sufficiently rapid motions. There might exist other sources of motion which can take the place of convection, but it will not here be of interest to speculate about them.

Thus convection and rotation (the Coriolis force) occurring simultaneously in an electrically conducting fluid of large linear dimensions constitute the prerequisites for dynamo action. Since the theory does not yet aim at over-all mathematical rigor, the question as to how far these conditions are necessary and sufficient for a dynamo cannot be answered in a simple way. No doubt, for the particular type of dynamo analyzed here, all three conditions are necessary, but there might be hydromagnetic dynamos operating on other principles where the system need not rotate. The observational indications are, however, in favor of the model developed here. It is hardly possible to detail the sufficiency of the conditions discussed; to do this one will have to wait for a considerably more advanced insight into the detailed dynamics of such systems.

2. THE INDUCTION EQUATION

Consider a cosmic fluid which has a conductivity σ and executes motions described by the velocity field $\mathbf{v}$. This velocity is for the present assumed given; the mechanical reactions of the field upon the fluid motion will be dealt with later. For simplicity we assume $\sigma=\text{const}$ throughout the fluid.

We shall use rationalized mks units; the quantities ϵ and μ will be assumed to be constant throughout space, including the conducting fluid. We then have the Maxwell equations

$$\nabla\times\mathbf{E}=-\partial\mathbf{B}/\partial t, \quad \nabla\cdot\mathbf{E}=\eta/\epsilon, \tag{2.1}$$

$$\nabla\cdot\mathbf{B}=0, \quad \nabla\times\mathbf{B}=\mu\mathbf{J}+\mu\epsilon\partial\mathbf{E}/\partial t, \tag{2.2}$$

where η and $\mathbf{J}$ are charge and current density. We now assume the most general expression for $\mathbf{J}$ in a homogeneous isotropic medium,

$$\mathbf{J}=\sigma\mathbf{E}+\sigma\mathbf{v}\times\mathbf{B}+\eta\mathbf{v}, \tag{2.3}$$

where the terms on the right represent, respectively, the conduction current, the induction current, and the convection current.

Let braces designate the order of magnitude of a quantity, e.g., let $\{\omega\}$ represent the order of magnitude of a reciprocal time. We note first that for cosmic fluids

$$\{v/c\}=\{\beta\}\ll 1. \tag{2.4}$$

We can therefore neglect relativistic terms of quadratic or higher order in β as compared to terms of the first order. This implies the assumption that velocities (other than c) corresponding to *electromagnetic* phenomena can be assimilated into the *mechanical* velocities of the fluid. We can for instance derive a quantity of the dimension of a velocity from the first of (2.1), say

$$\{E\}=\{v_{el}B\}, \tag{2.5}$$

and if the quantity so defined were $v_{el}\gg v$ we could not assert that (2.4) holds generally. We shall show later that in the problems discussed below

$$\{v_{el}\}\leq\{v\}, \tag{2.6}$$

which of course justifies the general application of (2.4). Furthermore, from (2.5) and (2.6) we obtain readily for the ratio of the electrical to the magnetic field energies

$$\{\epsilon E^2/\mu^{-1}B^2\}=\{E^2/c^2B^2\}\leq\{\beta^2\}. \tag{2.7}$$

It follows that whenever (2.6) is valid, the electrical part of the Maxwellian stress tensor may be neglected compared to its magnetic part: The electrical component, $\eta\mathbf{E}$, of the ponderomotive force which the field exerts upon the fluid is negligible compared to the magnetic component, $\mathbf{J}\times\mathbf{B}$.

We next show that the displacement current in (2.2) and the convection current in (2.3) are negligible compared to the conduction current, $\sigma\mathbf{E}$. The ratio of displacement to conduction current will be designated by γ, where

$$\{\gamma\}=\{\omega\epsilon/\sigma\}. \tag{2.8}$$

On the right of this expression, ω stands in the first place for the electromagnetic frequencies, say ω_{el}. Now $\{v\}=\{\omega\lambda\}$, where λ is a typical length. We assume that $\{\lambda\}$ is the same for the electromagnetic and for the mechanical processes. From (2.6) we then have $\{\omega_{el}\}\leq\{\omega\}$. Choosing the equality sign as the most unfavorable case we obtain (2.8) where ω now refers to the mechanical motions. It is readily seen that γ is exceedingly small: Let, for instance, $\sigma=10^5$, one hundredth of the conductivity of ordinary iron, which is a very low estimate for the conductivity of the earth's core (Elsasser, 1950). If we let $\omega=10^{-2}$ corresponding to periods of the order of minutes, which is certainly too large for most motions in cosmic fluids, we find $\gamma=10^{-18}$. For ionized cosmic gases the conductivity may be estimated from the kinetic formula (Kuiper, 1953, p. 537)

$$\sigma=n_e e^2/m_e\nu,$$

(in emu if e is in emu) where n_e and m_e are the number density and mass of the electrons and ν is the collision frequency. Now ν/n_e is independent of pressure; thus σ does not become small for rarefied gases. The conditions for σ to be small are low temperatures and small n_e, that is, low degree of ionization. It is easy to show that under all conditions which can reasonably be assumed to prevail in cosmic gases we have $\gamma\ll 1$.

Next, we compare the convection current to the conduction current. We see from (2.1) that $\{\eta\}=\{\epsilon E/\lambda\}$. This gives for the ratio of the two currents

$$\{\eta v/\sigma E\}=\{\epsilon v/\sigma\lambda\}=\{\gamma\}. \tag{2.9}$$

Thus displacement current and convection current are both negligible and we can write, from (2.2) and (2.3),

$$\nabla\times\mathbf{B}=\mu\mathbf{J}=\mu\sigma\mathbf{E}+\mu\sigma\mathbf{v}\times\mathbf{B}. \tag{2.10}$$

Next, we may eliminate $\mathbf{E}$ and obtain a differential equation in $\mathbf{B}$ only. Taking the *curl* of (2.10) and using (2.1) one obtains

$$\mu\sigma\partial\mathbf{B}/\partial t=\mu\sigma\nabla\times(\mathbf{v}\times\mathbf{B})-\nabla\times\nabla\times\mathbf{B}. \tag{2.11}$$

On account of $\nabla\cdot\mathbf{B}=0$ we can write this *induction equation* as

$$\partial\mathbf{B}/\partial t=\nabla\times(\mathbf{v}\times\mathbf{B})+\nu_m\nabla^2\mathbf{B}, \tag{2.12}$$

where the quantity

$$\nu_m=1/\mu\sigma, \tag{2.13}$$

will be designated as the *magnetic viscosity*. The term "viscosity" is used here by way of a formal analogy with mechanics and its meaning is apparent from (2.13). Some authors speak of magnetic viscosity in an entirely different sense: they refer to the mechanical stresses which a magnetic field exerts on the fluid; as is well known these are such that they tend to "straighten out" the lines of force and they therefore counteract eddy formation and suppress turbulence. The literature on these effects has been reviewed by us elsewhere (1955 and 1956). Here we shall use the term magnetic viscosity only in the sense defined by (2.13).

The quantity (2.8) is familiar to the student of metal optics. It is well known that if an electromagnetic wave penetrates into a metallic conductor, the displacement current is negligible on the inside. In hydromagnetic phenomena the essential processes take place in the *interior* of conductors and the displacement current is consistently negligible. As a consequence of this the electromagnetic processes are essentially *aperiodic*; this is apparent from the fact that only $\partial/\partial t$ and not the second derivative appears in (2.12): For $\mathbf{v}=0$ this equation reduces to the diffusion equation (2.16).

The reader might comment here that radio noise, which is a commonplace astrophysical fact, can certainly be idealized in terms of periodic processes, and that therefore the restriction to aperiodic electromagnetic phenomena does not at once appear justified for cosmic gases. Radio noise is in part due to free-free transitions in the atomic hydrogen spectrum, but such noise is also often attributed to plasma oscillations in ionized gas. Now it is possible to approach $\gamma\sim 1$, so as to make oscillations possible, provided σ is low enough and we let, say $\omega\sim 10^{11}$, corresponding to microwaves. But the corresponding linear dimensions cannot be much larger than the wavelengths involved, that is of the order of centimeters. Compare this to the linear dimensions of typical hydromagnetic phenomena: Observation shows magnetic fields of very large dimensions in the earth, sun, and many stars. It is likely that these fields have a fine structure, primarily due to eddy formation, but simple calculations show that, smaller eddies being damped out more rapidly than larger ones, the eddy spectrum effectively terminates at a scale length of many kilometers. Thus in the spectrum of lengths (and also of frequencies) there appears a broad gap between the hydromagnetic phenomena on the one hand and oscillatory, that is, radiation-producing electromagnetic processes of various types. This fact justifies our using the hydromagnetic approximation to represent *a distinct class* of observable phenomena, limited to large linear dimensions.

Let us return to the induction equation (2.12) and compare the relative magnitude of its three terms. The ratio of the first to the second term on the right is of order

$$\lambda v/\nu_m = R_m. \tag{2.14}$$

The dimensionless quantity R_m will be designated as the *magnetic Reynolds number*. The analogy to the conventional hydrodynamic Reynolds number is apparent: The latter is defined as $R=\lambda v/\nu$, where ν is the kinematic viscosity.

In large linear dimensions R_m tends to be numerically large. Taking again values for the earth's core, say $\sigma=10^5$, $\lambda=3\times10^6$ m, and $v=3\times10^{-4}$ m/sec (as inferred from the geomagnetic secular variation) we obtain $R_m=100$. In astrophysical applications R_m is often, say, 10^4–10^6 or larger. We shall see later on that a hydromagnetic dynamo cannot function unless R_m is at least moderately large. To bring out this point, let us split (2.12) into two equations

$$\partial\mathbf{B}/\partial t=\nabla\times(\mathbf{v}\times\mathbf{B}), \tag{2.15}$$

and

$$\partial\mathbf{B}/\partial t=\nu_m\nabla^2\mathbf{B}. \tag{2.16}$$

Dimensionally, both these expressions are of type ωB. Since R_m is the ratio of (2.15) to (2.16) we may write

$$\{R_m\}=\{\omega/\omega_{e1}\}, \tag{2.17}$$

where, as before, ω refers to the mechanical motions, ω_{e1} is a measure of the reciprocal free-decay time of electromagnetic modes. Large R_m then indicates that the fluid can be very much deformed before an electromagnetic field existing in it has spontaneously decayed. For $R_m\ll 1$ on the other hand, no dynamo could be maintained because the field decays too fast.

We can now justify the assumption (2.6) which we used to derive some of the preceding results. By virtue of (2.17) the relation (2.6) expresses simply $R_m\geq 1$ (provided we make the additional assumption that the characteristic linear dimensions of the electromagnetic phenomena are comparable to those of the mechanical motions; the truth of this may be deduced from a study of the solutions of the induction equation.) Hence we can now replace (2.6) by the condition $R_m\geq 1$ from which the other preceding results then follow. It may be verified on substituting numbers that this condition is amply fulfilled in all electrically conducting cosmic fluids, and hence all previous arguments apply to them.

Consider now Eq. (2.10). It is readily seen that the left-hand side is of order $1/R_m$ compared to the individual terms on the right. Thus for large R_m we find a remarkable balance between the electric field and the induced field:

$$\mathbf{E}\sim -\mathbf{v}\times\mathbf{B}. \tag{2.18}$$

3. ELECTRIC FIELDS: POTENTIALS

Consider a Lorentz transformation. If we retain only first-order terms in v/c it reduces to the Galilei transformation

$$\begin{gathered}\mathbf{r}'=\mathbf{r}-\mathbf{v}_0 t,\quad t'=t,\\ \nabla'=\nabla,\quad \partial/\partial t'=\partial/\partial t+\mathbf{v}_0\cdot\nabla,\end{gathered} \tag{3.1}$$

where $\mathbf{v}_0$ is the velocity of the primed system with respect to the unprimed one. We need only consider values of $\mathbf{v}_0$ of the general order of the fluid velocity $\mathbf{v}$; this justifies the neglect of terms of order $(v/c)^2$. In the same approximation the field vectors transform as

$$\mathbf{B}'=\mathbf{B}+\mathbf{v}_0\times\mathbf{B},\quad \mathbf{B}'=\mathbf{B}-\mathbf{v}_0\times\mathbf{E}/c^2.$$

Now for $R_m \gg 1$ we can use (2.18) for an order-of-magnitude estimate, whence

$$\{v_0E/c^2\}=\{Bvv_0/c^2\}=\{B\beta^2\},$$

and the last transformation reduces to

$$\mathbf{E}'=\mathbf{E}+\mathbf{v}_0\times\mathbf{B},\quad \mathbf{B}'=\mathbf{B}. \tag{3.2}$$

The current density transforms as $\mathbf{J}'=\mathbf{J}+\eta\mathbf{v}_0$. Since

$$\{\eta v_0/\sigma E\}=\{\gamma\}\quad\text{and}\quad\{\sigma E/J\}=\{R_m\},$$

by previous results, we see that $\eta\mathbf{v}_0$ is small of order γR_m. As a rule γ is so exceedingly small that γR_m is also small; hence

$$\mathbf{J}'=\mathbf{J},\quad \eta'=\eta, \tag{3.3}$$

where the second equation follows from general principles of relativity. Furthermore, the conductivity, σ, can be shown to be a relativistic invariant (von Laue, 1921). We see from these formulas that the induction equation (2.12) which contains only the magnetic field vector is invariant under a Lorentz transformation, and so is the ponderomotive force, $\mathbf{J}\times\mathbf{B}$. The only thing that changes is the electric field strength as reckoned to correspond to the time-dependent magnetic fields.

We next inquire into the magnitude of the space charges, η, which can occur in our systems. The equation of continuity for the charge gives, on using (2.3) and the second of (2.1),

$$-\frac{\partial\eta}{\partial t}=\nabla\cdot\mathbf{J}=\frac{\sigma}{\epsilon}\eta+\sigma\nabla\cdot(\mathbf{v}\times\mathbf{B})+\nabla\cdot(\eta\mathbf{v}).$$

The last term is small and may be neglected, leaving us with the differential equation

$$\dot{\eta}+(\sigma/\epsilon)\eta=\sigma f(t), \tag{3.4}$$

where

$$-f(t)=\nabla\cdot(\mathbf{v}\times\mathbf{B})=\mathbf{v}\cdot\nabla\times\mathbf{B}-\mathbf{B}\cdot\nabla\times\mathbf{v}, \tag{3.5}$$

(and this divergence does not in general vanish). The integral of (3.4) is

$$\eta(t)=\exp(-\sigma t/\epsilon)\int_0^t dt f(t)\exp(\sigma t/\epsilon).$$

Now if ω is again characteristic of the spectrum of the fluid motion (and of the corresponding slowly changing hydromagnetic fields) we have, by (2.4),

$$\{\sigma/\epsilon\}=\{\omega/\gamma\}\gg\{\omega\}. \tag{3.6}$$

On letting $f(t)=f(0)+tf'(0)$, the solution becomes, to within terms of the order of γ,

$$\eta(t)=\epsilon f(0)+\epsilon tf'(t).$$

Its meaning is as follows: The space charge is

$$\eta=-\epsilon\nabla\cdot(\mathbf{v}\times\mathbf{B}), \tag{3.7}$$

and, as the expression on the right changes with time, η follows this change quasistatically, to within terms of the order of γ, that is synchronously, for all practical purposes. All space charges in excess of the quasistatic equilibrium value (3.7) disappear with great rapidity, the reciprocal time being given by (3.6). The last result is well known from conventional electrodynamics for charges in the interior of a conductor (e.g., Stratton, 1941).

Thus in a hydromagnetic medium we have in general $\nabla\cdot\mathbf{E}\neq0$ and $\nabla\times\mathbf{E}\neq0$. But the effects of electric fields are small. It is true that the corresponding voltages, λE, can become very large when λ is large, e.g., for galactic dimensions. This might have implications for the study of cosmic-ray accelerations, but is not of much concern in the dynamics of hydromagnetic fluids. Since one can always, by a Lorentz transformation, make $\mathbf{v}=0$, locally, in a sufficiently small region of the fluid, the question has been raised by some authors as to whether the remaining $\mathbf{E}$ can give rise to discharge-like phenomena in an ionized gas. We can estimate the order of $\mathbf{E}$ from (2.18). As an example consider a typical sunspot with a field, $B=0.3$ mks ($=3000$ gauss) and assume $v=3$ km/sec. This gives $E=1$ volt/m. Such a field prevailing in the photosphere where the density is of the order of 10^{-7} to 10^{-8} g/cm^3 can hardly produce breakdowns. In the region of the sun or stars where the hydromagnetic dynamo effects are most pronounced the density is far larger, and it is unlikely that the electrical component of the hydromagnetic fields has any effect on the condition of the ionized gas such as the generation of a discharge. In the earth's core, the associated potentials amount to small fractions of a volt. We have already seen that, by virtue of (2.7), the electrical field exerts no appreciable mechanical forces, only the magnetic field does. Thus it is entirely legitimate for all *dynamical* questions to disregard electrostatic effects. In particular, the irrotational part of $\mathbf{E}$ may be ignored altogether, since by the Faraday relation (2.1) it has no influence on $\mathbf{B}$.

Consider now the ordinary electromagnetic potentials. Without approximations, we first assume in the usual way,

$$\mathbf{B}=\nabla\times\mathbf{A},\quad \mathbf{E}=-\partial\mathbf{A}/\partial t-\nabla\phi, \tag{3.8}$$

with the subsidiary condition

$$\nabla\cdot\mathbf{A}=0, \tag{3.9}$$

fulfilling identically the first equations (2.1) and (2.2). From the second of (2.1) we have

$$\eta/\epsilon=-\nabla^2\phi.$$

From (2.10) or from the second of (2.2) we now get

$$\partial\mathbf{A}/\partial t=\mathbf{v}\times(\nabla\times\mathbf{A})-\nabla\phi+\nu_m\nabla^2\mathbf{A}.$$

On taking the divergence there follows

$$-\eta/\epsilon=\nabla^2\phi=\nabla\cdot[\mathbf{v}\times(\nabla\times\mathbf{A})]. \tag{3.10}$$

Now according to the arguments just given we shall disregard (3.10) which is just the "longitudinal" (irrotational) component of $\mathbf{E}$. Hence we let

$$\partial\mathbf{A}/\partial t=[\mathbf{v}\times(\nabla\times\mathbf{A})]_{\mathrm{tr}}+\nu_m\nabla^2\mathbf{A}, \tag{3.11}$$

where the symbol $[\;]_{\mathrm{tr}}$ indicates that only the "transverse" (divergence-free) part of this expression is to be taken. The *curl* of (3.11) gives (2.12). Any vector field defined in all space can be uniquely decomposed into a longitudinal and a transverse part (Sommerfeld, 1950). For a finite, bounded volume this analysis becomes more difficult, but a generalization of this procedure can be established (see for instance, Parker, 1955a). An important practical case where the decomposition is automatic is the one of a set of orthogonal vector modes to which we shall revert later.

An alternate, sometimes more convenient form of the vector potential is obtained by dropping the divergence condition (3.9). We then use the available freedom to omit the scalar potential, setting $\phi=0$. This is perfectly satisfactory since we are not concerned with questions of invariance, and since electrostatic effects are negligible. Thus

$$\mathbf{B}=\nabla\times\mathbf{A},\quad \mathbf{E}=-\partial\mathbf{A}/\partial t, \tag{3.12}$$

and the induction equation becomes

$$\partial\mathbf{A}/\partial t=\mathbf{v}\times(\nabla\times\mathbf{A})+\nu_m\nabla^2\mathbf{A}. \tag{3.13}$$

The assumption (3.12) satisfies all conditions met with in hydromagnetism.

Since we are on the subject of electrical potentials we shall also deal with the effects of an *impressed* electromotive force in a hydromagnetic system. Such potentials have been postulated on various grounds, e.g., thermoelectric couples or pressure couples acting in the interior of the earth; potentials due to the diffusive separation of oppositely charged carriers have been assumed to arise in ionized cosmic gases. We shall show that the effects of such impressed emf's are generally negligible in cosmic fluids, the only exception being the quasi-potentials (Schlüter, 1950 and 1951) which are equivalent to the anisotropy of conduction (Cowling, 1932) produced by a magnetic field in an ionized, sufficiently rarefied gas.

Let V_0 be the impressed potential, giving rise to an impressed electric field $\mathbf{E}_0$. This will lead to a term $\mu\sigma\mathbf{E}_0$ on the right of (2.10). We assume $R_m\geq 1$ as usual, and for crude estimates we may omit the dissipative terms from the induction equation. With the extra term (2.12) becomes now

$$\frac{\partial\mathbf{B}}{\partial t}=\nabla\times(\mathbf{v}\times\mathbf{B})+\nabla\times\mathbf{E}_0.$$

The ratio of the first to the second term on the right is of order

$$\{vB/E_0\}=\{\lambda\omega B/E_0\}=\{\lambda^2\omega B/V_0\}.$$

Taking $V_0=1$ volt seems a fair enough estimate of order of magnitude. Furthermore, letting $\omega B\sim 1$ would be an overestimate for conditions prevailing in cosmic fluids. Even so, the effects of motional hydromagnetic induction will exceed those of an impressed emf by a factor λ^2 (where λ is in meters); hence the electric currents due to such an emf may be neglected in systems of large linear dimensions, unless it could be shown that the induction effects average out to zero, which is certainly not the case in a dynamo.

4. THE HYDROMAGNETIC EQUATIONS: WAVES

The induction equation (2.12) constitutes only half of the hydromagnetic equations. The other half is represented by the equation of motion of the fluid in which there appears the ponderomotive force exerted by the magnetic field. By well-known principles of electrodynamics, this force is, per unit volume,

$$\mathbf{F}=\mathbf{J}\times\mathbf{B}=\mu^{-1}(\nabla\times\mathbf{B})\times\mathbf{B}. \tag{4.1}$$

Putting this into the Navier-Stokes equations of hydrodynamics we have

$$\partial\mathbf{v}/\partial t+(\mathbf{v}\cdot\nabla)\mathbf{v}=-\nabla p/\rho+(\mu\rho)^{-1}(\nabla\times\mathbf{B})\times\mathbf{B}+\nu\nabla^2\mathbf{v}, \tag{4.2}$$

where ν is the conventional specific viscosity. In addition there is an equation of continuity for the fluid which we do not write down. We have omitted a term representing gravitational forces; we have also for now omitted a Coriolis term which is important since, as pointed out in the introduction, our hydromagnetic dynamos are essentially rotating systems. (The effect of compressibility on frictional dissipation has also been neglected.)

The combination of (2.12) and (4.2) constitutes the full hydromagnetic equations. We shall now give an application of these equations which, while it will not be used extensively later on, is very instructive. We shall assume in the present section that the fluid is *incompressible*. We note the well-known vector identity

$$(\nabla\times\mathbf{B})\times\mathbf{B}=(\mathbf{B}\cdot\nabla)\mathbf{B}-\tfrac{1}{2}\nabla(\mathbf{B}^2), \tag{4.3}$$

which we substitute in (4.2). We introduce the following symbols and abbreviations:

$$\mathbf{P}=\mathbf{v}+(\mu\rho)^{-\frac{1}{2}}\mathbf{B},\quad \mathbf{Q}=\mathbf{v}-(\mu\rho)^{-\frac{1}{2}}\mathbf{B}, \tag{4.4}$$

$$\begin{aligned}\psi&=p/\rho+(2\mu\rho)^{-1}\mathbf{B}^2,\\ &=p/\rho+\tfrac{1}{8}(\mathbf{P}-\mathbf{Q})^2,\end{aligned} \tag{4.5}$$

and furthermore,

$$2\nu_1=\nu+\nu_m,\quad 2\nu_2=\nu-\nu_m. \tag{4.6}$$

If we now first add and then subtract (2.12) and (4.2), we obtain after some simple rearrangements the following set of equations (Elsasser, 1950a and Lundquist, 1952)

$$\begin{aligned}\partial\mathbf{P}/\partial t+(\mathbf{Q}\cdot\nabla)\mathbf{P}&=-\nabla\psi+\nabla^2(\nu_1\mathbf{P}+\nu_2\mathbf{Q}),\\ \partial\mathbf{Q}/\partial t+(\mathbf{P}\cdot\nabla)\mathbf{Q}&=-\nabla\psi+\nabla^2(\nu_1\mathbf{Q}+\nu_2\mathbf{P}).\end{aligned}\tag{4.7}$$

These equations are remarkable for their symmetry, though they are perhaps not quite as useful as their aspect might lead one to believe. It has rightly been remarked that the combination (4.4) is somewhat artificial since $\mathbf{v}$ is a polar and $\mathbf{B}$ an axial vector. Also, efforts to extend the symmetrization to the case of compressible fluids have met with failure.

Since the mechanical and magnetic viscosities enter here symmetrically one might ask how much of the dissipation in a cosmic fluid is due to mechanical friction and how much to Joule's heat. (It is notable, by the way, that if electromagnetic dissipation preponderates, ν_2 becomes negative, a fact that has no analog in ordinary hydrodynamics.) The ratio

$$\nu/\nu_m=R_m/R=\mu\sigma\nu,\tag{4.8}$$

may be estimated from elementary formulas of kinetic theory for an ionized gas such as hydrogen (Elsasser, 1954). Assuming a collision diameter of 10^{-8} cm one obtains, numerically,

$$\mu\sigma\nu=2\cdot10^{-7}\alpha/\rho,\tag{4.9}$$

where α is the degree of ionization and ρ is expressed in cgs unit. By a convenient coincidence (4.9) approaches unity for densities characteristic of stellar photospheres; thus in the interior of the stars electromagnetic dissipation prevails, whereas for rarefied cosmic gases dissipation is essentially by mechanical friction.

The symmetrized equations (4.7) are most useful in perturbation (wave) theory. In hydromagnetism we meet a type of transverse waves which have no counterpart in ordinary hydrodynamics, the Alfvén waves. As Alfvén (1950) remarks, these transverse hydromagnetic waves are somewhat similar to mechanical waves moving along a taut string and might be interpreted in an analogous fashion: Given a homogeneous magnetic field, it is well known that the ponderomotive forces are equivalent to contractive stresses longitudinally and to expansive stresses transversally, relative to the field. If such a field is disturbed, the field lines being slightly bent locally, a restoring force appears that tends to bring the field lines back to parallelism. If the energy of the disturbance is small, it gives rise to Alfvén waves which travel along the field lines.

We shall ignore dissipation, setting $\nu_1=\nu_2=0$. We take the fluid to be at rest in the undisturbed state, but containing a large homogeneous field $\mathbf{B}_0$ which we assume in the x-direction. Let $\mathbf{C}$ be a vector in the x-direction, of magnitude

$$C=B_0/(\mu\rho)^{\frac{1}{2}}.\tag{4.10}$$

We then set in accordance with (4.4),

$$\mathbf{P}=\mathbf{C}+\mathbf{p},\quad \mathbf{Q}=\mathbf{C}+\mathbf{q}.\tag{4.11}$$

Inserting into (4.7), using (4.5) and omitting all terms quadratic in the small quantities we obtain

$$\begin{aligned}\frac{\partial\mathbf{p}}{\partial t}-C\frac{\partial\mathbf{p}}{\partial x}&=-\nabla(p/\rho)-\tfrac{1}{2}C\nabla(p_x-q_x),\\ \frac{\partial\mathbf{q}}{\partial t}+C\frac{\partial\mathbf{q}}{\partial x}&=-\nabla(p/\rho)-\tfrac{1}{2}C\nabla(p_x-q_x).\end{aligned}\tag{4.12}$$

We now study separately the components longitudinal and transverse, relative to $\mathbf{B}_0$. Considering the components, p_x and q_x only and adding the two equations (4.12) we obtain readily

$$\frac{\partial(p_x+q_x)}{\partial t}=2\frac{\partial v_x}{\partial t}=-2\frac{\partial p}{\rho\partial x}.$$

The second equality is a purely hydrodynamic relation which follows from the Euler equation by perturbation process. But in an incompressible fluid no longitudinal waves exist. We conclude that in the approximation considered, the same is the case in our incompressible hydromagnetic medium, and that all waves are purely transverse,

$$\mathbf{p}\cdot\mathbf{C}=\mathbf{q}\cdot\mathbf{C}=0.\tag{4.13}$$

This is intuitively plausible since a purely longitudinal displacement would not deform the lines of force of the homogeneous field and therefore does not evoke a ponderomotive reaction.

Hence we drop the last term in both of (4.12). The two lines of (4.12) are then equal to each other, but since $\mathbf{p}$ and $\mathbf{q}$ are arbitrary and mutually independent, being subject only to the transversality condition (4.13), it follows that $\nabla(p/\rho)$ must be a constant. A uniform pressure gradient is not of interest and we might equate it to zero, leaving

$$\begin{aligned}\partial\mathbf{p}/\partial t-C\partial\mathbf{p}/\partial x&=0,\\ \partial\mathbf{q}/\partial t+C\partial\mathbf{q}/\partial x&=0.\end{aligned}\tag{4.14}$$

The solutions are waves

$$\mathbf{p}=\mathbf{p}(x+Ct),\quad \mathbf{q}=\mathbf{q}(x-Ct),\tag{4.15}$$

traveling to the left and right, respectively, with a velocity given by (4.10). These waves have no dispersion. For further discussion of hydromagnetic waves see Alfvén's book (1950) or Lundquist (1952). The preceding is a crude sketch of the fact proved in detail by Parker (1955a) that any disturbance of a large homogeneous field can, in a first-order approximation, be represented as a linear superposition of Alfvén waves. Unfortunately, so far, no observational data on such waves exist.

5. CONSERVATION THEOREMS

In conventional hydrodynamics, the Helmholtz-Kelvin vorticity conservation theorem holds for a frictionless fluid. A far-reaching analogy exists between the vorticity field in an ordinary fluid and the magnetic field of hydromagnetic systems (Elsasser, 1946 and 1947): For simplicity let us confine ourselves to a fluid not necessarily incompressible, but in which dp/ρ is assumed a complete differential, whence

$$\nabla\times(\nabla p/\rho)=0. \tag{5.1}$$

We define the vorticity as $\mathbf{w}=\nabla\times\mathbf{v}$ and write from the identity (4.3)

$$(\mathbf{v}\cdot\nabla)\mathbf{v}=\mathbf{w}\times\mathbf{v}-\tfrac{1}{2}\nabla(\mathbf{v}^2). \tag{5.2}$$

If now we take the *curl* of the Navier-Stokes equation (4.2), with $\mathbf{B}=0$, we obtain, in view of (5.1),

$$\partial\mathbf{w}/\partial t=\nabla\times(\mathbf{v}\times\mathbf{w})+\nu\nabla^2\mathbf{w}. \tag{5.3}$$

This equation is identical in form with the induction equation (2.12), the vorticity playing the same role here as the magnetic field, **B**, there. All ensuing classical hydrodynamic theory which does not again make use of the fact that **w** is the *curl* of **v** can thus at once be applied to (2.12). One such deduction is the vorticity conservation theorem (e.g., Sommerfeld, 1950): We see that there exists a conservation theorem for the magnetic flux in any hydromagnetic system if $\nu_m=0$. This theorem was early discovered by T. G. Cowling (see for instance Cowling, 1953). We shall now give an explicit proof. No restrictions about compressibility need be made.

Integrating the induction equation in the form (2.11) over a surface bounded by a contour C, fixed in space, and converting by Stokes' formula we obtain

$$(\partial/\partial t)\int B_n dS=\int(\mathbf{v}\times\mathbf{B})\cdot d\mathbf{C}-\nu_m\int(\nabla\times\mathbf{B})\cdot d\mathbf{C}.$$

If the first integrand on the right is written $\mathbf{B}\cdot(d\mathbf{C}\times\mathbf{v})$, the integral can be given a simple geometrical meaning: it becomes $-\int B_n dS$ extending over the strip swept out by the contour C during dt as this contour partakes in the motion of the fluid. Since $\int B_n dS=0$ for any *closed* surface, it is readily seen that we can write the preceding relation

$$(d/dt)\int B_n dS=-\nu_m\int(\nabla\times\mathbf{B})\cdot d\mathbf{C}, \tag{5.4}$$

where the substantial derivative on the left refers as usual to motion with the fluid particles. The ratio of the left to the right member of (5.4) is of order R_m; thus for large R_m the right-hand side becomes small. In the limit of an ideal conductor, or else for very large linear dimensions, (5.4) reduces to

$$(d/dt)\int B_n dS=0. \tag{5.5}$$

Hence *the magnetic flux is carried bodily with the fluid* or, as it is often expressed, the lines of force are "frozen" into the fluid. An alternate expression of the conservation theorem (5.5) is in terms of the vector potential,

$$(d/dt)\int\mathbf{A}\cdot d\mathbf{C}=0. \tag{5.6}$$

One sometimes finds in elementary books the statement that the field lines representing a divergence-free vector field must be closed. This is not so. Individual field lines can terminate in singular points or lines (where $\mathbf{B}=0$) or they can be "ergodic" (McDonald, 1954). To give a simple example of ergodic field lines, consider an electric line current i_1 flowing along the z-axis together with another line current i_2 flowing in a circular loop in the xy-plane centered on the origin. A magnetic field line in the neighborhood of i_2 will spiral around this circular loop, but it will not return upon itself, except for a special set of values of i_1/i_2 (which form a manifold of measure zero).

As is well known, the Helmholtz theorem holds with suitable modifications for a compressible fluid, and (5.5) also holds in this case. Sometimes it is convenient to exhibit compressibility more clearly (Truesdell, 1950). We use the induction equation in the form (2.15). Using a well-known vector identity we can write this equation

$$\partial\mathbf{B}/\partial t+(\mathbf{v}\cdot\nabla)\mathbf{B}=d\mathbf{B}/dt=(\mathbf{B}\cdot\nabla)\mathbf{v}-\mathbf{B}(\nabla\cdot\mathbf{v}), \tag{5.7}$$

which may be further simplified from the equation of continuity,

$$\nabla\cdot\mathbf{v}=\rho d(\rho^{-1})/dt,$$

giving finally

$$d(\rho^{-1}\mathbf{B})/dt=(\rho^{-1}\mathbf{B}\cdot\nabla)\mathbf{v}. \tag{5.8}$$

This is another form of the induction equation, equivalent of course to the integral theorem (5.5). We shall use it here to prove, by way of a short digression, the well-known fact that on the basis of purely electromagnetic measurements one cannot distinguish between a state of uniform rotation of, say the earth, and a state of rest. By (3.1) the transformation required for our purposes is a Galilei-type transformation with relativistic terms neglected; we have in cylindrical coordinates, s, φ, z,

$$s=s_0,\quad \varphi=\varphi_0-\Omega t_0,\quad z=z_0,\quad t=t_0, \tag{5.9}$$

where the subscript 0 refers to the nonrotating system and Ω is the angular velocity of the rotating system. It follows that $\partial/\partial\varphi=\partial/\partial\varphi_0$, and similarly for the other two spatial coordinates; hence ∇^2 is invariant under the transformation. We shall confine ourselves to the dissipationless equation (5.8). In the latter we replace $\rho^{-1}\mathbf{B}$ by **B** for the convenience of notation. Now we have

$$d\mathbf{B}/dt=(d\mathbf{B}/dt)_0-\mathbf{\Omega}\times\mathbf{B}, \tag{5.10}$$

a kinematical identity proved in textbooks on mechanics. It derives directly from the definition of the substantial derivative, and such a formula is valid for any vector field whatever. On applying it to $\mathbf{v}=d\mathbf{r}/dt$,

$$\mathbf{v}=\mathbf{v}_0-\mathbf{\Omega}\times\mathbf{r}, \tag{5.11}$$

(note that $\mathbf{r}=\mathbf{r}_0$). On substituting (5.10) and (5.11) into (5.8) and using the identity

$$(\mathbf{B}\cdot\nabla)(\mathbf{\Omega}\times\mathbf{r})=\mathbf{\Omega}\times\mathbf{B},$$

we find that the invariance follows. This proves it for $\mathbf{B}$; we shall omit the extension of the proof to $\mathbf{E}$.

Returning now to the discussion of conservation theorems, we next note that vorticity is not conserved in the presence of magnetic fields. This is fairly obvious from the existence of such technical devices as induction motors. In a later section we discuss at some length the hydromagnetic equivalent of an induction motor, and the occurrence of vorticity transfer will be apparent. We shall here omit the formalism of vorticity exchange because so far no significant simplifications have been proposed; the rate of change of vorticity is of course simply the *curl* of (4.1).

We finally come to the conservation of energy. We first consider the equation of motion (4.2), obtaining as usual the energy change per unit volume on scalar multiplication by $\rho\mathbf{v}$. The rate of work done on the fluid per unit volume and unit time is from (4.1),

$$\mathbf{v}\cdot\mathbf{F}=-\mu^{-1}(\nabla\times\mathbf{B})\cdot(\mathbf{v}\times\mathbf{B}). \tag{5.12}$$

We next show from the induction equation that the work done on the magnetic field is just the negative of (5.12). On scalar multiplication of (2.15) by $\mu^{-1}\mathbf{B}$ and transformation by a well-known vector identity one obtains

$$(2\mu)^{-1}\partial\mathbf{B}^2/\partial t=\mu^{-1}\nabla\cdot[(\mathbf{v}\times\mathbf{B})\times\mathbf{B}] +\mu^{-1}(\mathbf{v}\times\mathbf{B})\cdot\nabla\times\mathbf{B}. \tag{5.13}$$

The last term on the right is seen to be the negative of (5.12). The first term can be interpreted as follows (Skabelund, 1956). Since we have set $\nu_m=0$, the relation (2.18) is rigorously valid, and the square bracket in (5.13) is nothing but the Poynting vector

$$\boldsymbol{\pi}=\mu^{-1}(\mathbf{E}\times\mathbf{B}). \tag{5.14}$$

On integrating (4.13) over a volume fixed in space we obtain for the rate of change of the magnetic energy

$$(\partial/\partial t)\int m dV=\int \pi_n dS-\int \mathbf{v}\cdot\mathbf{F}dV, \tag{5.15}$$

where m is the space density of magnetic energy. Thus even for an ideal conductor the energy flow can be expressed in terms of a Poynting vector, even though in the hydromagnetic case this flow is due to mechanical displacements of the fluid, not to radiation. It appears that the definition of the Poynting vector as representing radiation only, sometimes found in textbooks, is too narrow in the presence of mechanical motions. In order to trace the flow of energy in the most general hydromagnetic case we must carry the dissipation terms as well as take account of the purely hydrodynamic transport of energy across a fixed boundary. The formalism is straightforward and need not be dealt with in detail.

6. TURBULENCE: EDDY STRESSES

One cannot deal with the physics of cosmic fluids without encountering at almost every turn the problems of turbulence. The mathematical theory of turbulence is still in a far from satisfactory state. Even ignoring the more elaborate theories we still must sketch briefly the implications of turbulent conditions for our dynamo models. In fluids of large dimensions not only is R_m large, but the conventional Reynolds number is large, so that the fluid motion is necessarily turbulent. Thus dynamos in which a well-ordered fluid motion is assumed can only represent a crude approximation to real systems. Consider a simple example of what is implied by the presence of turbulence. Suppose we have a magnetic field in a fluid at rest. This field is subject to dissipation and "diffusion," as described by (2.16). A stationary state, $\partial\mathbf{B}/\partial t=0$, will be possible only for a homogeneous field under proper boundary conditions. Any inhomogeneities will be smoothed out in a time inversely proportional to ν_m, that is by (2.13), proportional to the conductivity, σ. In a turbulent fluid the transport of properties characteristic of the fluid particles, such as momentum, entropy, the concentration of solutes, etc., is effected, not so much by molecular diffusion, but by eddy diffusion which is very much more rapid. The latter is characterized by a "mixing length" giving the mean distance of travel of a fluid parcel before it loses its identity. Now we have seen that the magnetic field is carried along bodily by the fluid; if therefore we think of eddy diffusion in terms of transport of macroscopic fluid parcels over a finite distance, we must assume that the magnetic field also is transported at a rate not given by (2.13), but by a much larger eddy diffusion rate. The mathematics of such a diffusion mechanism for the magnetic field have not, apparently, been worked out, but one might, in an extremely crude approximation, let the eddy diffusivity be $\nu_m'=R_m\nu_m$ which would exceed ν_m by the more, the larger the linear dimensions of the system. If one uses (2.16) to calculate the free-decay time of a magnetic field in a large cosmical conductor (see Sec. 8) one finds for instance for the sun as a whole decay times of the order of 10^{10} years (Cowling, 1945). This seems very hard to reconcile with the nature of the hydromagnetic phenomena observed in the sun and stars. There is no reason to doubt that in stellar hydromagnetism the transport rates and free-decay coefficients are increased by a tremendous factor due to eddy diffusion.

For hydromagnetic dynamo models this argument may be given a general and readily understandable form. In a dynamo such as the terrestrial one the pattern of the magnetic field lines must clearly be stationary in the mean (in the case of solar magnetic fields it must be periodic with the period of the sunspot cycle). Now the dynamo operates by shearing and twisting the field lines in such a way that energy is pumped into the field, that is, the field lines must in the average be bundled closer together. To make this process compatible with stationarity in the mean we must, as Bullard (1949 and 1954) has remarked, assume that the decay terms are of an order of magnitude comparable to the motional induction terms. Thus in a dynamo mechanism based on (2.12) the two terms on the right of this equation would have to be of comparable magnitude.

A particularly simple case of hydromagnetic turbulence arises when there is a *weak* magnetic field in a turbulent fluid. This has been discussed by Batchelor (1950) and also by Biermann and Schlüter (1950 and 1951). In this case the ponderomotive force which the field exerts upon the fluid is small and the corresponding term may be omitted from the equations of motion (4.2). Batchelor proceeds from the complete formal analogy of the vorticity equation (5.3) with the induction equation (2.12). Wind-tunnel experiments show that in a turbulent regime the vorticity lines are being drawn out in the statistical average. There is then good reason to think that the same happens to the magnetic field lines, at least when the field is small. In formulas, one may obtain from (2.12) the relation

$$\tfrac{1}{2}(d/dt)[|\mathbf{B}|^2]_{\mathrm{Av}}=[|\mathbf{B}|^2(\partial\mathbf{v}/\partial x_B)_B]_{\mathrm{Av}} - \nu_m[\Sigma|\nabla\mathbf{B}|^2]_{\mathrm{Av}}, \quad (6.1)$$

where the subscript B indicates the component in the direction of the field, and Σ indicates a summation of the three Cartesian components. The last term is essentially negative, as one should expect. Now an entirely analogous relationship has long been known to hold, from (5.3), for the square average of the vorticity $\mathbf{w}$, and in this case it has been shown experimentally that the first term on the right is always positive. Assuming the same for (6.1) Batchelor concludes that for turbulent motions on a scale where the last, frictional term is small, the rms value of the magnetic field strength increases with time.

Biermann and Schlüter have gone one step farther; they point out that the induction equation (2.15) is a generalization of the scalar equation $\partial B/\partial t=FB$ where F is some linear operator. The integral of the last equation is $B=B(0)\exp(Ft)$; they infer from this that a field amplified by random motions should also in the mean increase exponentially. This conclusion seems less secure than the more general one that the mean field will increase with time.

It appears from these arguments that there is no serious difficulty with regard to the *initial apparition* of magnetic fields in a conducting fluid of sufficiently large dimensions. Any minute stray field will, in the average, be amplified. Such a process should not be confounded with dynamo action as the term is understood here. There is a great deal of observational evidence to the effect that cosmic magnetic fields, while they contain random components (as they must if the fluid motion is turbulent) are on the whole fairly well ordered (e.g., the earth's dipole field); thus they can hardly be the result of a random process of amplification.

If the magnetic energy of a small field increases statistically, a state of statistical equilibrium between motion and field must ultimately be reached. Just what this state is cannot as yet be said in general. A relationship which presents itself readily, on dimensional grounds, is *equipartition*,

$$\mu^{-1}\mathbf{B}^2=\rho\mathbf{v}^2, \quad (6.2)$$

and this has been proposed by a large number of authors. It is readily seen that when (6.2) is fulfilled the ponderomotive force term is of the same order as the $(\mathbf{v}\cdot\nabla)\mathbf{v}$ term in the equations of motion. Batchelor claims that equipartition should prevail for the smaller eddies of the turbulence spectrum, whereas for the largest eddies where the transfer of energy into the magnetic field is just beginning, the field remains below the value (6.2). His arguments in favor of this point do not seem fully conclusive. There is a certain amount of empirical evidence to the effect that the magnetic field in the earth's core is very strong, much stronger than would correspond to equipartition value (see later). Whether the Coriolis force acts as a constraint so as to produce such a deviation is an interesting point for speculation, but nothing is known mathematically.

We next consider the *mechanical* effects of a magnetic field such as produced by turbulence. It is convenient to start from the classical derivation of the mechanical Reynolds stresses (see for instance Sommerfeld, 1950). We shall use tensor notation with *summation convention*, but assume a Cartesian system, so that there is only one kind of tensor component. We assume incompressibility and set $\nu=0$ in the remainder of this section.

The Euler equations of motion of the fluid are

$$\partial v_i/\partial t+v_k(\partial v_i/\partial x_k)=-\rho^{-1}\partial p/\partial x_i. \quad (6.3)$$

Using the incompressibility condition, $\partial v_i/\partial x_i=0$, we readily verify the identity

$$(\partial/\partial x_k)(v_iv_k)=v_k(\partial v_i/\partial x_k)+v_i(\partial v_k/\partial x_k)=v_k(\partial v_i/\partial x_k).$$

We now set

$$v_i=v_i^0+v_i^1, \quad (6.4)$$

where v^0 designates the "smooth" and v^1 the "turbulent" part of the velocity. As usual we choose this decomposition so that for the linear averages

$$[v_i]_{\mathrm{Av}}=v_i^0, \quad [v_i^1]_{\mathrm{Av}}=0,$$

and hence

$$[v_iv_k]_{\mathrm{Av}}=v_i^0v_k^0+[v_i^1v_k^1]_{\mathrm{Av}}=v_i^0v_k^0-S_{ik}, \quad (6.5)$$

which defines the Reynolds stress tensor, S_{ik}. Substituting (6.4) and (6.5) into (6.3) we find

$$\partial v_i^0/\partial t + v_k^0(\partial v_i^0/\partial x_k) = -\rho^{-1}\partial p/\partial x_i + \partial S_{ik}/\partial x_k, \quad (6.6)$$

which is Reynolds' formula, interpreted by saying that the mean effect of the turbulent motion can be represented as the divergence of a stress tensor, in complete analogy to ordinary viscous forces.

In the elementary theory of turbulence it is shown how the magnitude of these eddy stresses can be put in evidence from an almost purely dimensional argument (essentially constructed by analogy with kinetic theory). One introduces a characteristic length λ_0, the "mixing length" which is representative of the mean distance a parcel of the fluid travels before losing, its identity. Assuming a fair approximation to isotropy, the last term in (6.6) which is nothing but the mean viscous force exerted by the eddies is then of order

$$\{F\} = \{v^2/\lambda_0\}. \quad (6.7)$$

The ponderomotive force (4.1) can be expressed as the divergence of a Maxwell stress tensor (for instance, Stratton, 1941). Referring to unit mass,

$$\rho^{-1}F_i = \partial T_{ik}/\partial x_k, \quad T_{ik} = (\rho\mu)^{-1}(B_iB_k - \tfrac{1}{2}B^2\delta_{ik}), \quad (6.8)$$

where δ_{ik} is the usual Kronecker symbol. Thus our equation of motion (4.2) becomes

$$\partial v_i^0/\partial t + v_k^0(\partial v_i^0/\partial x_k) = \rho^{-1}\partial p/\partial x_i + (\partial/\partial x_k)(S_{ik} + T_{ik}). \quad (6.9)$$

It is of course possible to split the magnetic field,

$$B_i = B_i^0 + B_i^1, \quad (6.10)$$

with the same conditions as before regarding linear averages; in this way the turbulent components could be more effectively exhibited. The magnetic field is the equivalent of an added mechanical "stiffness" (elasticity) with the attendant consequences such as suppression of eddies, reduction of instability, delay of the onset of turbulence. Such effects have been investigated quantitatively, by Chandrasekhar and others; we shall here merely mention the list of references given elsewhere (Elsasser, 1955). We remind the reader that this effect of magnetic fields upon mechanical viscosity is to be distinguished from the dissipative terms in the induction equation which, in the present review, are taken as expressing "magnetic viscosity" in a different sense.

If next we apply turbulence considerations to the induction equation, we write it first, from (2.15), in the form

$$\partial B_i/\partial t = (\partial/\partial x_k)(v_iB_k - v_kB_k), \quad (6.11)$$

the parenthesis on the right being an *antisymmetrical* tensor. Hence if we wanted to split off a term representing the divergence of a turbulence tensor, this tensor would be antisymmetrical. Clearly, the dissipative term in (2.12) like all other terms of that sort represents the divergence of a symmetrical tensor, and hence the analogy between eddy effects and molecular dissipation seems to break down for the induction equation.

This difficulty seems serious; it has not apparently been studied in detail. The situation can be improved by referring the turbulent averages to a system moving with the mean velocity of the fluid. To show this, let us confine ourselves to the incompressible case and write the induction equation in the form (5.8):

$$dB_i/dt = B_k(\partial v_i/\partial x_k). \quad (6.12)$$

Going through the same procedure as used in deriving the Reynolds stress tensor, one obtains on the right of (6.12) a term which is the divergence of a tensor, say,

$$I_{ik} = B_iv_k,$$

which is clearly neither symmetrical nor antisymmetrical; but at least it has a symmetrical part, analogous to other eddy-friction effects. The problem deserves further investigation. We remark that Chandrasekhar (1950) has studied the correlation of vectors pertaining to two different points in a homogeneous hydromagnetic turbulence field, in generalization of the well-known kinematical method of von Karman and Howarth used in nonmagnetic turbulence theory.

We have remarked before that a dynamo can function properly only if the dissipation term and the induction term are comparable in magnitude, and that this is no doubt achieved in real dynamos through a sufficiently large eddy diffusivity. We shall, however, continue to write ν_m in the sequel, where it is implicitly understood that what we really mean is the eddy diffusivity.

7. THE CAUCHY INTEGRAL: AMPLIFICATION

This section and the next are in part devoted to the development of some formal apparatus of the theory. Since the machinery is rather conventional we have omitted in this review technicalities that are not of the essence for the main line of reasoning and which any theoretician setting out to deal with these problems can readily supply for himself.

Cauchy was the first to show that in the absence of friction the Helmholtz vorticity equation (5.3) can be fully integrated in terms of the Lagrangian variables of hydrodynamics. This classical method is extensively described for instance by Brand (1947) who uses vector symbolism. The application to the hydromagnetic flux-conservation theorem due to Lundquist (1952). Here we shall use tensor notation. No special familiarity of the reader with Lagrangian hydrodynamics is assumed. It will suffice to say that the superscript zero will be used below to indicate *initial conditions*, the variables x_i, $v_i = dx_i/dt$, B_i taken at the time $t=0$ being designated by x_i^0, v_i^0, B_i^0. The functional relations are

$$x_i = x_i(x_k^0, t), \quad B_i = B_i(x_k, t) = B_i(x_k^0, t).$$

In Lagrangian hydrodynamics a material particle of the fluid is *labeled* by the parameters x_i^0 which indicate its position at $t=0$. We shall start out with the assumption of incompressibility which will be lifted later on. For $\nu_m = 0$ the induction equation is given by the preceding formula, (6.12). Now note the identity,

$$0 = \frac{d}{dt}\left(\frac{\partial x_i}{\partial x_j^0}\frac{\partial x_j^0}{\partial x_k}\right) = \frac{\partial v_i}{\partial x_j^0}\frac{\partial x_j^0}{\partial x_k} + \frac{\partial x_i}{\partial x_j^0}\frac{\partial v_j^0}{\partial x_k}$$

which follows from the fact that the parenthesis on the left is nothing but the constant, δ_{ik}. By virtue of this relation (6.12) can be written

$$\frac{dB_i}{dt} = B_k \frac{\partial v_i}{\partial x_j^0}\frac{\partial x_j^0}{\partial x_k} = -B_k \frac{\partial v_j^0}{\partial x_k}\frac{\partial x_i}{\partial x_j}.$$

Multiplying by $\partial x_l^0/\partial x_i$ with summation over i one obtains

$$\frac{dB_i}{dt}\frac{\partial x_l^0}{\partial x_i} + B_k \frac{\partial v_l^0}{\partial x_k} = \frac{d}{dt}\left(B_i \frac{\partial x_l^0}{\partial x_i}\right) = 0. \quad (7.1)$$

[The form (7.1) of the flux conservation theorem is intimately related to the integral (5.5) into which it may be transformed directly.] Integrating (7.1) we have

$$B_i(\partial x_l^0/\partial x_i) = B_l^0, \quad (7.2)$$

which is inverted on multiplication by $\partial s_j/\partial x_l^0$ and summation over l, giving the desired integral,

$$B_j = B_l^0(\partial s_j/\partial x_l^0). \quad (7.3)$$

An entirely similar transformation can be carried out for the vector potential.§ We start from (3.13), setting $\nu_m = 0$, thus

$$\partial \mathbf{A}/\partial t = \mathbf{v} \times (\nabla \times \mathbf{A}), \quad (7.4)$$

which we rewrite as

$$d\mathbf{A}/dt = \mathbf{v} \times (\nabla \times \mathbf{A}) + (\mathbf{v} \cdot \nabla)\mathbf{A}. \quad (7.5)$$

In tensor notation (7.5) becomes simply

$$dA_i/dt = v_k(\partial A_k/\partial x_i). \quad (7.6)$$

Multiplying by $\partial s_i/\partial x_l^0$ and summing over i,

$$\frac{dA_i}{dt}\frac{\partial x_i}{\partial x_l^0} = v_k\left(\frac{\partial A_k}{\partial x_l^0} - \frac{\partial A_k}{\partial t}\right) = v_k \frac{\partial A_k}{\partial x_l^0},$$

since $v_k(\partial A_k/\partial t) = 0$, as proved by scalar multiplication of (7.4) by $\mathbf{v}$. Now let

$$\psi = A_i v_i, \quad \varphi = \int \psi dt.$$

§ These results were obtained by Dr. William L. Bade whom the author wishes to thank for permission to publish them here.

We then have the identity,

$$\frac{d\psi}{\partial x_l^0} = \frac{\partial A_k}{\partial x_l^0} v_k + A_k \frac{\partial v_k}{\partial x_l^0},$$

and the previous equation becomes

$$\frac{dA_i}{dt}\frac{\partial x_i}{\partial x_l^0} = -A_k \frac{\partial v_k}{\partial x_l^0} + \frac{\partial \psi}{\partial x_l^0},$$

or else

$$\frac{d}{dt}\left(A_i \frac{\partial x_i}{\partial x_l^0}\right) = \frac{\partial \psi}{\partial x_l^0}. \quad (7.7)$$

Integrating (7.7) we have

$$A_j(\partial x_j/\partial x_l^0) - A_l^0 = \partial \varphi/\partial x_l^0, \quad (7.8)$$

and finally, on multiplying by $\partial x_l^0/\partial x_i$ and summing over l,

$$A_i = A_l^0(\partial x_l^0/\partial x_i) + \partial \varphi/\partial x_i, \quad (7.9)$$

which is the desired formula. Here the gradient term is irrelevant and may be omitted if we are not interested in electrical but only in magnetic fields. We then, by the way, find from (7.3) and (7.9) that $\mathbf{A} \cdot \mathbf{B}$ is a constant of the motion.

It is not difficult to extend these expressions to the case of a compressible fluid. There is no reference to compressibility in the formulas for the vector potential, and hence the last derivation can be retained without change. Next, we see from (5.8) that in the case of a compressible fluid we need merely write B_i/ρ everywhere in place of B_i. Thus (7.3) becomes

$$B_j/\rho = (B_l^0/\rho^0)(\partial x_j/\partial x_l^0). \quad (7.10)$$

The partials on the right are simply the coefficients of the strain tensor for strains of finite magnitude. It is not difficult to generalize (7.10) to non-Cartesian coordinates, but we shall forego a more general proof. Instead, we merely rewrite (7.10) in vector symbols:

$$\mathbf{B}/\rho = [(\mathbf{B}^0/\rho^0) \cdot \nabla_0]\mathbf{r}, \quad (7.11)$$

where $\mathbf{r}$ is the vector with components x, y, z, and ∇_0 refers of course to differentiation with respect to x^0, y^0, z^0. For curvilinear *orthogonal* coordinates (7.11) can be expressed as (Morse and Feshbach, 1953, Chapter 1)

$$B_i = \frac{B_k^0}{h_k}\frac{\partial x_i}{\partial x_k^0} + \frac{x_k}{h_i h_k}\left(B_i^0 \frac{\partial h_i}{\partial x_k^0} - B_k^0 \frac{\partial h_k}{\partial x_i^0}\right), \quad (7.12)$$

where summation is over k throughout, but where the parenthesis on the right vanishes for $k=i$. From the fact that the differentiation ∇_0 refers to the x_i^0, it is clear that the metric coefficients h_i must be considered as functions of the x_i^0. For cylindrical coordinates, s, φ, z,

$$h_1 = 1, \quad h_2 = s^0, \quad h_3 = 1,$$

and for spherical polar coordinates, r, ϑ, φ,

$$h_1=1, \quad h_2=r^0, \quad h_3=r^0 \sin\vartheta^0.$$

We shall now make some simple applications of the preceding formulas which will illustrate hydromagnetic amplifying processes. Let us first comment on effects of compression. In Lagrangian hydrodynamics compression or expansion of the fluid can be transcribed in terms of the Jacobian determinant, namely,

$$\rho^0/\rho = \|\partial(x_i)/\partial(x_k{}^0)\|. \tag{7.13}$$

Consider a homogeneous initial field, $B_z{}^0$. By means of (7.10) and (7.13) it is readily ascertained that an arbitrary velocity field v_z in the direction of the magnetic field does not affect the latter. Again, if the motion is in planes normal to the field, we have for an element of area dS in such a plane

$$BdS=B^0dS^0.$$

This result might be obtained from (7.10) but follows more directly from the integral theorem (5.5). For *isotropic* compression it follows that B increases as $\lambda^{\frac{2}{3}}$ where λ designates the linear dimensions, and the magnetic energy density as $\lambda^{4/3}$. One suspects that for sufficiently strong fields compression or expansion will tend to be anisotropic. These effects are of great interest in astrophysical problems where the formation of stars from dispersed matter and the ejection of tenuous gas from stars involves tremendous changes in density. They are of less importance for the dynamo theory. Such dynamo models as have been studied to date can be expressed in terms of the motion of incompressible fluids. For this reason we shall later on assume that our fluids are incompressible.

Consider next amplificatory processes confined to *two dimensions*. For Cartesian coordinates Eqs. (7.3) are

$$\begin{aligned} B_x &= B_x{}^0(\partial x/\partial x^0)+B_y{}^0(\partial x/\partial y^0),\\ B_y &= B_x{}^0(\partial y/\partial x^0)+B_y{}^0(\partial y/\partial y^0). \end{aligned} \tag{7.14}$$

Assume for simplicity that the initial field is homogeneous, $B_x=0$, $B_y=\text{const}$ [Fig. 1(a)]. Since a y-component of the flow does not affect this field we assume motion in the x-direction with a linear shear, say,

$$v_x=ay+b, \quad v_y=0,$$

which gives for the Lagrangian variables

$$x=v_xt+x^0, \quad y=y^0,$$

or in terms of the initial values

$$x=ay^0t+bt+x^0, \quad y=y^0,$$

whence (7.14) takes the form

$$B_x=atB_y{}^0, \quad B_y=B_y{}^0.$$

As one might expect, the lines of force become stretched in the y-direction [Fig. 1(b)]. The magnetic energy density is

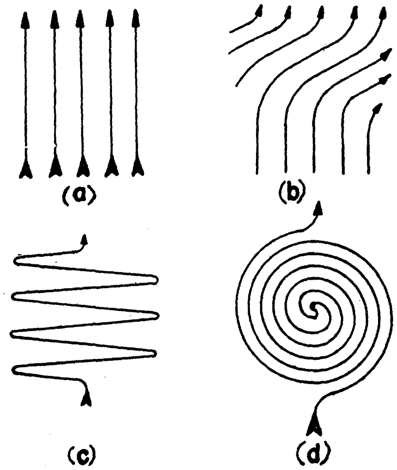

FIG. 1. Ways of stretching lines of force in two dimensions. (a) Homogeneous field; (b) linear shear normal to field; (c) alternating linear shear; (d) circular shear.

$$m=(2\mu)^{-1}(B_y{}^0)^2(1+a^2t^2).$$

Thus if the region considered is infinite along the y-axis the magnetic energy can be increased indefinitely. This is clearly not possible for a *finite* two-dimensional region. We may therefore ask what are the limits of amplification for such a region. (Finiteness seems more important than boundedness; the arguments given below appear to apply equally to an unbounded but finite area, e.g., the surface of a sphere.) As a general rule, amplification corresponds to a *stretching* of the magnetic field lines. Two ways of doing this in a finite two-dimensional area are shown in Fig. 1(c) and Fig. 1(d), the former producing its result by translatory motions of alternate sign and the latter by a rotation. The drawback of these schemes is apparent. There are always field vectors of opposite directions close to each other; thus even a small diffusion term will suffice to cancel most of this field. No amplification schemes in finite two-dimensional regions have been found which are not beset by this difficulty.

On these grounds we conjecture the existence of a theorem which we have not, however, proved formally. It applies to a field, $B_x{}^0$, $B_y{}^0$, say, defined in a two-dimensional finite domain, D. By a finite deformation corresponding to an incompressible fluid motion this is transformed into B_x, B_y. Instead of having a diffusion term in the induction equation we carry out an average over a small region: If σ is a circular area centered at ξ, η, we define

$$\beta_x(\xi,\eta)=\int_\sigma B_x dxdy, \quad \beta_y(\xi,\eta)=\int_\sigma B_y dxdy.$$

We then claim that the quantity

$$\int_D (\beta_x{}^2+\beta_y{}^2)d\xi d\eta,$$

integrated over the domain D (the integral being a

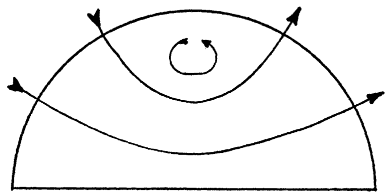

FIG. 2. Illustrating Cowling's theorem: Existence of a neutral point for a field whose lines are confined to meridional planes.

measure of the energy) is bounded for any fixed value of the small area σ. This is the conjectured theorem.

A similar statement does certainly not hold in three dimensions. In Sec. 10 we shall study amplification processes in finite three-dimensional regions for which the magnetic energy is not bounded in the above sense. If, however, by symmetry restrictions, the fluid particles are bound to move on two-dimensional surfaces similar limitations appear. Here belongs a theorem proved by Cowling (1934) regarding the impossibility of certain dynamo mechanisms. Cowling's conclusions refer to stationary dynamos only (it may be noted that in the previous arguments there was no need to assume stationarity). Consider a fluid motion confined to the meridional planes of a rotationally symmetrical figure. The magnetic field is also confined to these planes and hence remains so confined under the inductive action of the fluid motion. The problem is whether there exist types of fluid motion of this symmetry which can keep such a magnetic field stationary. Let the fluid be within an envelope of finite size, for example a sphere as in Fig. 2. Since any line of force issuing from this boundary returns to it, this being true both for the outside and for the inside of the fluid, it follows readily that there must at least be one "neutral point," $\mathbf{B}=0$, in each meridional half-plane. From $\nabla\cdot\mathbf{B}=0$ it follows that in the neighborhood of the neutral point the lines of force form closed curves surrounding the latter. Also it is readily deduced from the structure of the external field that all neutral points lie inside the fluid. If the field has preponderantly a dipole structure, as in Fig. 2, there is only one neutral point.

For stationary operation the left-hand side of the induction equation (2.12) vanishes and, after removing one *curl* by an integration, it becomes

$$\mathbf{v}\times\mathbf{B}=\nu_m\nabla\times\mathbf{B}. \tag{7.15}$$

We next integrate this over a small region containing the neutral point:

$$\int(\mathbf{v}\times\mathbf{B})\cdot d\mathbf{S}=\nu_m\int\mathbf{B}\cdot d\mathbf{C}.$$

Now if the "singularity," $\mathbf{B}=0$, is of the first order then, if we shrink the region, the right-hand side of this expression vanishes as the linear dimensions of the circuit whereas the left-hand side goes to zero quadratically. A similar discrepancy may readily be shown to exist for higher-order singularities. Hence (7.15) cannot be fulfilled in the neighborhood of a neutral point. Since the existence of such a point is essential for the dynamo, it follows that a stationary dynamo of the symmetry described cannot exist. The dynamos which we shall study later on are essentially three-dimensional; they do not have neutral points of the type considered and the restrictions imposed by Cowling's theorem do not apply to them.

8. TRANSVERSE MODES OF THE SPHERE

In the absence of motion the magnetic field in our conductors obeys the differential equation (2.16)

$$\nabla^2\mathbf{B}-\mu\sigma\partial\mathbf{B}/\partial t=0, \tag{8.1}$$

with suitable boundary conditions. We obtain normal modes in the usual way by postulating that the field decays without changing shape,

$$\mathbf{B}(\mathbf{r},t)=\mathbf{B}(\mathbf{r})\exp(-\Lambda t), \tag{8.2}$$

and define k by

$$\Lambda=k^2/\mu\sigma=k^2\nu_m. \tag{8.3}$$

We assume both Λ and k real. Now (8.1) becomes

$$\nabla^2\mathbf{B}+k^2\mathbf{B}=0. \tag{8.4}$$

For the time of free decay of a mode we have from (8.3)

$$\{\Lambda^{-1}\}=\{\lambda^2/\nu_m\}, \tag{8.5}$$

where λ is again a typical length. Since as a rule ν_m is of the general order of unity (very roughly) we can use this formula to estimate the order of free-decay times of astrophysical objects. As pointed out before, these times are fictitious since in actual fact we must substitute a suitable eddy viscosity in place of ν_m. They do provide a measure of R_m, however.

A few comments on the mathematical difficulties associated with the vector wave equation (8.4) are useful. The trouble is that one cannot simply extend the familiar boundary-value theory of the *scalar* wave equation

$$\nabla^2\psi+k^2\psi=0, \tag{8.6}$$

to the vectorial analog (8.4). As is well known, it is possible to construct a set of orthonormal modes by imposing on the solutions of (8.6) linear boundary conditions for a boundary of essentially arbitrary shape. A similar general theory for (8.4) has not been given and appears difficult of construction if at all feasible. Some of the scalar technique can be generalized, thus Stratton (1941) derives a vectorial analog of Green's theorems. On the whole, however, the theory of boundary-value problems of the vector wave equation (8.4) is essentially in a stage of, as it were, mathematical experimentation. The formalism for cylindrical and spherical vector waves is well worked out and is found in Stratton's book. Here, we shall omit proofs of orthogonality, etc. We confine ourselves to spherical waves. The case treated by Stratton is that of oscillatory solutions of Maxwell's equations for spherical boundary conditions as developed in the early years of the century by Mie and Debye. The *aperiodic* free modes which are solutions of (8.1) have been known somewhat longer; they were discovered around 1880 by Lord Kelvin and Horace Lamb (for the poloidal and toroidal modes, respectively).

It is easy to construct longitudinal solutions of (8.4), given a solution of the scalar equation (8.6). Such a vector is

$$U=\nabla\psi, \tag{8.7}$$

where the operator in (8.4) has the following meaning: $\nabla^2=\text{grad div}$. To obtain transverse solutions of (8.4) we first note that in the transverse case $\nabla^2=-\textit{curl curl}$. Furthermore, it is clear that with any solution of (8.4) its *curl* is also a solution. It is readily found from this that the transverse solutions can be constructed in pairs which are each other's *curl*. Letting $\nabla\cdot\mathbf{S}=\nabla\cdot\mathbf{T}=0$ we set

$$k\mathbf{T}=\nabla\times\mathbf{S},\quad k\mathbf{S}=\nabla\times\mathbf{T}, \tag{8.8}$$

whence by elimination it readily follows that both $\mathbf{S}$ and $\mathbf{T}$ obey the vector wave equation (8.4).

Let ψ be a solution of the scalar wave equation (8.6); we then set

$$\mathbf{T}=\nabla\times(\psi\mathbf{r})=\nabla\psi\times\mathbf{r}, \tag{8.9}$$

where again $\mathbf{r}$ is the vector with components x, y, z. From (8.8) we find now after a straightforward calculation

$$\mathbf{S}=k\psi\mathbf{r}+k^{-1}\nabla(\partial\psi/\partial r). \tag{8.10}$$

We take the scalar generating functions in the form

$$\begin{aligned}\psi&=N_{ns}{}^{m}J_n(k_{ns}r)Y_n{}^m(\vartheta,\varphi),\\ j_n(x)&=(\pi/2x)^{\frac{1}{2}}J_{n+\frac{1}{2}}(x),\\ Y_n{}^m&=P_n{}^m(\cos\vartheta)\begin{pmatrix}\cos m\varphi\\ \sin m\varphi\end{pmatrix},\end{aligned} \tag{8.11}$$

where $N_{ns}{}^m$ is a normalization factor, and where otherwise the symbols have their conventional meaning.

In components we have for (8.7)

$$\begin{aligned}\mathbf{U}_{(r)}=\partial\psi/\partial r,\quad \mathbf{U}_{(\vartheta)}=r^{-1}\partial\psi/\partial,\\ \mathbf{U}_{(\varphi)}=(r\sin\vartheta)^{-1}\partial\psi/\partial\varphi.\end{aligned} \tag{8.12}$$

This type of mode is purely longitudinal. Next from (8.9)

$$\begin{aligned}\mathbf{T}_{(r)}=0,\quad \mathbf{T}_{(\vartheta)}=(\sin\vartheta)^{-1}\partial\psi/\partial\vartheta,\\ \mathbf{T}_{(\varphi)}=-\partial\psi/\partial\vartheta.\end{aligned} \tag{8.13}$$

This type of transverse mode will be designated a *toroidal*. Again, from (8.10)

$$\begin{aligned}\mathbf{S}_{(r)}&=kr\psi+k^{-1}\partial^2(r\psi)/\partial r^2=n(n+1)(kr)^{-1}\psi,\\ \mathbf{S}_{(\vartheta)}&=(kr)^{-1}\partial^2(r\psi)/\partial r\partial\vartheta,\\ \mathbf{S}_{(\varphi)}&=(kr\sin\vartheta)^{-1}\partial^2(r\psi)/\partial r\partial\varphi.\end{aligned} \tag{8.14}$$

This type of transverse mode will be designated as *poloidal*. There exists a simple relation between these poloidal modes and the longitudinal modes (8.12). It applies in the limit $k=0$ when the wave equation goes over into Laplace's equation. Then

$$\lim_{k=0}(k\mathbf{S})=\mathbf{U}. \tag{8.15}$$

An important special case is that of full rotational symmetry. Then $\mathbf{T}_{(r)}=\mathbf{T}_{(\vartheta)}=0$ and $\mathbf{S}_{(\varphi)}=0$. If we let these vectors represent, say, magnetic fields we may state this specialization as follows: *In a toroidal field of rotational symmetry the field lines are circles about the axis; in a poloidal field of rotational symmetry the field lines lie in the meridional planes.*

We now relate these vector fields to the solutions of the electromagnetic field equations. If we confine ourselves to transverse modes we can set $\nabla\cdot\mathbf{A}=0$ and we may use the vector potential and the electric field vector almost interchangeably; we have

$$\mathbf{E}=\mathbf{J}/\sigma=\Lambda\mathbf{A}=k^2\nu_m\mathbf{A}, \tag{8.16}$$

by (3.12) and (8.3). We can write the field equations as

$$\mathbf{B}=\nabla\times\mathbf{A},\quad k^2\mathbf{A}=\nabla\times\mathbf{B}. \tag{8.17}$$

We can therefore fulfill (8.8) in two ways, namely,

$$\begin{aligned}\mathbf{B}/k=\mathbf{T},\quad \mathbf{A}=\mathbf{S},\\ \mathbf{B}/k=\mathbf{S},\quad \mathbf{A}=\mathbf{T},\end{aligned} \tag{8.18}$$

the first choice giving the toroidal magnetic modes, the second the poloidal magnetic modes.

We next come to the boundary conditions for the electromagnetic fields, assuming, say, that a homogeneous metallic sphere is surrounded by vacuum. For $\mu=\mu_0$ throughout we have continuity of all three components of $\mathbf{B}$. Moreover, there is continuity of the tangential component of $\mathbf{E}$, but not of the radial component since there can be a surface charge on the boundary. It is readily found that no toroidal solution exists in the limit $k=0$; this again, together with the boundary condition for $\mathbf{B}$, leads to the conclusion that *the field of the toroidal magnetic modes vanishes identically in outer space.* By (8.11) and (8.13) this leads to the condition for the toroidal magnetic modes, at the surface $r=R$ of the sphere,

$$j_n(k_{ns}{}^tR)=0. \tag{8.19}$$

For the poloidal modes (8.14) shows that in outer space the magnetic field may be expressed as the gradient of a scalar by (8.12); solutions of the Laplace equations of this type are nothing but the familiar multipole fields. From (8.11), (8.12), and (8.14) we find on applying the electromagnetic boundary conditions to the poloidal magnetic modes,

$$j_{n-1}(k_{ns}{}^pR)=0. \tag{8.20}$$

(For details of the calculations see Stratton, 1941, and for the aperiodic modes in particular, Elsasser, 1946 and 1947.)

The well-known method of solving differential equations by development into a series of orthonormal functions may now be applied to the induction equation. We shall merely sketch the procedure. It is convenient to use the equation in the form (ignoring dissipation)

$$\partial \mathbf{A}/\partial t = \mathbf{v}\times\mathbf{B}. \tag{8.21}$$

Now we may develop $\mathbf{A}$ into a series, say

$$\mathbf{A} = \Sigma c_\nu \mathbf{A}_\nu,$$

where the subscript ν goes over all "quantum numbers" and both types of transverse modes. The corresponding development of $\mathbf{B}$ involves the same coefficients c_ν; it is related to the development of $\mathbf{A}$ by (8.18). Furthermore one carries out a development of $\mathbf{v}$ in terms of an orthogonal system of functions; these latter are subject to slightly different boundary conditions as compared to the magnetic field.

If these developments are entered into (8.21) it becomes an infinite system of coupled linear differential equations for the c_ν containing $\dot{c}_\nu$ on the left and a linear combination of the c_ν on the right-hand side. The coefficients ("matrix elements") are of the form

$$\int \mathbf{A}\cdot\mathbf{v}\times\mathbf{B}dV, \tag{8.22}$$

where each of the three vectors appearing in the integrand is a normal-mode field. The expression (8.22) is invariant under any permutation of an even order of the three vectors. These matrix elements have been evaluated (Elsasser, 1946 and 1947) for all vectors of dipole and quadrupole character ($n\leq 2$). "Selection rules," that is instances in which the elements (8.22) vanish, can thus be obtained. General selection rules for any n, m are given by Bullard and Gellman (1954). By means of such rules one establishes which couplings of modes vanish; this provides significant limitations on models of dynamos. Consider a few cases in which all fields have *rotational symmetry* about an axis. The Cowling dynamo discussed at the end of the previous section represents a "primary" poloidal field producing a "secondary" poloidal field (in this case itself) by means of a poloidal fluid motion. Later on we shall see that a toroidal fluid motion acting upon a poloidal magnetic field as the primary produces a secondary magnetic field which is toroidal. If the primary magnetic field is toroidal, a toroidal fluid motion has no effect (since $\mathbf{v}\times\mathbf{B}=0$). A poloidal fluid motion acting upon the toroidal field merely rearranges the circular lines of force, so that the secondary is again toroidal. With regard to the amplification by this latter process the same type of argument may be used that was applied in the preceding section to the amplification of a field in a limited two-dimensional domain. We shall here abstain from reproducing the matrix formalism in detail since it is quite lengthy and since the essential mathematical properties of dynamos can be described largely without it.

9. MECHANICS OF A ROTATING FLUID

At this point we may pause briefly to take stock of our progress toward a dynamo theory. In the introduction we mentioned three characteristics of our dynamo models: large linear dimensions, convection, and rotation. We have dealt at length with the problem of linear dimensions and have established the condition $R_m > 1$ in order that any hydromagnetic amplifying system be effective. Now a dynamo is a process rather than a static state; convection is merely the machinery that keeps the process going, we may expect it to be nonspecific. It does drive the dynamos in celestial bodies, from the earth to the sun to the stars. More will be said about convection in the earth's core in Sec. 12. The sun is known to have a convective outer layer whose depth is estimated at 15 to 20% of the solar radius; other stars in which magnetic fields have been observed have similar convective layers. We cannot go into details here (see also Elsasser, 1955 and 1956). On the other hand, whatever the driving mechanism that generates and presumably maintains the magnetic field in our spiral arm of the galaxy, there is little likelihood that it can be classified as convection.

The existence of a driving force does not suffice, however, for the form of the motions must be essentially three-dimensional and convection by itself will not insure this. In a nonrotating system convective motions would tend to take place in the meridional planes, and by Cowling's argument this cannot give rise to an effective dynamo. Other kinds of motion, say for instance tidal motion in an otherwise stable fluid sphere do not give rise to dynamos for similar reasons. But if the fluid sphere or layer *rotates* rapidly enough the type of motion shifts towards the three dimensional: The Coriolis force, if strong enough, tends to produce a lowering of the symmetry of flow. As a consequence of this, new types of inductive couplings between various transverse modes become possible. These lead to effective dynamos, as we shall see.

We now proceed to study the effects of the Coriolis force, first for the purely hydrodynamical case and later with regard to its effect on magnetic fields. What follows in this section is a survey of dynamical facts familiar to geophysicists but which are relatively little known outside of this specialized group.

If we leave out magnetic effects and ignore mechanical friction, the equations of motion of the fluid in a system rotating with angular velocity Ω are

$$\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}+2\mathbf{\Omega}\times\mathbf{v}=-\frac{\nabla p}{\rho}+\nabla\phi_c, \tag{9.1}$$

where $\phi_c=\frac{1}{2}\Omega s^2$ is the potential of the centrifugal force. We shall omit this last term since it is a purely static effect expressed by the flattening of the earth's figure.

We first consider the relative magnitude of the Coriolis term. We define a dimensionless parameter C as the ratio of the third to the second term in (9.1),

$$\{C\}=\{\lambda\Omega/v\}=\{\Omega/\omega\}, \tag{9.2}$$

where ω is representative of the angular frequencies in the spectrum of the motion, and where λ as before characterizes linear dimensions. If we let $\lambda=R$, the radius of the sphere, this becomes

$$\{C\}=\{V/v\}, \tag{9.3}$$

where V is the linear velocity of rotation. For smaller systems, such as eddies, C is correspondingly smaller and (9.3) no longer holds. For the earth's core, taking $v=0.03$ cm/sec as deduced from the observed geomagnetic secular variation, we find $C\sim10^6$. The Coriolis effects will predominate in eddies of the core whose dimensions are in excess of a few meters.

Consider conditions on the sun: The velocities directly observed at the upper boundary of the convection zone, near the bottom of the photosphere, are of order 1 km/sec, giving $C\sim1$. But farther down in the convection zone the velocities must be smaller by several powers of ten; hence C is larger in about the same ratio. This may be shown as follows: It is known that just below the photosphere the convective transport of heat outward is comparable to the radiative transport, thus it represents the total transport in order of magnitude. But the total transport per unit area is independent of depth apart from a small curvature factor. For fixed total convective transport the rms convective velocity varies roughly as ρ^{-1}. Now the density increases with extreme rapidity and by many powers of ten as we go downwards in the convection zone (Kuiper, ed., 1953). This verifies the statement that C is numerically large farther down in the convection zone.

In the earth $C\gg1$ and, as we have just seen, for the major part of the solar convection zone $C\gg1$ also. Furthermore, the stars in which magnetic fields have been observed are relatively early types which, by evidence of the Doppler broadening of their spectral lines, rotate rapidly. Magnetic fields do, however, appear only in those star types, from type F onwards, that are assumed on more general astrophysical grounds to have convective layers (the earlier types, O and B, apparently do not possess such layers). The characteristic parameters of magnetic stars do not differ from those of the sun by many powers of ten; hence it is rather plausible that $C\gg1$ characterizes *empirical* dynamos in general. The traditional branch of hydrodynamics dealing with $C\gg1$ is dynamical meteorology. We shall borrow some of the standard terminology from there.

Ordinarily (excepting very special conditions such as, say, the formation of a jet) the first and second term of (9.1) are of comparable order of magnitude. We can then neglect these two terms compared to the Coriolis

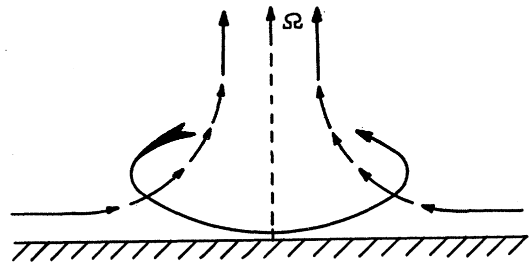

FIG. 3. Creation of cyclonic circulation by convergence of the fluid in a plane normal to the axis of rotation.

term and obtain

$$2\mathbf{\Omega}\times\mathbf{v}+\nabla p/\rho=0. \tag{9.4}$$

Thus in the absence of accelerations the steady state is not a static equilibrium, but a stationary pattern of motion known as geostrophic flow. The velocity is perpendicular to the pressure-gradient force. The prime examples of such flow are the major eddies of the atmosphere (say 200 to 2000 km in diameter) the cyclones and anticyclones. These may be schematized by a stationary two-dimensional model: In a cyclone we have a pressure minimum at the center and the force, $-\nabla p/\rho$, is directed radially inwards; it is balanced by a circulation which is counterclockwise on a map of the northern hemisphere, clockwise on a map of the southern hemisphere. In an anticyclone there is a pressure maximum at the center and the sense of the circulation is the opposite. It is customary to define the terms cyclonic and anticyclonic by comparing the circulation (or vorticity) with the sign of $\mathbf{\Omega}$: The circulation in an eddy (or else a vorticity field) is termed *cyclonic* if it has the same sense as the rotation of the earth; it is termed *anticyclonic* if its sense is the opposite.

We now return to (9.1) and consider time-dependent motion. Since the acceleration terms are small we can use a perturbation technique say, on letting $\mathbf{v}_0$ be the unperturbed velocity, $\mathbf{v}_1$ the perturbation field,

$$d\mathbf{v}_1/dt=-2\mathbf{\Omega}\times\mathbf{v}_0-\nabla p/\rho. \tag{9.5}$$

On integrating this over a time Δt such that $\mathbf{v}_0$ changes but little, we find

$$\Delta v=-2\mathbf{\Omega}\times\Delta\mathbf{r}-(\nabla p/\rho)\Delta t. \tag{9.6}$$

Consider a flow converging toward the origin in the xy-plane and flowing out of this plane in the z-direction (Fig. 3) where it may be assumed that $\mathbf{\Omega}$ is normal to the lower boundary shown in Fig. 3. Ignoring the last term of (9.6) we see that the effect of any inflow or outflow is the creation of a circulation about the origin. For the earth, $v\sim10^{-4}\Delta r$, and this is independent of the size of the system. To get $v=3\times10^{-2}$ cm/sec we require $\Delta r=3$ meters, a very small displacement indeed. The sense of the circulation produced is *cyclonic on convergence, anticyclonic on divergence.*

Now consider the earth's core as a whole. Assume that a force field acts on the fluid which is poloidal and rotationally symmetrical (that is confined to meridional planes). It does not take much imagination to conceive of, e.g., a thermal field of this type owing to the ellipticity of the earth. The net effect of such forces is not a meridional but a *zonal circulation*, that is nonuniform rotation, $\omega=\omega(r,\vartheta)$ where ω is the local angular velocity.

To proceed to more formal results, the Helmholtz vorticity theorem may readily be generalized to apply to a rotating system. We write (9.1) using (4.3) for $\mathbf{v}$,

$$\partial\mathbf{v}/\partial t+(\mathbf{w}+2\mathbf{\Omega})\times\mathbf{v}=-\nabla p/\rho+\tfrac{1}{2}\nabla\mathbf{v}^2,$$

where $\mathbf{w}$ is again the vorticity. Taking the *curl* of this we obtain after some straightforward calculations

$$d/dt[\rho^{-1}(\mathbf{w}+2\mathbf{\Omega})]=[\rho^{-1}(\mathbf{w}+2\mathbf{\Omega})\cdot\nabla]\mathbf{v}+\rho^{-1}\boldsymbol{\tau}, \quad (9.7)$$

where $\boldsymbol{\tau}$ is the vorticity-generating force field, conventionally known as the *solenoidal field*,

$$\boldsymbol{\tau}=-\nabla\times(\nabla p/\rho)=\nabla p\times\nabla(\rho^{-1}). \quad (9.8)$$

For $\boldsymbol{\tau}=0$, (9.7) admits of a Cauchy integral which we need not write down. We mention without proof the corresponding integral-conservation (Kelvin) theorem

$$\frac{d}{dt}\int(\mathbf{w}+2\mathbf{\Omega})_n dS=-\oint\frac{dp}{\rho}, \quad (9.9)$$

where the integration on the right is along the bounding contour. Let the area S be plane and let it make an angle α with the equatorial plane. Then (9.9) may be written

$$\frac{d}{dt}\int\mathbf{v}\cdot d\mathbf{C}=-2\mathbf{\Omega}\frac{d}{dt}(S\cos\alpha)-\int\frac{dp}{\rho}. \quad (9.10)$$

We see from this that there are two main processes for changing the circulation of an eddy by means of the Coriolis force. One way consists in changing S, the area carried bodily with the fluid; this produces cyclonic circulation on convergence, anticyclonic circulation on divergence. The second process consists in changing the tilt of S. Whereas we shall use the former process in our hydromagnetic considerations, the tilting mechanism has not found such a use. It does have considerable importance in meteorology. We need hardly say that since the Coriolis force does no work ($\mathbf{v}\cdot\mathbf{\Omega}\times\mathbf{v}=0$) the driving power for acceleration must be supplied from other sources, specifically of course from the solenoidal field (more commonly known as buoyancy).

Consider then a parcel of relatively lighter fluid tending to rise from the lower boundary as in Fig. 3. Cyclonic circulation is generated near the bottom; higher up where the rising fluid tends to push away the surrounding mass anticyclonic circulation is generated. It follows readily from the equation of continuity (assuming an incompressible fluid) that in this simple, cylindrically symmetrical case the net circulation integrated over the height vanishes, provided again $\mathbf{\Omega}$ is normal to the lower boundary. But this need not be the case for more general models. Thus assume that in the upper levels the fluid, instead of flowing out with rotational symmetry, is accelerated in one direction, say to the east. At some location eastward it must then descend again in order to satisfy the equation of continuity for the upper level as a whole. The point we wish to make is that in the atmosphere (for dynamical reasons that are none too well understood) the convergence and divergence effects *do not cancel*, but that as a general rule a correlation between cyclonic circulation and upward velocity, and between anticyclonic circulation and downward velocity holds throughout the troposphere. Such a lack of cancellation, clearly, represents a nonlinear effect which is very difficult to analyze.

Returning now to our model, Fig. 3, we have so far assumed that $\mathbf{\Omega}$ is normal to the boundary. If it is inclined relative to the latter, as it is in the case of a spherical mass, the whole eddy will be tilted and the conditions on the vorticity become more involved; but the basic correlation described should persist. Note finally that in our eddy the light fluid ceases to rise as soon as the geostrophic equilibrium (9.4) is established. Owing to the peculiar accelerating mechanism acting normal to the force applied, this will in general not correspond to exhaustion of the potential energy of the solenoidal field. But further lifting can take place only when the intensity of the circulation is decreased by (eddy) friction. Thus a convective regime for $C\gg1$ is quite different from that in a nonrotating system. In the latter case most of the potential energy of buoyancy can almost at once be converted into kinetic energy of the rising parcel.

We conclude this brief outline with some remarks on turbulence for a system in which $C\gg1$. No analytical studies of this type of turbulence seem to exist, but there are some rather intriguing theoretical problems. Consider a fluid parcel which travels to or from the axis of rotation. If it travels outward it will suffer an acceleration in the direction of $-\varphi$, if it travels inward in the direction of $+\varphi$. As these accelerations are communicated to the mass of fluid surrounding the displaced parcel, the outer part of the fluid will have its angular velocity reduced and the inner part will have it increased. In a stationary turbulent regime, then, *the local angular velocity decreases outwards*, the average vorticity is anticyclonic throughout. We have seen an analogous situation (though not confined to anticyclonic motion) in the case where a system of purely meridional forces acts on the fluid, but the last-mentioned result is more remarkable: Its significance lies in the fact that in the rotating system of reference *a set of small-scale irregular motions engenders a mean large-scale velocity field*, contrary to all the conventional postulates of turbulence theory, where one invariably assumes that the flow of energy in momentum space is unidirectional, namely, from the larger to the smaller eddies.

10. THE TOROIDAL FIELD

We next deal with rotationally symmetrical magnetic fields. We have seen that in terms of transverse vector modes there are two types, poloidal and toroidal. For the former the lines of force lie in the meridional planes, for the latter the lines of force are circles about the axis. Cowling's result as given in Sec. 7 affirms that there exists no stationary dynamo which is purely poloidal; the same is true in all probability for a dynamo that is only stationary in the mean. Similarly we cannot construct a dynamo from a purely toroidal field: any amplification process which leaves the field toroidal amounts merely to a reshuffling of the circular lines of force. It is likely, moreover, that even in the absence of rotational symmetry no dynamo can be constructed involving only modes of one type. We presume that any dynamo must involve *both types* of modes. Our next step is therefore to consider couplings between the two types. Here, one is tempted to compare the two types of modes with the two windings of a technical generator, the field coils and the armature. In this simile the poloidal field would correspond to the field in the field coil and the toroidal field would correspond to the field produced by the currents flowing in the armature. Just as is the case in the industrial machine, a part of the current induced must be diverted in order to maintain the "primary" field, in our case the poloidal field. The whole constitutes a two-stage feedback cycle. There is a good deal of empirical evidence to the effect that this two-stage scheme is realistic, and on general grounds the occurrence of more complicated feedback systems must seem unlikely, provided only the two-stage system can successfully compete with higher order ones.

Since the feedback system forms a closed cycle we shall "cut it open," as it were, at one point, whereupon we expect to return to the starting point and to justify our initial assumptions after having gone through the full cycle. We postulate first the existence of a poloidal field of rotational symmetry. We now introduce the nonuniform rotation, $\omega(r,\vartheta)=\omega(r^0,\vartheta^0)$ discussed in Sec. 9. This draws out the lines of force along circles of latitude (Fig. 4) and thus produces a toroidal field. From (7.3) we have

$$B_\varphi=(\partial\varphi/\partial r^0)B_r+(\partial\varphi/r^0\partial\vartheta^0)B_\vartheta, \qquad (10.1)$$

and the two other components of the "secondary" field vanish. Since $\varphi=\omega t$ we can write this as

$$B_\varphi=(\partial\omega/\partial r^0)tB_r+(\partial\omega/r^0\partial\vartheta^0)tB_\vartheta, \qquad (10.2)$$

which shows that the energy of the induced toroidal field increases as t^2. In contradistinction to the two-dimensional mechanisms studied earlier, this process of amplification is not bounded. Provided the nonuniform rotation is maintained against the ponderomotive forces of the magnetic field, amplification will stop only when the primary field ultimately decays. If we set B_r, $B_\vartheta\sim\exp(-\alpha t)$ we see that B_φ increases linearly in t for a time of order α^{-1} and then levels off; asymptotically for large t we have decay, $B_\varphi\sim t\exp(-\alpha t)$. Thus such a mechanism while providing a powerful means of amplification is not self-sustaining.

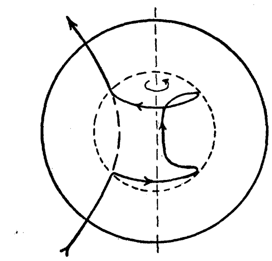

FIG. 4. Illustrating the generation of a toroidal field from a poloidal one by nonuniform rotation of the fluid sphere.

The amplification of the toroidal field (Elsasser, 1947) has been investigated in detail and a full analytical solution including dissipation given by Bullard (1949). He schematizes nonuniform rotation by means of a solid sphere rotating inside a solid spherical shell, the two being in metallic contact over the whole boundary. The presence of the inner boundary does not give rise to any more serious mathematical difficulties than a cusp of the toroidal field strength. Owing to the presence of a diffusion term, a stationary solution, $\partial B_\varphi/\partial t=0$, can be found, expressibly in terms of spherical eigenfunctions (8.11) and (8.13) of a complex argument.

An inspection of Fig. 4 shows that the toroidal field is antisymmetrical with respect to reflection at the equatorial plane. If the primary, poloidal field is a pure dipole it may readily be shown by means of selection rules such as follow from (8.21) that the induced toroidal field is a pure quadrupole. More generally the following may be shown: Note that a poloidal field which is antisymmetrical under reflection is composed of harmonics of odd n; a toroidal field which is antisymmetrical under reflection consists of harmonics of even n. Assume now that the toroidal fluid motion, the nonuniform rotation, is symmetrical about the equatorial plane (so that its harmonic components all have odd n). It may then be shown rigorously that for an antisymmetrical poloidal field (odd n) the toroidal field generated is also antisymmetrical (even n). This is the case occurring in the earth and the sun. With the same assumption about the symmetry of $\mathbf{v}$, there exists a second set of solutions where all magnetic fields are symmetrical with respect to reflection. All couplings of this second system with the first vanish under the symmetry assumptions made for $\mathbf{v}$. No cogent reason has so far been advanced why the second system cannot be realized in Nature, but no case is known, the stellar data being at present too inconclusive to decide such a question on empirical grounds. Furthermore, still assuming that $\mathbf{v}$ has symmetry of rotation and of reflection, the above theorems can be generalized to hold for magnetic modes with $m\neq 0$.

We have so far altogether ignored the ponderomotive forces (4.1). Although a fully dynamical theory of hydromagnetic systems taking account of these forces has not yet been given, this is a convenient place to discuss at least some of their features. In order to investigate the ponderomotive reaction to the amplifying process (10.2), let

$$\mathbf{B}=\mathbf{B}_p+\mathbf{B}_t, \qquad (10.3)$$

representing the poloidal and toroidal components, respectively. We now write (4.1) as

$$\mathbf{F}=\mathbf{F}_{pp}+\mathbf{F}_{tp}+\mathbf{F}_{pt}+\mathbf{F}_{tt}, \qquad (10.4)$$

depending on whether the first factor $\nabla\times\mathbf{B}$, or the second factor $\mathbf{B}$ is poloidal or toroidal. Clearly $\mathbf{F}_{pp}$ is not of interest here; it would be the reactive force of a Cowling-type amplifying mechanism. The second term is entirely in the φ direction; its absolute value is, in polar coordinates

$$\mu F_{tp}=\frac{B_\vartheta}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}(\sin\vartheta B_\varphi)+B_r\frac{\partial}{\partial r}(rB_\varphi). \qquad (10.5)$$

This is the force which tries to stop the nonuniform rotation as a result of the transformation of kinetic energy into field energy.

The last two components appearing in (10.4) are poloidal vector fields confined to meridional planes. Since all forces are bilinear in $\mathbf{B}$ the last term of (10.4) will preponderate if $\mathbf{B}_t>\mathbf{B}_p$. Thus we are confronted with a very complicated system of forces which, however, are still only small perturbations on the geostrophic equilibrium (9.4) as will appear presently. The ponderomotive force will change the vorticity of the fluid unless

$$(\nabla\times\mathbf{B})\times\mathbf{B}=\nabla\psi, \qquad (10.6)$$

where ψ is some scalar. Fields of this last type have been extensively investigated by Lüst and Schlüter (1954). There is much to be said for assuming that (10.6) is rarely fulfilled under actual astrophysical conditions (except for those indicated by the authors quoted).

The numerical magnitude of the toroidal field and its forces in the earth is of interest. Since the rms dipole field is $\sim\frac{1}{2}$ gauss at the earth's surface, it is near 4 gauss at the core's boundary. Thus an rms value over the interior of the core of, say, 6 to 8 gauss for the poloidal field seems appropriate. The amplifying mechanism for the toroidal field is extremely powerful, and it is undoubtedly the process by which energy is fed from the fluid motion into the field; the feedback process described later is unlikely to contribute much to the field energy. Hence it was for some time assumed that in a dynamo the toroidal field is large compared to the poloidal field. Recently Rikitake (1955) showed that such a large field might lead to instability and the true condition is perhaps more nearly $\mathbf{B}_t\sim\mathbf{B}_p$. Thus an rms value of, say, 10 gauss for the total field in the core should be not too far from a lower limit.

The ponderomotive forces may be compared to the $(\mathbf{v}\cdot\nabla)\mathbf{v}$ term in the equations of motion; the equality of the two terms corresponds to equipartition, (6.2). On the other hand, the ratio of the Coriolis force to this magnetic force is

$$\{\rho\Omega v\lambda/\mu B^2\}=\{C\rho v^2/\mu B^2\}, \qquad (10.7)$$

C times larger than equipartition. Now entering into (6.2) with the value, $v=0.03$ cm/sec, observed for the secular variation near the core's surface, we obtain $B=0.1$ gauss, which is too small by a factor 10^2. It follows that either $\mathbf{v}$ must increase rapidly with increasing depth in the core, or else $\mathbf{B}$ is far above equipartition in the terrestrial dynamo. Again, to make (10.7) equal to unity, a field near 300 gauss would be required, for larger than the observed field. This question is of considerable interest, and we shall return to it later.

The presence of a toroidal field in the earth cannot be established by direct observation since all toroidal modes vanish outside the conducting sphere (Sec. 8) but we shall later on find indirect evidence for such a field. There is a great deal of indirect evidence for a *solar* toroidal field, from sunspots. All sunspots are magnetic, the field direction being normal to the solar surface in the middle of the spot. On going from there toward the edges of the spot the field diverges laterally (this refers to the relatively thin photosphere where the Zeeman observations are made). For small spots the mean field strength increases proportional to the area of the umbra (the dark region) but for larger spots it reaches a saturation value of about 2500±500 gauss. Sunspots appear frequently in the form of a grouping of smaller spots, but the most common form is the so-called *bipolar group*. These are two spots, or else one spot and a group, or two spot groups, separated from each other by a small interval and oriented roughly east-west relative to each other (in solar co-ordinates). Invariably the leader spot or group (forward in the sense of solar rotation) has one magnetic polarity, *the same for all* leaders appearing during a sunspot cycle, and the follower spot or group has the opposite magnetic polarity. In the southern solar hemisphere the polarities of the bipolar groups are the opposite from those in the northern hemisphere. There can be little doubt that bipolar groups are derived from strands of the solar toroidal field which by some mechanism have been lifted to the solar surface, as schematized in Fig. 5. This is corroborated by Babcock's (1955) observation that when there is only one spot, or a group of one polarity, there is often a wide, weakly magnetized area through which, as it were, most of the magnetic flux returns into the sun. The question arises as to whether the weak field of one of Babcock's regions is more closely related to the toroidal fields farther down in the sun, so that the sunspot areas result from a secondary concentration of flux, or whether the converse is true. For various reasons the former alternative seems more likely (Parker, 1955b). The darkness of the spot is no doubt a phenomenon secondary to the magnetic field, related to the fact that the ponderomotive forces (4.1) approach the general order of magnitude of the hydrostatic pressure in the photosphere of a spot and thus tend to expand the gas containing a field.

At the beginning of a sunspot cycle spots appear at irregular locations around $\pm 30°$ of solar latitude. As the sunspot cycle progresses, spots appear at lower and lower latitudes until the last spots of the cycle are seen near latitude $\pm 5°$. Then no more new spots of this cycle appear; at the same time new spots of the next cycle begin to break through around $\pm 30°$, but the bipolar groups from now on have the *opposite polarity*. Thus, magnetically speaking, the entire sunspot cycle lasts, not $11\frac{1}{2}$ but 23 years. The impression that toroidal fields, first generated in middle latitudes, wander in wave-like fashion toward the equator is very strong. We shall revert to this problem later.

11. THE FEEDBACK MECHANISM

The system described so far does not constitute a dynamo. The primary, poloidal field decays and, as we have seen, this implies that the secondary, toroidal field must also ultimately decay, no matter how strong it becomes during some finite time. The poloidal field cannot be made to maintain itself without support from the toroidal field, as this would imply the possibility of a purely poloidal self-sustained dynamo, contrary to our previous results. We must now find a coupling mechanism that acts in the inverse direction from the one described in the previous section: Given a toroidal field as the primary, a motion is to be found which produces a toroidal field as the secondary. The two processes together will form a two-stage feedback cycle.

Here we run into a remarkable asymmetry of the inductive couplings. One finds that for rotational symmetry of both **B** and **v** no mechanism of the desired type exists: A purely toroidal fluid motion does not interact with the primary toroidal field, and a purely poloidal fluid motion merely rearranges the circular lines of force without generating a poloidal field. The general rotationally symmetrical motion is a linear combination of these two types. Hence for rotational symmetry no

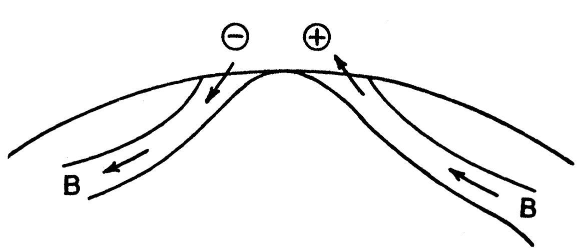

Fig. 5. Showing a strand of the solar toroidal field lifted locally and giving rise to a bipolar sunspot group.

feedback process exists, and with a slight generalization: there can be no dynamo of full rotational symmetry. We shall be able to show, however, that the asymmetrical components of the motion and field can be represented as small-scale eddies and that hence it is possible to construct dynamos which are rotationally symmetrical *in the mean*.

Now the dissipative term $\nabla^2\mathbf{B}$ in the induction equation (2.12) cannot produce a field in a direction in which there was none to begin with. Hence our discussion can be based on the frictionless equation (2.15). This equation involves only the dimensions of length and time and contains no material constant of the medium. It is clear, then, that any suitable fluid motion can be scaled down to arbitrarily small dimensions; the time of the process can be kept constant by making **v** correspondingly larger. In thus reducing the linear scale we can dispense with any curvature effects of the over-all spherical model and can conveniently adapt the motions to Cartesian coordinates. A concrete model of the local eddies has been developed by Parker (1955c) and will now be explained.

Consider a fluid layer heated from below. The stratification becomes unstable and at some places a local upsurge will occur (Fig. 3). But the associated lateral convergence at once produces a cyclonic circulation. It was shown in Sec. 9 that for $C \gg 1$ the velocity of this circulation will soon become large compared to the velocity of inflow. We therefore disregard the radial component of the motion. The vertical component, although of the same order as the radial one, cannot be neglected for reasons that will become clear presently.

We introduce a local Cartesian system with x pointing to the east, y to the north along the meridian circle, and z vertically upwards (in the northern hemisphere). For simplicity $\mathbf{\Omega}$ will for the time being be assumed in the z-direction. Let $\mathbf{B}_x$ be the initial toroidal field in the $+x$-direction. If this field is subjected to both a cyclonic twist and a lifting motion, a new field will be created which has a nonvanishing projection upon the y-z (meridional) plane (Fig. 6). Essentially this new field may be represented by a set of field lines forming closed loops in that plane. If there are many such sets of loops at different places inside the conducting sphere, they will eventually coalesce by diffusion and thus form an overall poloidal field. It will be convenient later on to make use of a vector potential, $\mathbf{A}_x$, having the same direction as the toroidal field; clearly a "hill" of $\mathbf{A}_x$ represents a magnetic field whose projection upon the y-z-plane forms closed loops. (It is true that this local vector potential has a nonvanishing divergence. This is not important, however, since ultimately, on letting the hills coalesce, we shall obtain a toroidal **A** which is divergence free.)

We proceed now to establish analytically the intuitive result of Fig. 6; we shall show that a local motion of the type described acting upon an originally homo-

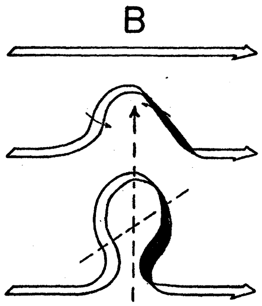

FIG. 6. A strand of toroidal field (top) is lifted (middle) and twisted (bottom) giving rise to a loop in the meridional plane normal to the original field.

geneous $\mathbf{B}_x$ produces a "hill" of $\mathbf{A}_x$. Parker showed this in two ways, firstly in terms of the Cauchy integral of the induction equation and secondly by means of a perturbation method. We use here this second procedure, not only because it is simpler, but also because it is largely independent of special parameters that might characterize the shape of the local eddy. Let s, φ, z be a local system of cylindrical coordinates and let

$$\mathbf{v}(s,z)=\mathbf{v}_\varphi(s,z)+\mathbf{v}_z(s,z),$$

the two terms on the right corresponding to circulation and lift, respectively. We assume this velocity field to exist during the interval $0\leq t\leq \tau$ and assume that there is no motion before or after. The first-order perturbation field is now found by integrating (2.15) with respect to the time,

$$\mathbf{B}_1=\tau\nabla\times[\mathbf{v}_\varphi\times\mathbf{B}_x+v_z\times\mathbf{B}_x], \qquad (11.1)$$

where $\mathbf{B}_x$ is the zero-order homogeneous field. To obtain the second-order approximation we use the induction equation in the form (8.20), namely,

$$\partial\mathbf{A}/\partial t=\mathbf{v}\times\mathbf{B},$$

which gives on substitution of (11.1) on the right and on a second integration,

$$\mathbf{A}_2=\tau^2[\mathbf{v}_\varphi\times\nabla\times(\mathbf{v}_z\times\mathbf{B}_x)]+\tau^2[\mathbf{v}_z\times\nabla(\mathbf{v}_\varphi\times\mathbf{B}_x)]. \quad (11.2)$$

Altogether there should be four brackets on the right of (11.2), but the one containing $\mathbf{v}_\varphi$ twice and the other containing $\mathbf{v}_z$ twice are clearly irrelevant since they do not represent a combination of circulation and lift. We need only the x-components of $\mathbf{A}_2$. (It may furthermore be shown that the other two components average out on integration over the hill.) Straightforward calculation gives

$$(\mathbf{A}_2)_x=\tau^2B_x\cos^2\varphi\left(v_\varphi\frac{\partial v_z}{\partial s}-v_z\frac{\partial v_\varphi}{\partial s}\right) -\tau^2B_x\sin^2\varphi(v_\varphi v_z/s). \quad (11.3)$$

Now for small s we clearly have $v_\varphi\sim s$ whereas v_z may be assumed to decrease monotonically from a maximum on the axis. If the two velocity components vary with s as shown schematically in Fig. 7, then all three terms of (11.3) are negative over practically the whole region; we have a hill of negative $\mathbf{A}_x$. The corresponding magnetic loops form a left-handed screw if combined with a positive $\mathbf{B}_x$ (pointing eastward, applicable to the northern hemisphere). After coalescence of all the loops in the fluid sphere this yields a field which near the earth's axis points from north to south (as does the real geomagnetic field). This in turn gives rise, by virtue of the nonuniform rotation discussed in the previous section, to a toroidal field pointing eastwards in the northern hemisphere. Thus the feedback is regenerative. Similar arguments show that the feedback is regenerative in the southern hemisphere where the toroidal field points in the opposite direction.

In the above derivation we have ignored the obliquity of $\mathbf{\Omega}$ relative to the local z-axis if the eddy occurs in middle latitudes. This should modify things quantitatively, but can hardly be expected to be fatal for the feedback scheme. Another effect ignored is the tilting to the west as the eddy rises, owing to the action of the Coriolis force upon $\mathbf{v}_z$. This gives rise to the nonuniform rotation of the fluid as a whole, as discussed before. Again we may assume that the feedback loops are not obliterated by this distortion of the eddy.

The condition of regenerative feedback is *that a cyclonic circulation be associated with rising motion, or else an anticyclonic circulation with sinking motion.* In an alternate form this is equivalent to saying that there be in either case a positive correlation between v_z and w_z (where $\mathbf{w}$ is again the vorticity). Consider now the entire fluid sphere with numerous local eddies. In a coordinate system whose origin is at the center of this sphere we can express this condition (on ignoring again the obliquity of $\mathbf{\Omega}$ relative to the rising eddy) by saying that there must be a net positive correlation between v_r and w_r on integrating over the sphere. The nature of such a mean correlation is not obvious. If we accept our model of feedback loops as correct, this requirement constitutes a necessary condition for the feedback to be regenerative. We can, however, give it tentatively a more physical interpretation.

Consider a fluid layer, for simplicity flat, between two parallel boundaries. Convection can either originate at the lower boundary, through specifically lighter fluid being created there and then rising; or else at the upper boundary, through specifically heavier fluid being created there and then sinking. It is true enough that a perfectly stationary convective regime could readily be conceived as being symmetrical with respect to the

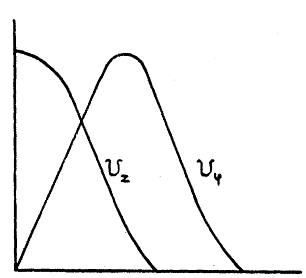

FIG. 7. Assumed distribution of velocity components in an eddy.

two boundaries, but naturally occurring convection is often highly asymmetrical. An example of this is the atmosphere. In it the air is heated at the bottom by contact with the ground; higher up it loses heat by infrared radiation into space. Observation shows that in this regime the required velocity correlation is universally present: cyclones contain rising air and anticyclones sinking air. The influence of the layers near the ground is strong enough so that this correlation prevails, at least up to the bottom of the stratosphere. It seems appropriate to advance the suggestion that the thermally convective layers of the sun and stars are sufficiently similar in structure and dynamics to the atmosphere, so that their motions are driven by eddies rising from below. The problem of the earth's core is more complicated and the physical data rather uncertain. We shall discuss this case in Sec. 12 and shall see that with plausible assumptions it may also be subsumed under our model. The dynamics of a convective fluid in which $C\gg 1$ is not yet very well understood and we must proceed by arguments which are none too quantitative. Our presumption is that in the observed hydromagnetic dynamos the convectively induced circulation is *controlled from below;* the rising eddies which originate by convergence at the bottom have cyclonic circulation and the corresponding sinking motions spreading out near the bottom have anticyclonic circulation. For reasons which we do not yet fully understand, this regime is not, or not fully balanced by sets of eddies arising at the top of the layer, which latter would exhibit an opposite correlation. Clearly, this problem needs much further dynamical study.

There is probably a second effect within the earth's core that tends to maintain the correlation required for the production of regenerative feedback. This is the ponderomotive force of the toroidal field itself. Consider the function $B_\varphi(r)$. It vanishes at the center and at the upper boundary and must have a maximum in between. Now there is some evidence that the toroidal field reaches its maximum higher than halfway up from the center (see Bullard and Gellman, 1954). Let us assume in accordance with the foregoing that the convectively controlled dynamo mechanism is mainly located below this maximum, that is in a region where $\partial B_\varphi/\partial r>0$ (for further justification of this assumption see Sec. 12). The descending material will then move into a region where the toroidal field is weaker than at the level whence the matter came; thus the toroidal field carried downwards will exert an expansive pressure (an expansive motion in turn leads to anticyclonic circulation). It is quite possible that this effect might overcome any tendency to cyclonic circulation appearing in the early formation of such an eddy. Conversely, the rising material will be kept together for some time by the pressure of the surrounding toroidal field, giving enough time for the feedback loops to diffuse rather than be destroyed at once by immediate expansion. Clearly, by the equation of continuity, there must be a balance of ascending and descending matter. The gist of our argument is that if an ascending eddy stops at some level, the compensating descending matter comes from *somewhere else*, instead of resulting from an immediate expansion of the rising eddy, which would destroy its cyclonic twist.

A glance at Fig. 6 will show that regenerative feedback loops are most effective when the angle described by a fluid particle in its motion about the origin is $\sim 90°$. In fact if this angle exceeds 180° the feedback effect is reversed. (The perturbation method used above implies of course that all displacements are small.) Now the dynamics of atmospheric cyclones shows that such a dynamical system travels as a whole relative to the material medium, and there is no reason to doubt that the same applies to the core. This reduces the time a given particle is displaced by any one cyclone; hence it reduces the length of the trajectory which, from the flux-conservation theorem, determines the deformation of the magnetic field lines. Furthermore, the ponderomotive forces of the magnetic field act so as to resist any deformation of the field lines. The problem does not readily lend itself to quantitative treatment, but we might say that the required limitation of the trajectories is at least plausible.

We next consider coalescence of the individual loops. The detailed calculations are given by Parker (1955c). The local "hills" of vector potential will now act as *sources* of a toroidal $\mathbf{A}$. Ignoring all fluid motion, this vector obeys the equation

$$\partial\mathbf{A}/\partial t-\nu_m\nabla^2\mathbf{A}=\sum_\nu \mathbf{H}_\nu(\mathbf{r}-\mathbf{r}_\nu,\, t-t_\nu), \qquad (11.4)$$

where $\mathbf{H}_\nu$ is the vector potential produced by the νth hill. We now develop $\mathbf{A}$ into a set of rotationally symmetrical normal modes

$$\mathbf{A}(\mathbf{r},t)=\sum_{ns} c_{ns}(t)\mathbf{A}_{ns}(\mathbf{r}). \qquad (11.5)$$

For the components $\mathbf{A}_{ns}$, Eqs. (8.1) to (8.3) hold. Substituting (11.5) into (11.4), multiplying by $\mathbf{A}_{ns}$ and integrating we get

$$\partial c_{ns}(t)/\partial t+k_{ns}\nu_m{}^2c_{ns}(t)$$
$$=\sum_\nu \int \mathbf{H}_\nu\cdot\mathbf{A}_{ns}dV=\sum_\nu u_\nu(t-t_\nu), \qquad (11.6)$$

say. The integral of (11.6) is

$$c_{ns}(t)=\sum_\nu \int_{-\infty}^{t} u_\nu(t')\exp[k_{ns}\nu_m{}^2(t'-t_\nu)]dt'. \qquad (11.7)$$

This equation calls for a statistical treatment, by evaluating the averages of $c_{ns}(t)$ and $c_{ns}{}^2(t)$, etc. Let us now assume that the lifetime of an individual eddy is short; we set, using the Dirac function

$$u_\nu=a_\nu\delta(t-t_\nu).$$

If there are many eddies, the higher moments will be small and we can confine ourselves to the linear average which we shall designate simply by c_{ns}:

$$c_{ns}=[c_{ns}(t)]_{\mathrm{Av}}=N[a_\nu]_{\mathrm{Av}}, \qquad (11.8)$$

where N is the number of feedback loops appearing per unit time. We are thus reduced to a steady state and we replace (11.5) by

$$\mathbf{A}(\mathbf{r})=\sum c_{ns}\mathbf{A}_{ns}(\mathbf{r}). \qquad (11.9)$$

Next, we must make an assumption about the spatial distribution of the loops. We shall assume that they appear with uniform probability per unit volume of the sphere. Thus,

$$A_\varphi=\mathrm{const}\cdot r^2 \sin\vartheta,$$

and the other two components of $\mathbf{A}$ may be assumed to vanish. If we substitute this into (11.9) the coefficients c_{ns} can be calculated. For reasons of symmetry $c_{ns}=0$ for even n. The lengthy calculations are again due to Parker. After summing over s for constant n he finds for the ratio of the octupole field to the dipole field at the surface of the sphere,

$$(B_r)_{\mathrm{oct}}/(B_r)_{\mathrm{dip}}=0.16P_3/P_1, \qquad (11.10)$$

where the quantities on the left are radial components of the field and the P's on the right are Legendre functions.

The preceding model, although grossly simplified, should not be too far from the conditions for the earth's core. Observations show that the octupole field at the surface of the earth is about 2% of the dipole field; hence at the surface of the core it would be about 8% of the latter. The observed higher harmonics of the earth's field are, however, almost certainly caused by eddies occurring in the surface layers of the core: there is no discontinuity between even and odd n as there should be if these harmonics were directly connected with the dynamo mechanism producing the main field (see Elsasser, 1950). Hence we should expect the factor in (11.10) to be smaller than 0.08, say. Now Parker has investigated a more general model where the feedback loops appear only inside a sphere of radius $r_0<R$, where R is the radius of the core. He shows that as soon as r_0 becomes somewhat smaller than R, the coefficient on the right of (11.10) becomes rapidly small. In connection with our previously made assumption that the convective eddies are driven from below, this seems a physically plausible assumption and seems to provide a satisfactory explanation for the preponderance of the earth's dipole term (see also Sec. 12).

The assumption of numerous small feedback loops, on the other hand, does not appear to be a very good approximation: The observed inclination of the magnetic dipole axis by $11\frac{1}{2}°$ relative to the geographical axis implies fluctuations in the dynamo mechanism which point to a rather small number of such loops. It is noteworthy that according to the paleomagnetic investigations of Graham (1949) the mean angle of inclination of the dipole axis in the geological past seems to have been considerably smaller than the present value, perhaps only one-half to one-third of the latter. Thus the present dipole field would represent a relatively large fluctuation. A moderate number of feedback loops existing at any one time might produce a variation of the magnetic axis by a few degrees. It is hardly necessary to add that the treatment of a limited number of feedback eddies, each of a finite size, would introduce almost insuperable analytical difficulties.

12. MOTIONS IN THE EARTH'S CORE

Having surveyed the dynamical principles that enter into a theory of the geomagnetic dynamo we shall conclude by relating this knowledge to some additional geophysical data. We shall confine ourselves to data referring more or less directly to the existence of mechanical motions in the core.

The geomagnetic secular variation indicates the existence of motions which, on any geological time-scale, are rapid indeed. The velocity of 0.03 cm/sec quoted before may be derived directly, say from the rate at which lines of constant $\mathbf{B}$ are displaced; it has thus a minimum of hypothetical connotations. As a look upon the maps of the secular variation (Vestine, 1947 and 1948, and Elsasser, 1950) shows, the hills and dales in the landscape of $d\mathbf{B}/dt$ grow, move about, and disappear with periods of the general order of, say 30 to 200 years. While on the whole irregular, the motions have one systematic component, a general mean displacement of all the hills and dales toward the west, the so-called *westward drift*. Bullard and his associates (1950) have investigated this phenomenon by studying the time dependence of the nonsymmetrical (tesseral) harmonics in the series for the geomagnetic potential. Some 8 harmonic analyses have been made in the last 125 years, since the time of Gauss. If one now tries to combine the $\sin m\varphi$ and $\cos m\varphi$ terms in the form $\cos m(\varphi+\alpha t)$, it is found that α does not vary much either during the time span considered or from one harmonic to the other, and its mean value is

$$\alpha=0.18°/\mathrm{year}, \qquad (12.1)$$

omitting the dipole from the average (about which, see later). No interpretation other than a difference in angular velocity between the mantle and the top layers of the core has been proposed, the mantle rotating *faster* than these layers. It is difficult to attribute this effect to a purely mechanical cause since, as we have seen, any mechanical coupling must almost certainly lead to a minimum of angular velocity for the outermost layer. An explanation in terms of ponderomotive forces of the magnetic field was given by Bullard (1950). He extended his two-shell model by adding a weakly conducting third shell on the outside and showed that the ponderomotive forces of the field penetrating into the latter are such as to make it rotate faster than the underlying shell. Elsasser and Takeuchi (1955) have investigated the hydromagnetic conditions in the top layers of the core adjacent to the mantle and in this way derived an approximative fluid model of the westward drift. It is necessary in any event to assume that the mantle has a finite conductivity, otherwise there can of course be no differential ponderomotive torque.

In order to show that a torque on the mantle exists we write down the φ component of the ponderomotive force (4.1) already given in (10.5), namely

$$\mu F_\varphi = \frac{B_r}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}(\sin\vartheta B_\varphi) + \frac{B_r}{r}\frac{\partial}{\partial r}(rB_\varphi). \quad (12.2)$$

By the boundary conditions the toroidal field, B_φ, vanishes outside of conductors. Now assume that only a layer of moderate thickness at the bottom of the mantle is a fair conductor; then B_φ will become small at the top of this layer. Under these conditions the second term on the right is small compared to the first, and approximately

$$\mu F_\varphi = B_r(\partial B_\varphi/\partial r).$$

If the layer is thin, B_r will be sensibly constant across it, and on integrating over its thickness

$$\mu\int F_\varphi dr = -[B_r B_\varphi]_{r=R},$$

the fields on the right to be taken at the boundary of the core. Letting now $B_r = b_r P_1$, and $B_\varphi = b_\varphi \partial P_2/\partial\vartheta$ where P_1 and P_2 are Legendre functions, we obtain the torque by integration, giving

$$\mu\tau = -(2/5)R^3 b_r b_\varphi. \quad (12.3)$$

Now for the earth b_r is negative and b_φ positive, and hence τ is positive, the acceleration of the mantle is to the east. It might be remembered that the toroidal field owes its existence to a nonuniform rotation which is slower in the outer layers. By Lenz' law such a field in turn tends to accelerate the outer layers.

In the stationary state the torque (12.3) must be balanced by friction. In all probability this friction is again not purely mechanical but results from the ponderomotive forces of the magnetic fields caused by eddies near the surface of the core. While the main components, B_r and B_φ, are independent of φ, these local fields move relative to the mantle, and it is readily ascertained that they produce the required forces.

A geophysical phenomenon closely related to the westward drift is the irregularity of the earth's rate of rotation. Given an astronomical record of the earth's angular velocity as a function of time, the uniform deceleration usually attributed to tidal friction must first be deducted. What remains is an irregular curve representing changes that might on occasion amount to as much as a deviation of the earth clock by 1 second per year (Brouwer, 1951). It seems that the causes of this fluctuation cannot be attributed to any changes occurring in the outer parts of the earth. The atmosphere has too small an angular momentum to be significant, and any change in the crust or ocean, icecaps, etc., would appear in the first instance as a change in moment of inertia. Unless this change had rotational symmetry about the earth's axis to a highly improbable degree it would produce a displacement of the geographical pole much larger than is observed (Munk and Revelle, 1952), Vestine (1953) has determined deviations of the geomagnetic drift from a mean, steady-state value. During the period 1900 to 1940 where adequate data exist there is evidence for a negative correlation between drift variation and clock variation. The effect is not much beyond the limits of observational accuracy; it does indicate the action of a time-dependent torque between mantle and core. Moderate fluctuations, say in B_φ, of the order of a few tenths of a gauss can produce the required variable torques (Elsasser and Takeuchi, 1955). Or else, the effect may be due to a variation in eddy friction. In order that an exchange of angular momentum between mantle and core be possible, a layer of the core of appreciable depth must participate, otherwise the moment of inertia of the part of the core involved would be too small. The required "stiffness" of the core must again be attributed to the ponderomotive forces of the magnetic fields which resist deformation.

Let us now turn our attention to the fluid of the core as a whole. We have already encountered a significant dynamical problem: if the velocities deeper down in the core are of the same order as those observed near the top, then **B** exceeds the equipartition value by a factor of about a hundred. The only way, if any, whereby one might obtain a strong increase of velocity with depth would seem by means of a thermally stable stratification in the top layers of the core. Farther down the stratification must necessarily be near-adiabatic in order to permit true convection and three-dimensional dynamo effects. If the upper part of the core is not thermally stable, the eddies produced by the Coriolis force tend to be approximately isotropic (Takeuchi and Elsasser, 1954). There would then be no reason to assume an appreciable change of mean velocity with depth. While a definite decision might perhaps be premature there exist fairly definite empirical arguments in favor of a low mean velocity. These are found in the remarkable stability of the dipole part of the earth field as compared to the higher harmonics from the quadrupoles on up. In contradistinction to the rapid changes of the latter, the inclination of the dipole axis relative to the earth axis has not changed measurably since the time of Gauss' first determination (1830). The longitude of the dipole axis has changed but slightly and seems to have remained nearly constant since about 1880. Now if a fluid particle traveled at a rate of a hundred times the velocities near the surface, say 3 cm/sec, it would traverse the core in about 20 years. The individual feedback eddies, much smaller than the core, should then change fairly rapidly and the dipole field should be much less stable than observed. The very fact that a relatively large deviation of the dipole axis from the geographical axis can maintain itself for over a century indicates that the lifetime of some major eddies must be fairly large (it must by the way be comparable to the free decay time of the volume occupied by an eddy if the eddy is to be effective in feedback). This time is then likely to be of the order of several hundred years. This seems hard to reconcile with velocities very much higher than those observed at the core's surface.

We have already had several indications to the effect that the main dynamo mechanism is located at some depth inside the core. The relative stability of the inclination of the magnetic axis favors this view. Next, as pointed out before, the excess of the dipole terms over the other spherical harmonics (which by themselves form a fairly uniformly converging series) can best be explained if one assumes that the contributions of the dynamo to the higher harmonics are suppressed in the diffusion through the upper layers, and that the observed higher harmonics are essentially caused by eddies quite near the surface ("visible" down to only about 50 to 100 km, from skin-effect calculations). Finally we have seen that the correlation between cyclonic circulation and rising motion needed for regenerative feedback is best achieved when the convective regime is driven from below. While none of these arguments would seem too convincing just by itself, their combination lends some credence to the view that the dynamo mechanism operates at a considerable depth.

Seismic data show that the fluid core surrounds an *inner core* of a radius of 1300 km, about one-third of the radius of the core. The available evidence makes it highly probably that this body is solid. Among its chemical constituents iron should be preponderant, since the cosmic abundance data of nuclei show clearly that there is not enough material heavier than iron to fill a fractional volume of the earth of the size required. We are then inclined to seek for the agency driving the convection at the boundary of this inner core.

It has usually been assumed that convection in the core is of thermal origin. There are serious drawbacks to this view since the mantle surrounding the core should be a worse conductor of heat than the core, and no evidence of plastic-flow convection in the mantle has been forthcoming. Furthermore, Urey (1955) has been able to demonstrate that the presence of even small amounts of uranium and thorium in the core, sufficient to produce thermal convection, is most unlikely on chemical grounds. The only radioelement whose presence cannot yet be ruled out is potassium, but it would be rather forcing an issue to assume thermal convection on so slender a basis. A solution to this difficulty may be found in the assumption that convection in the core is of *chemical* rather than thermal origin. Urey (1952) has proposed that the interior of the earth is not at rest but that gradual chemical differentiation and slow relative displacement of different constituents is taking place all the time. These views are appealing since it is becoming more and more evident that near the earth's surface the gradual growth of continental blocks during geological time and the associated processes of mountain building are not so much caused by convective "overturning" as formerly believed by many, but seem to be largely the result of progressive separation, with the lighter constituent moving upwards, the heavier down. Although these processes are not strictly molecular (the diffusion coefficient being too small) but represent macroscopic displacements, they are still made up of many local components, each on a scale small compared to the gigantic convective "cells" assumed by some geological theories.

To test the applicability of ideas of this kind to the core we make an order-of-magnitude estimate. We start from the barometric equation, $dp = -g\rho dr$. We write this in order of magnitude, if λ is again a representative length, $\{\Delta p\} = \{g(\Delta\rho)\lambda\}$. But from the equation of motion

$$\{\Omega v\} = \{\Delta p/\lambda\rho\} = \{g(\Delta\rho/\rho)\}, \qquad (12.4)$$

which gives $\Delta\rho/\rho \sim 2\cdot 10^{-9}$, small indeed. There is no serious obstacle to the view that very slow chemical differentiation at the boundaries of the core can maintain convection (Elsasser, 1957).

It remains to be shown that the processes at the lower rather than at the upper boundary of the core can preponderate in determining the convective motions. Now at the upper boundary we have the systematic motion of the westward drift which moves the material of the core past the mantle at the relatively high speed of (12.1). Under these circumstances it is understandable that density differences cannot develop over a large enough area of the boundary layer to give rise to sizeable eddies. The conditions at the lower boundary are quite different. Under the high pressures prevailing it is likely that the physical properties, especially electrical conductivity, of the solid and liquid phases are fairly similar; no large change of σ would be involved. Furthermore, calculations show (Bullard and Gellman, 1954) that the main poloidal and toroidal field will vary but slowly with depth in the central parts of the core. Thus there is no reason to assume strong differential rotation in these regions. It is therefore conceivable that the boundary layer adjacent to the inner core is far more quiescent than the boundary layer at the top and that there is time enough for large eddies to develop.

13. APPENDIX: PERIODIC DYNAMOS, REVERSALS

The observations of solar and stellar hydromagnetic fields have helped us to elucidate the nature of terrestrial magnetism. These solar and stellar fields have one outstanding characteristic in common which we have not considered, namely, their *variability*. A number of the stars studied by Babcock reverse their magnetic polarity with periods of a few days; others show irregular variations without outright reversals. The curve of magnetic field *versus* time is usually rather complex and is in no case simple sinusoidal. The existence of a nonvariable magnetic star has not yet been established. There is furthermore the phenomenon of the sunspot cycle in which the bipolar groups reverse their polarity every half-cycle of $11\frac{1}{2}$ years.

In the last few years a remarkable phenomenon has been deduced from paleomagnetic data: The earth's magnetic dipole field appears to have reversed its polarity a considerable number of times during the geological past (Runcorn, 1955). This is not the place to go into a scrutiny of the evidence which by now is quite convincing. These reversals occur intermittently, apparently at irregular intervals. It is fairly well established that no reversal has occurred during the Quaternary, that is for several hundred thousand years past. On the other hand, from the theory we can assert that all Fourier components occurring in the dynamo mechanism represent times shorter than the free-decay times of the fundamental magnetic modes of the core. The decay time of the dipole mode is at best 20 000 years (that is not counting eddy diffusion but only molecular diffusion) and the decay time of the toroidal quadrupole is a quarter of that of the dipole. Thus one cannot expect the reversal to take more than a few thousand years. The earth's dipole moment has been observed to have decreased since the time of Gauss at an average rate of 5% per century; on linear extrapolation it would then take 4000 years to obtain a reversed dipole of the same magnitude as the present one. The agreement of these estimates is good enough so that we may take 3 to 4000 years as a representative value of the mean time of reversal. The individual reversals can hardly scatter about this value by more than a factor of 2–3. This time is very short compared to the mean geological interval between reversals. Again, the possibility exists in principle, that in the past the dipole might on occasion have reversed itself several times in succession before settling down to a relatively fixed direction. In any event, the analogy between the irregular terrestrial reversals and the periodic reversals of the solar and stellar fields is striking enough.

Although the hydromagnetic dynamo theory appears to explain in a fairly satisfactory way the observed features of the present-day earth's field, we are yet far from a theoretical understanding of nonstationary dynamos. Some developments and ideas that would contribute to a future dynamical theory, fragmentary as they may be, are given in this appendix.

Some astrophysical authors have suggested that mechanical oscillations play a part in the reversal of stellar fields. Motions in the photosphere will drag the field along, and if these motions are reversed the component of the field parallel to the star's surface can be reversed. But hydromagnetic dynamos occur only in objects that rotate fairly rapidly and that have outer convective zones. The combined action of Coriolis force and eddy viscosity in this zone is likely to suppress efficiently the higher harmonics, leaving us with only two fundamental vibrations: the radial one (cepheids) which is not likely to reverse the field, and the fundamental of incompressible motion where the eccentricity of the ellipsoid of revolution oscillates about a mean value. It is doubtful whether such a model could explain more than a small fraction of the stellar data, and its application to the sun seems out of place. Again, it has been suggested that the surface of some stars is magnetically "patchy" and that as the star rotates it offers various magnetic aspects to the observer. This would require extraordinarily large local fields in order that an oscillation of the mean field of amplitude several thousand gauss may result. Here, we shall confine ourselves to some comments on hydromagnetic reversals proper.

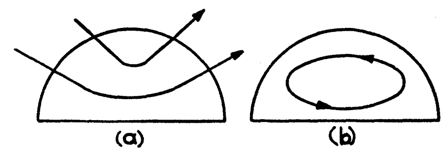

FIG. 8. (a) Poloidal field formed by coalescence of feedback loops. (b) Field which might be formed if diffusion is suppressed in the outer layers and might lead to reversal of the toroidal and ultimately of the poloidal field.

There seem to be *two* basic mechanisms whereby one can conceive reversals to occur. One of them is clearly a reversal of the correlation between circulation and lift which we had to postulate in order to obtain regenerative feedback. We have so far assumed that this inverse correlation would lead to a degenerative dying out of the dynamo, but it may be conducive to oscillatory solutions if there is a time lag between the two inductive processes constituting the full feedback cycle. A second possible mechanism of reversal is illustrated by Fig. 8, here referring to the earth's dynamo, but of more general applicability. We saw previously that in the process of coalescence of loops the dipole field preponderates [Fig. 8(a)]. Suppose now that for some reason the (eddy) diffusion near the outer boundary is hampered; we might then end up with the type of closed loops shown in Fig. 8(b). Now the outer segment of this loop points in a direction roughly opposite to that of the dipole field of Fig. 8(a). On nonuniform rotation a reversed toroidal field would result in the outer parts.

Parker (1955c) has constructed what he calls a *migratory dynamo* which is somewhat related to the mechanism outlined in Fig. 8; it operates on a time lag caused by diffusion between the two components of the feedback cycle. Consider the conditions in the solar convection zone in low latitudes where the sunspot belt is observed to migrate toward the equator. Ignoring curvature effects, introduce a local Cartesian system as in Sec. 11, with the x-axis pointing east, the y-axis north along the meridian circle, and the z-axis pointing vertically upwards. Let the nonuniform rotation be described by v_x where we let $\partial v_x/\partial y=0$ and $\partial v_x/\partial z=-H$, a constant. We express the poloidal field in terms of its vector potential, $A_x=A$, say. After some calculations the induction equation (2.12) for the generation of the toroidal field, $B_x=B$, reduces to

$$\partial B/\partial t=H(\partial A/\partial y)+\nu_m\nabla^2 B. \qquad (13.1)$$

We next assume that the rate of creation of feedback loops, expressed in terms of sources of A, is given by $-\Gamma B$ where Γ is a constant. Then our feedback equation (11.4) may be written

$$\partial A/\partial t=-\Gamma B+\nu_m\nabla^2 A. \qquad (13.2)$$

This pair of equations may be designated as the *dynamo equations*. Following Parker, we now seek particular integrals of these equations that represent waves traveling in the direction of the y-axis, say

$$B=B_0e^{i\omega t+iky}, \quad A=A_0e^{i\omega t+iky}.$$

By straightforward calculations the characteristic equation of the system (13.1), (13.2) is then found to be

$$i\omega=(i)^{\frac{1}{2}}(k\Gamma H)^{\frac{1}{2}}-\nu_m k^2. \qquad (13.3)$$

This expression has both a real and an imaginary part of comparable magnitude, assuming k to be real. This means that the wave is *being amplified* (or else deamplified, depending on the choice of sign in 13.3) as it travels along. The ratio of the amplitudes of the two vector fields is now found to be

$$A/B=A_0/B_0=(-i\Gamma/kH)^{\frac{1}{2}},$$

which shows that there is a phase shift of 45° between A and B. The field consists of alternating strands of toroidal field interlaced with alternating loops of poloidal field, the whole traveling along the y-axis with velocity ω/k. This type of particular solution of the dynamo equations illustrates their yet unexplored potentialities in dealing with time-dependent dynamo problems.

The dynamo equations, being linear, can be handled by known methods; they do not of course inform us how the excitation whose propagation they describe originates. A few comments on the nature of such excitation might be in place, even though they are of necessity very qualitative. One phenomenon that any theory of the sunspot cycle must eventually explain is the approximately simultaneous appearance of new spots in middle latitudes ($\sim\pm 30°$) at the beginning of a half-cycle, in *both* hemispheres. (There are, however, variations of a few percent, both in the length of the half-cycle and in the relative onset in the two hemispheres.) So far as the present author can see, there seems to be no means of effecting synchronization at so large a distance other than by action of the ponderomotive stresses which the field itself exerts. One may assume that these forces vary periodically over the sunspot belt and that when they exceed a certain value at any one place the toroidal field becomes unstable and "sheds" strands toward the solar surface.

At this place we might remark about a significant difference between the terrestrial dynamo and the stellar ones. In the earth's core the density varies by about 20% from the bottom to the top of the fluid layer. The major feedback eddies are relatively large. In stellar convective zones the density varies tremendously with depth (in the solar photosphere for instance the scale height, that is the height over which the density varies by a factor e, is of order 100 km). The linear dimensions of a convective eddy can hardly be of much larger order than the scale height, and this is borne out by the observed solar granulation which is a direct expression of the convective eddies. Thus we have another nondimensional parameter characterizing the hydromagnetic convective regime, namely, $(\lambda/\lambda_e)^2$, where λ is the depth of the convective layer and λ_e the mean size of an eddy (the square being taken because the free-decay time are proportional to λ^2). For the earth this number is only moderate whereas for stellar convective layers it is very large. One is induced to interpret this as meaning that in the earth's core coalescence of the loops into the lowest free mode, the dipole, can readily take place, whereas in stars the fields are more localized. This agrees with the solar observations which do not indicate a preponderance of the fundamental modes. Perhaps the occurrence of the periodic reversals in the sun and stars is also related to this fact.

We cannot fail to mention here that bipolar sunspot groups have as a rule an anticyclonic twist: The follower spot or group is not only to the solar west of the leader but also slightly displaced toward the pole. The angle which the bipolar group axis makes with the east-west direction depends on latitude and other parameters and can be as high as 20° (see Kiepenheuer's article in *The Sun*, edited by Kuiper, 1953). Now this is the opposite of the correlation required for regenerative feedback. We can only conclude that the rising matter expands sufficiently, probably as a result of the expansive stresses of the magnetic field it contains, so that an anticyclonic circulation results. This brings to mind the remark made in Sec. 11, that a positive $\partial B_\varphi/\partial r$ tends to maintain the regenerative correlation whereas a negative sign will have the opposite effect. Clearly, it will not pay to speculate about the dynamical properties of a delicately balanced gyroscopic system with infinitely many degrees of freedom. We must be guided by whatever further observation will reveal.

A word, finally, about the reversals of the earth's field. The difficulty here is that the model is not unambiguous; a little reflection will show that several possibilities of effecting reversals present themselves in principle. It seems premature to try to find a definitive model at the present time. The external features of the reversal, as it would appear to an observer at the earth's surface are, however, well determined. The physicist can indicate to the geologist that the process of reversal should not require more than a few thousand years, and that it does not consist in a migration of the dipole axis from the northern to the southern hemisphere, but in a vanishing of the dipole field which continues into the appearance of a field of opposite polarity.

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