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Induction Effects in Terrestrial Magnetism*

Part I. Theory

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The paper deals with the electromagnetic effect of motions in the earth's core, considered as a fluid metallic sphere. On the basis of simple estimates the electric conductivity of the core is assumed of the same order of magnitude as that of common metals. The mathematical treatment follows Hansen and Stratton: three independent vector solutions of the vectorial wave equation are introduced; two of these have vanishing divergence, and they are designated as toroidal and poloidal vector fields. The vector potential and electric field are toroidal, whereas the magnetic field is poloidal. These vectors, expressed in terms of spherical harmonics and Bessel functions, possess some notable properties of orthogonality which are briefly discussed. The theory of the free, exponentially decaying current modes is then given, leading to decay periods of the order of some tens of thousands of years. Next, the field equations in the presence of mechanical motions of the conducting fluid are set up. The field is developed in a series of the fundamental, orthogonal vectors, and the field equations are transformed into a system of ordinary differential equations for the coefficients of this development. The behavior of the solutions depends on the symmetry of the "coupling matrix" that arises from the term of the field equations expressing the induction effects. In order to evaluate this matrix the velocity field is developed into a series of the fundamental vectors similar to the series for the electromagnetic field. It is then shown that when the velocity is a toroidal vector field the coupling matrix is antisymmetrical. When the velocity field is poloidal, the coupling matrix is neither purely symmetrical nor purely antisymmetrical. For stationary fluid motion the linear differential equations can be integrated in closed form by a transformation to new normal modes, whenever the matrix of the system is either symmetrical or antisymmetrical. In the latter case the eigenvalues are purely imaginary and the coefficients of the new normal modes are harmonic functions of time, representing oscillatory changes in amplitude of the field components. For a symmetrical matrix the eigenvalues are real and the time factors of the new normal modes are real exponentials representing amplification or de-amplification as the case may be, depending on the sign of the velocity. For a matrix without specific symmetry, normal modes do not, as a rule, exist but similar, somewhat less stringent results can be derived in special cases. In the case of toroidal flow, in particular, the oscillatory changes of the field components are superposed upon the slow exponential decay characteristic of the free modes.

THE existence of a fluid metallic core in the interior of the earth seems sufficiently well established[1] as a result of seismic observations to be used as a starting point for the explanation of some of the more outstanding phenomena of terrestrial magnetism. From this viewpoint the metallic core is the place where the electric currents flow that are the sources of the field. The secular variation of the field is interpreted as a modification of the current system caused by inductive interaction between mechanical motions of the fluid and the magnetic field. This interaction is expressed mathematically by the differential Eq. (32), below, which can be solved by the methods of electromagnetic wave theory.

One need not make any particular assumptions about the magnitude and form of the fluid motions but can try to determine these as far as possible from the analysis of the secular variation of the field. This attitude is no doubt somewhat unsatisfactory as one might expect that information about the character of the motions could be derived from general dynamical principles. Unfortunately, physical hydrodynamics is still a rudimentary subject in many respects. A few remarks on this topic will be found in the appendix immediately following Part II of this paper. We shall also abstain from speculations about the cause of the motions. The magnitude of the velocities derived from the secular variation is indicated below; the corresponding kinetic energy is an exceedingly small quantity when compared to any thermodynamical energy of physical interest. The rate at which this energy is converted into heat by turbulence is probably

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[1] B. Gutenberg, ed., *The Internal Constitution of the Earth* (McGraw-Hill Book Company, Inc., New York, 1939).

extremely slow and the minute power needed to maintain such motions can be supplied by a number of possible processes (radioactive or thermodynamical).

Among the various forces acting upon the fluid in the earth's core there is one that cannot be overlooked; that is the mechanical reaction (magneto-mechanical force) of the electric currents upon the fluid. If, as we shall see later, amplification of the field by the fluid motion can occur then, with sufficiently strong amplification, the field must increase exponentially in time. The "braking" action necessary to slow down and eventually stop such an increase does not arise from the inductive process itself, but will be brought about by the magneto-mechanical forces.

Although the electric conductivity of the core is probably a function of depth, it does not seem plausible that this variation introduces phenomena that modify the theory in a fundamental way, and for the sake of simplicity the conductivity will be assumed constant throughout the core. We shall use a value, $\sigma=10^6$ mhos/meter which is one-tenth of the conductivity of iron under ordinary laboratory conditions. Geochemists have brought forth a considerable amount of plausible argument to the effect that most of the matter in the core is metallic iron. The high temperature decreases, but the compression increases the conductivity, as will be shown in the appendix following Part II where a justification of the numerical value adopted will be given. The results of the theory presented here are not substantially changed, however, should the conductivity be several times smaller or larger than the value assumed.

The magnetic susceptibility of the core will be taken as $\mu=\mu_0$, the susceptibility of space.

We might remark at this point that in the matter of units and dimensions we have consistently adopted a rationalized m.k.s. or Giorgi system, following Stratton.[2]

THE FUNDAMENTAL VECTORS

Maxwell's equations, neglecting the displacement current, are

$$\nabla\times\mathbf{E}+\partial\mathbf{B}/\partial t=0,\quad \nabla\times\mathbf{B}-\mu\sigma\mathbf{E}=0. \tag{1}$$

On introducing the vector potential, **A**, by

$$\mathbf{B}=\nabla\times\mathbf{A},\quad \mathbf{E}=-\partial\mathbf{A}/\partial t,\quad \nabla\cdot\mathbf{A}=0, \tag{2}$$

and using the abbreviation[2]

$$\nabla^2\mathbf{A}=\nabla\nabla\cdot\mathbf{A}-\nabla\times\nabla\times\mathbf{A}, \tag{3}$$

one obtains in the usual way the differential equation

$$\nabla^2\mathbf{A}-\mu\sigma\,\partial\mathbf{A}/\partial t=0. \tag{4}$$

Outside the conducting sphere this reduces to

$$\nabla^2\mathbf{A}=0. \tag{5}$$

Numerically, we have in the m.k.s. system with the value of σ given before

$$\mu\sigma=1.26 \text{ sec./meter}^2.$$

It is a well-known fact that freely decaying currents can exist in a conducting sphere,[3] and these will be derived below. On putting

$$\mathbf{A}=\mathbf{A}_0e^{-\Lambda t},\quad \Lambda\mu\sigma=k^2, \tag{6}$$

Eq. (4) reduces to

$$\nabla^2\mathbf{A}_0+k^2\mathbf{A}_0=0, \tag{7}$$

which is identical in form with the familiar wave equation.

In order to obtain solutions of (7) we apply a method due, in this form, to Hansen.[4,5] In the field of terrestrial magnetism similar methods have been used to treat induction in the earth's crust.[6] Let ψ be a scalar subject to the wave equation

$$\nabla^2\psi+k^2\psi=0. \tag{8}$$

There are three solutions of the vectorial wave Eq. (7) corresponding to any given solution of (8). They can be constructed as follows. The first solution is

$$\mathbf{U}=R\nabla\psi, \tag{9}$$

where R is a constant of the dimension of a length which will be taken later as the radius of

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[2] J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Company, Inc., New York, 1941), Chap. 1.

[3] W. R. Smythe, *Static and Dynamic Electricity* (McGraw-Hill Book Company, Inc., New York, 1939), Chap. 11.

[4] W. W. Hansen, Phys. Rev. **47**, 139 (1935).

[5] See reference 2, Chap. 7.

[6] S. Chapman and T. T. Whitehead, Trans. Camb. Phil. Soc. **22**, 463 (1923); S. Chapman and A. T. Price, Phil. Trans. **229**, 427 (1930); A. T. Price, Proc. London Math. Soc. **31**, 217 (1930); **33**, 233 (1932); B. N. Lahiri and A. T. Price, Phil. Trans. **237**, 509 (1938).

the earth's core. Since (9) is an irrotational vector it is not represented in the magnetic field. The second solution is

$$\mathbf{T} = \nabla \times \mathbf{r}\psi = \nabla\psi \times \mathbf{r}, \tag{10}$$

where $\mathbf{r}$ designates the radius vector from the origin. The third solution, finally, is

$$\mathbf{S} = R\nabla \times \nabla \times \mathbf{r}\psi. \tag{11}$$

The last two solutions are connected by the identities[5]

$$\mathbf{S} = R\nabla \times \mathbf{T}, \quad \mathbf{T} = (Rk^2)^{-1}\nabla \times \mathbf{S}. \tag{12}$$

It should be noted that vectors of the types (9)–(11) may be derived from an arbitrary scalar, not necessarily fulfilling the wave Eq. (8). The second relation (12) no longer holds in the general case. We shall, however, not use completely general expressions for ψ but shall always assume the dependence upon the polar angles in the form of a spherical harmonic. Fundamental vectors of this type will be used to represent the fluid motion.

We shall now introduce names for these three types of vectors. They will be designated as *scaloidal* ($\mathbf{U}$), *toroidal* ($\mathbf{T}$), and *poloidal* ($\mathbf{S}$) vector fields. The electric field and vector potential pertaining to a poloidal magnetic field are toroidal, and *vice versa.*

On writing down components in spherical coordinates one finds

$$\mathbf{T}_{(r)} = 0, \quad \mathbf{T}_{(\theta)} = (\sin\theta)^{-1}\,\partial\psi/\partial\varphi, \quad \mathbf{T}_{(\varphi)} = -\partial\psi/\partial\theta, \tag{13}$$

and

$$\mathbf{S}_{(r)} = -(R/r\sin\theta)\left\{\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial\psi^2}{\partial\varphi^2}\right\} = n(n+1)R\psi/r \quad \text{(for } \psi \sim n\text{th order harmonic)}, \tag{14}$$

$$\mathbf{S}_{(\theta)} = (R/r)\partial^2(r\psi)/\partial r\partial\theta,$$

$$\mathbf{S}_{(\varphi)} = (R/r\sin\theta)\partial^2(r\psi)/\partial r\partial\varphi.$$

The function ψ from which these vectors are derived fulfills, in the case of the electromagnetic field vectors, the wave equation (8). Accordingly we introduce[5] a system of fundamental solutions of (8) whose component functions are Bessel functions of r and spherical harmonic functions of θ and φ.

$$\psi_{sn}{}^m = \text{const}\cdot r^{-\frac{1}{2}}J_{n+\frac{1}{2}}(k_{sn}r)\,Y_n{}^m(\theta, \varphi). \tag{15}$$

As will be shown in a later section, the k_{sn} are determined by the condition

$$J_{n-\frac{1}{2}}(k_{sn}R) = 0, \tag{16}$$

where R is the radius of the sphere considered (radius of the earth's core).

ORTHOGONALITY AND NORMALIZATION. COMPLEX VECTORS

This section contains some mathematical enlargements on the subject of the orthogonality of the fundamental vectors and some related problems which will be found useful later on. The normalization of the toroidal vectors is given in formulas (21)–(24). Thereafter the more physical aspects of the problem will be resumed in the next section.

Orthogonality

The scalar functions (15) form an orthogonal set for the interior of the sphere of radius R. The vectors $\mathbf{U}$, $\mathbf{T}$, $\mathbf{S}$ derived from ψ in the way described also have notable properties of orthogonality.[5] We shall in the first line be interested in orthogonality on integration over the polar angles θ and φ.

It is convenient to introduce temporarily an operator ∇_2 that is the projection of the operator ∇ upon the surface of the unit sphere. Thus, if Y is a scalar function of θ and φ

$$(\nabla_2 Y)_{(\theta)} = \partial Y/\partial\theta, \quad (\nabla_2 Y)_{(\varphi)} = (\sin\theta)^{-1}\partial Y/\partial\varphi,$$

and if $\mathbf{X}$ is a two-dimensional vector in the surface of the sphere

$$\nabla_2\cdot\mathbf{X} = (\sin\theta)^{-1}\partial(\sin\theta\,\mathbf{X}_{(\theta)})/\partial\theta + (\sin\theta)^{-1}\partial\mathbf{X}_{(\varphi)}/\partial\varphi$$

and identically for any vector $\mathbf{X}$, on integrating over the sphere

$$\int \nabla_2\cdot\mathbf{X}\,d\sigma = 0 \tag{17}$$

which is the two-dimensional form of Gauss' theorem applied to a closed surface. From (17) we can derive two-dimensional cases of Green's formulas, for instance (where α and β stand for two sets of indices)

$$0=\int \nabla_2\cdot(Y(\alpha)\nabla_2 Y(\beta))d\sigma$$
$$=\int \nabla_2 Y(\alpha)\cdot\nabla_2 Y(\beta)d\sigma$$
$$-n_\beta(n_\beta+1)\int Y(\alpha)Y(\beta)d\sigma. \quad (18)$$

Similarly one obtains the identity which will be used later

$$\int Y(\alpha)\nabla_2 Y(\beta)\cdot\nabla_2 Y(\gamma)d\sigma$$
$$+\int Y(\beta)\nabla_2 Y(\alpha)\cdot\nabla_2 Y(\gamma)d\sigma$$
$$=n_\gamma(n_\gamma+1)\int Y(\alpha)Y(\beta)Y(\gamma)d\sigma. \quad (19)$$

Later on we shall make use of complex spherical harmonics as defined in (22). These are orthogonal in the Hermitian sense; i.e., when the asterisk designates the conjugate complex quantity

$$\int Y(\alpha)Y^*(\beta)d\sigma=0, \quad \text{if } \alpha\neq\beta.$$

On applying the identity (18) one can show that any two different toroidal vectors are orthogonal. One finds readily from (13) and (15)

$$\int \mathbf{T}(\alpha)\cdot\mathbf{T}^*(\beta)d\sigma=f(r)\int \nabla_2 Y(\alpha)\cdot\nabla_2 Y^*(\beta)d\sigma$$
$$=f(r)n_\beta(n_\beta+1)\int Y(\alpha)Y^*(\beta)d\sigma, \quad (20)$$

where $f(r)$ stands as an abbreviation for the product of the two radial functions.

The orthogonality of the radial functions follows from the formula proved in the theory of Bessel functions,

$$(k_\alpha^2-k_\beta^2)\int J_{n_\alpha+\frac{1}{2}}(k_\alpha r)J_{n_\beta+\frac{1}{2}}(k_\beta r)rdr$$
$$=k_\beta rJ_{n_\alpha+\frac{1}{2}}(k_\alpha r)J_{n_\beta-\frac{1}{2}}(k_\beta r)$$
$$-k_\alpha rJ_{n_\alpha-\frac{1}{2}}(k_\alpha r)J_{n_\beta+\frac{1}{2}}(k_\beta r)$$

in conjunction with (16).

The orthogonality of any two different poloidal vectors, **S**, with respect to integration over the angles θ and φ can be proved by a procedure analogous to the one just applied. Orthogonality of the radial functions of **S** will not be of importance later on so that we need not enter into an analysis of the behavior of these functions which is somewhat more complicated. The orthogonality of two different scaloidal vectors, **U**, with respect to integration over the angles θ and φ may also readily be demonstrated.

Any vector **S** is orthogonal to any vector **T**; from (13) and (14) the integral over the sphere has the form

$$\int \mathbf{S}(\alpha)\cdot\mathbf{T}^*(\beta)d\sigma$$
$$=f(r)\int\left[\frac{\partial Y(\alpha)}{\partial\theta}\frac{\partial Y^*(\beta)}{\partial\varphi}-\frac{\partial Y(\alpha)}{\partial\varphi}\frac{\partial Y^*(\beta)}{\partial\theta}\right]\frac{d\sigma}{\sin\theta},$$

and the integral on the right-hand side may be shown to vanish by means of integrations by parts. There is, however, no full orthogonality[5] of the vectors **S** and **U**, but we shall not be concerned with this question later on.

Normalization

The toroidal vectors, **T**, will be normalized with respect to integration over the interior of the conducting sphere:

$$\int \mathbf{T}\cdot\mathbf{T}^*dV=1. \quad (21)$$

The vectors **S** are derived from **T** by (12), and are therefore in general not normalized simultaneously with the **T**. We shall use complex spherical harmonics and set:

$$Y_n^m(\theta,\varphi)=\left[\frac{2n+1}{4\pi n(n+1)}\cdot\frac{(n-m)!}{(n+m)!}\right]^{\frac{1}{2}}$$
$$\times P_n^m(\cos\theta)e^{im\varphi}. \quad (22)$$

Here, m goes from $-n$ to $+n$. The normalization factor appearing here is slightly different from the one used to normalize the scalar functions, in order to take account of the factor $n(n+1)$ in (20). The radial functions, normalized to unity, are

$$Z_n(k_s r) = 2^{\frac{1}{2}} R^{-1} [J_{n+\frac{1}{2}}(k_s R)]^{-1} r^{-\frac{1}{2}} J_{n+\frac{1}{2}}(k_s r), \quad (23)$$

and thus the normalized function ψ becomes

$$\psi_{sn}{}^m = Z_n(k_s r) Y_n{}^m(\theta, \varphi). \quad (24)$$

It may be noted here that for $n=0$ the vectors **T** and **S** vanish, so that the dipole, $n=1$ is the lowest spherical harmonic that appears in these vectors.

Complex Vectors; Symmetry

The use of complex quantities in electromagnetic theory is of course commonplace, but in view of the special use which will be made of them below, some remarks about Hermitian symmetry might seem justified. The relation between a complex vector and its real counterparts may be written symbolically

$$\mathbf{T}_n{}^{\pm m} = \mathbf{T}_n{}^{mc} \pm i \mathbf{T}_n{}^{ms},$$

where the upper indices c and s refer to the *cosine* and *sine* functions, respectively. If an arbitrary toroidal vector is developed into a series, $\sum c_n{}^m \mathbf{T}_n{}^m$, the relation between the real and complex coefficients is

$$c_n{}^{\pm m} = c_n{}^{mc} \mp i c_n{}^{ms}.$$

We shall, in particular, need developments of vectors of the form $\mathbf{Q}\mathbf{T}_{n'}{}^{m'}$ where $\mathbf{Q}$ is a (vectorial) operator. The coefficients of development are of the form

$$c(n, m; n'm') = F \int Y_n{}^{*m} \mathbf{Q} Y_{n'}{}^{m'} d\sigma,$$

where F designates the integral over the radial functions. These coefficients are linear combinations of four coefficients of the corresponding real vectors, which are readily derived from the preceding formulas. These coefficients c form a matrix, and the symmetry properties of such a matrix within the Hermitian symmetry will turn out to be of considerable importance. Within the Hermitian symmetry a matrix M will be called *symmetrical* when

$$M(\alpha, \beta) = M^*(\beta, \alpha)$$

and *antisymmetrical* when

$$M(\alpha, \beta) = -M^*(\beta, \alpha),$$

where α stands for n,m and β for n',m'. To any antisymmetrical matrix there is related a symmetrical matrix by

$$M_{\text{sym}} = i M_{\text{ant}}. \quad (25)$$

The eigenvalues of a symmetrical matrix are all real numbers; consequently the eigenvalues of an antisymmetrical matrix are all purely imaginary numbers.

FREE MODES

We shall now determine the freely decaying current modes in a conducting sphere.[3] The vector potential is a toroidal vector, and accordingly we set in the *interior* of the sphere, for an individual mode

$$(\mathbf{A}_{sn}{}^m)^{(i)} = c_{sn}{}^m (\mathbf{T}_{sn}{}^m)^{(i)} \exp(-\Lambda_{sn} t), \quad (26)$$

where, by (6)

$$\Lambda_{sn} = k_{sn}{}^2 / \mu\sigma. \quad (27)$$

The generating scalar is given by (24); the $c_{sn}{}^m$ are constants. Similarly we set in the space *external* to the sphere

$$(\mathbf{A}_{sn}{}^m)^{(e)} = C_{sn}{}^m (\mathbf{T}_{sn}{}^m)^{(e)} \exp(-\Lambda_{sn} t), \quad (28)$$

where $\mathbf{T}^{(e)}$ is derived from a scalar

$$\psi_{sn}{}^m = r^{-n-1} Y_n{}^m(\theta, \varphi).$$

The magnetic and electric field are given by (2); for the internal field they are (on dropping for simplicity the index i)

$$\left. \begin{aligned} \mathbf{B}_{sn}{}^m &= (c_{sn}{}^m / R) \mathbf{S}_{sn}{}^m \exp(-\Lambda_{sn} t) \\ \mathbf{E}_{sn}{}^m &= c_{sn}{}^m \Lambda_{sn} \mathbf{T}_{sn}{}^m \exp(-\Lambda_{sn} t) \end{aligned} \right\},$$

and similar expressions for the external fields.

The boundary conditions require continuity of

$$\mathbf{B}_{(r)}, \quad \mathbf{B}_{(\theta)}/\mu, \quad \mathbf{B}_{(\varphi)}/\mu, \quad \mathbf{E}_{(\theta)}, \quad \mathbf{E}_{(\varphi)}$$

at the boundary of the sphere, $r=R$. $\mathbf{E}_{(r)}$ vanishes according to (13) so that no boundary condition for this component is required. We assume that there is no discontinuity of μ at the boundary. On writing down the relations expressing equality of the internal and external components of the field at the boundary, one is led to the condition

$$\partial/\partial r [r^{\frac{1}{2}} J_{n+\frac{1}{2}}(k_{sn} r)]_{r=R} - n r^{-\frac{1}{2}} J_{n+\frac{1}{2}}(k_{sn} R) = 0. \quad (29)$$

This is the characteristic equation whose roots are the k_{sn}; after some transformation it can be put into the simple form (16). Thus we have, using (27),

$$J_{n-\frac{1}{2}}(x_{sn})=0, \quad k_{sn}=x_{sn}/R, \quad \Lambda_{sn}=x_{sn}^2/R^2\mu\sigma. \tag{30}$$

For $n=1$, the roots of (30) are $x_s=s\pi$. The longest period of decay for the earth's core ($R=3.5\times10^6$ meters) is

$$(\Lambda_{11})^{-1}\sim 50{,}000 \text{ years}$$

with the value of σ adopted here. The length of this period is proportional to σ. An appropriate physically significant average over the lowest decay periods will be several times smaller than this value and will therefore be of the order of some tens of thousands of years. As will be shown in Part II, the intervals of time characteristic of the secular variation of the earth's field (more precisely, the periods of its predominant Fourier components) are of the order of magnitude of a few hundred years. Hence the periods of free decay are (for the lower spherical harmonics) usually large compared to the periods of the secular variation.

The solutions given here, representing a slowly decaying magnetic field, are distinct from the well-known "magnetic" modes of electromagnetic oscillations of a sphere. The difference resides not so much in the fact that the displacement current is neglected here, but in the boundary conditions at infinity which require that the field vanishes in our case, whereas in the case of oscillations an outgoing wave is required. The lowest oscillatory modes correspond to wave-lengths that are comparable with the diameter of the core; the frequencies are of the order of some ten cycles per second.

A second system of solutions may formally be constructed by assuming **B** toroidal, **A** and **E** poloidal (corresponding to the "electric" modes of oscillations). If one tries to satisfy the boundary conditions, however, it appears that they cannot be simultaneously fulfilled. One is probably justified in concluding that exponentially damped, free toroidal modes cannot exist inside a conducting sphere. It is perhaps premature to infer that no magnetic field of the toroidal type can exist or can be generated in the sphere by agents operating *inside* it. In this paper the possibility of a toroidal magnetic field is not considered, but the subject seems to be open to further investigation.

THE DIFFERENTIAL EQUATION OF INDUCTION

We shall now proceed to the analysis of the effects of fluid motion upon the magnetic field. If the motion is described by a velocity vector **v**, the second of Maxwell's Eqs. (1) is replaced by[7]

$$\nabla\times\mathbf{B}-\mu\sigma\mathbf{E}=\mu\sigma\mathbf{v}\times\mathbf{B}. \tag{31}$$

On retaining (2) for the vector potential we have

$$\nabla^2\mathbf{A}-\mu\sigma\,\partial\mathbf{A}/\partial t=-\mu\sigma\mathbf{v}\times(\nabla\times\mathbf{A}). \tag{32}$$

Before entering into the mathematical analysis we may obtain an estimate of the order of magnitude of the velocities required for an effective inductive action. A figure representing a lower limit may be gained by setting the term on the right-hand side of (32) equal to the value that each of the terms on the left-hand side has for the lowest free mode. This gives for the order of magnitude

$$\mathbf{v}\sim L\Lambda_{11},$$

where L has the dimension of a length. On putting $L=10^6$ meters,

$$\mathbf{v}\sim 6.5\times10^{-7}\text{ m/sec.}\sim 0.2\text{ mm/hour.}$$

Since, however, the periods of the secular variation of the magnetic field are, in general, several hundred times shorter than the lowest period of free decay, the actual velocities prevailing in the core will be several hundred times larger than this value, or of the order of several centimeters per hour.

In order to integrate Eq. (32) we develop the vector potential

$$\mathbf{A}=\textstyle\sum_\gamma c_\gamma(t)\mathbf{T}_\gamma, \tag{33}$$

where the symbol γ stands as an abbreviation for the triple s, n, m. On introducing this into (32), multiplying with $\mathbf{T}_\gamma{}^*$ and integrating over the interior of the conducting sphere there follows by (21) and (27), since **T** fulfills the wave

[7] The field equations for this case may be derived from the field equations for moving systems (Minkowski's equations) to be found in textbooks on relativity.

Eq. (7),

$$\Lambda_\gamma c_\gamma + dc_\gamma/dt = \sum_\beta c_\beta \int \mathbf{v} \times (\nabla \times \mathbf{T}_\beta) \cdot \mathbf{T}_\gamma^* dV.$$

This will be re-arranged to read

$$dc_\gamma/dt + \Lambda_\gamma c_\gamma = \sum_\beta c_\beta R^{-1} \int \mathbf{v} \cdot \mathbf{S}_\beta \times \mathbf{T}_\gamma^* dV. \quad (34)$$

The physical behavior of the solutions is determined by the character of the *coupling matrix* on the right-hand side. In general the velocity $\mathbf{v}$ will be a function of time and the integration of the system of differential Eqs. (34) is then quite difficult. If the fluid motion is stationary so that $\mathbf{v}$ is independent of time, the matrix elements become constants. The integration can then be performed by introducing normal coordinates, i.e., linear combinations of the c's which transform the coupling matrix into a diagonal matrix while retaining the diagonal form of the left-hand side. The transformation to normal coordinates will be taken up in the last section.

THE COUPLING MATRIX

In studying the coupling matrix we will have reference to symmetry of the Hermitian type as explained previously. We now develop the velocity field in a series

$$\mathbf{v} = \sum_\alpha (v_\alpha \mathbf{S}(\alpha) + w_\alpha \mathbf{T}(\alpha) + u_\alpha \mathbf{U}(\alpha)). \quad (35)$$

The components $\mathbf{U}$ are carried along here for the sake of completeness; they do not vanish since the fluid in the core of the earth is known to be compressible. According to seismic observations the density ρ of the core increases appreciably from the boundary downwards. The equation of continuity, $\nabla \cdot (\rho \mathbf{v}) = 0$ gives readily

$$\nabla \cdot \mathbf{v} = -\mathbf{v}_{(r)} \partial\rho / \rho \partial r.$$

Since $\nabla \cdot \mathbf{T} = \nabla \cdot \mathbf{S} = 0$, this shows that the $\mathbf{U}$-components actually appear in the series (35). We shall, however, assume that the effects of compressibility are small and shall confine our attention primarily to the $\mathbf{T}$ and $\mathbf{S}$ components of the velocity.

Now introduce the abbreviation

$$\int \mathbf{v} \cdot \mathbf{S} \times \mathbf{T}^* dV = [\mathbf{v} \cdot \mathbf{S} \times \mathbf{T}^*]. \quad (36)$$

There are three types of elements of the coupling matrix, namely,

$$\begin{gathered} [\mathbf{T}(\alpha) \cdot \mathbf{S}(\beta) \times \mathbf{T}^*(\gamma)], \\ [\mathbf{S}(\alpha) \cdot \mathbf{S}(\beta) \times \mathbf{T}^*(\gamma)], \\ [\mathbf{U}(\alpha) \cdot \mathbf{S}(\beta) \times \mathbf{T}^*(\gamma)], \end{gathered} \quad (37)$$

which will be considered in turn. Here and later, (α) refers to the fluid motion, (β) to the "primary" magnetic field, and (γ) to the "secondary" or induced magnetic field.

It is convenient to use uniform mathematical expressions for the three vectors involved in each of the matrix elements (37); thus all vectors will be derived from generating functions of the form (24). This imposes an unnecessary restriction upon the velocity vector since the scalar (24) is subject to the wave Eq. (8), whereas no such condition is required for the velocity field. It is convenient to use the same spherical harmonics for the angle functions of the velocity vector as for the electromagnetic field, but the radial functions of the velocity vector are not subject to the Bessel differential equation. Notwithstanding their outward similarity to the field components these radial functions, $Z(\alpha)$, are to be considered as arbitrary functions, vanishing at the boundary.

Toroidal Flow

We consider the first of the three types of matrix elements in which the velocity is a toroidal vector. On writing this element out in spherical coordinates we find from (13) and (14)

$$\begin{aligned} [\mathbf{T}(\alpha) \cdot \mathbf{S}(\beta) \times \mathbf{T}^*(\gamma)] &= [\mathbf{S}(\beta) \cdot \mathbf{T}^*(\gamma) \times \mathbf{T}(\alpha)] \\ &= -n_\beta(n_\beta+1) F \int Y(\beta) \left[\frac{\partial Y(\alpha)}{\partial \theta} \frac{\partial Y^*(\alpha)}{\partial \varphi} - \frac{\partial Y(\alpha)}{\partial \varphi} \frac{\partial Y^*(\alpha)}{\partial \theta} \right] \frac{d\sigma}{\sin\theta} \\ &= n_\beta(n_\beta+1) F \left[i m_\gamma \int \frac{\partial Y(\alpha)}{\partial \theta} Y(\beta) Y^*(\gamma) \frac{d\sigma}{\sin\theta} + i m_\alpha \int Y(\alpha) Y(\beta) \frac{\partial Y^*(\gamma)}{\partial \theta} \right] \frac{d\sigma}{\sin\theta}, \end{aligned} \quad (38)$$

where

$$F=R\int Z(\alpha)Z(\beta)Z(\gamma)r^2dr. \tag{39}$$

Now we must have

$$m_\alpha+m_\beta=m_\gamma, \tag{40}$$

otherwise the integrals vanish ("selection rule" for the index m). The first integral in (38) may be transformed by means of an integration by parts; if $x=\cos\theta$ we have

$$\int\frac{\partial Y(\alpha)}{\partial x}Y(\beta)Y^*(\gamma)dx=|Y(\alpha)Y(\beta)Y^*(\gamma)|_{-1}^{+1}$$
$$-\int Y(\alpha)\frac{\partial Y(\beta)}{\partial x}Y^*(\gamma)dx-\int Y(\alpha)Y(\beta)\frac{\partial Y^*(\gamma)}{\partial x}dx.$$

It is seen from (38) that the integral considered only appears when $m_\gamma\neq 0$; in this case $Y^*(\gamma)=0$ at the boundaries, so that the contribution of the boundaries in the last formula vanishes. By use of this formula and (40) the matrix element can be written

$$[\mathbf{T}(\alpha)\cdot\mathbf{S}(\beta)\times\mathbf{T}^*(\gamma)]=n_\beta(n_\beta+1)F$$
$$\cdot 2\pi i\left[m_\gamma\int Y(\alpha)\frac{\partial Y(\beta)}{\partial x}Y^*(\gamma)dx\right.$$
$$\left.+m_\beta\int Y(\alpha)Y(\beta)\frac{\partial Y^*(\gamma)}{\partial x}dx\right]. \tag{41}$$

The factor $n_\beta(n_\beta+1)$ which disturbs the symmetry of this formula can be made symmetrical by the substitution, in the equations of motion (34), of new coefficients

$$d_\gamma=[n_\gamma(n_\gamma+1)]^{\frac{1}{2}}c_\gamma,$$

whereby this factor changes to

$$[n_\beta(n_\beta+1)n_\gamma(n_\gamma+1)]^{\frac{1}{2}}.$$

The matrix composed of the elements (41) then becomes antisymmetrical in the Hermitian sense: when the indices β and γ of the primary and secondary field are interchanged the matrix elements (which are purely imaginary) remain unchanged, i.e., they change into the negative of their conjugate complex. We thus have the result: *For toroidal fluid motion the coupling matrix is antisymmetrical.*

The matrix element (41) has some other properties of interest. It vanishes unless the integrands are even functions of $x=\cos\theta$. Since Y_n^m is a polynomial in x of order $(n-m)$, the integrands are polynomials in x of the order

$$n_\alpha+n_\beta+n_\gamma-m_\alpha-m_\beta-m_\gamma-1.$$

By virtue of (40) this order is even or odd according to whether $n_\alpha+n_\beta+n_\gamma$ is odd or even. Hence: *In the matrix elements of toroidal flow either two of the indices n are even and one is odd, or all three are odd.*

There are other selection rules for the indices n which we shall mention without going into details. These rules may be illustrated by the special case where $Y(\alpha)=P_1(x)=x$. It can then be shown from the recurrence relations of the spherical harmonics, and by taking account of the rule just stated, that the only non-vanishing matrix elements are those where $n_\beta=n_\gamma$. Similarly, when for instance $Y(\alpha)=P_2(x)$, the only non-vanishing elements are those where $n_\beta=n_\gamma\pm 1$. We need not elaborate on these and similar relations here.

Poloidal Flow

We consider now the second type of the matrix elements (37) where the velocity vector is poloidal. On writing the element out in spherical coordinates we find by (13) and (14)

$$[\mathbf{S}(\alpha)\cdot\mathbf{S}(\beta)\times\mathbf{T}^*(\gamma)]$$
$$=-n_\alpha(n_\alpha+1)G_\beta\int Y(\alpha)\nabla_2 Y(\beta)\cdot\nabla_2 Y^*(\gamma)d\sigma$$
$$+n_\beta(n_\beta+1)G_\alpha\int Y(\beta)\nabla_2 Y(\alpha)\cdot\nabla_2 Y^*(\gamma)d\sigma, \tag{42}$$

where

$$G_\alpha=R^2\int\left[\frac{d}{dr}rZ(\alpha)\right]Z(\beta)Z(\gamma)dr, \tag{43}$$

and similarly for G_β. Applying the identity (19) to the second integral in (42) it is found that the matrix element can be written

$$[\mathbf{S}(\alpha)\cdot\mathbf{S}(\beta)\times\mathbf{T}^*(\gamma)]=-[n_\alpha(n_\alpha+1)G_\beta$$
$$+n_\beta(n_\beta+1)G_\alpha]\int Y(\alpha)\nabla_2 Y(\beta)\cdot\nabla_2 Y^*(\gamma)d\sigma$$
$$+n_\beta(n_\beta+1)n_\gamma(n_\gamma+1)\int Y(\alpha)Y(\beta)Y^*(\gamma)d\sigma. \tag{44}$$

Here, everything is symmetrical with respect to the interchange of the indices β and γ except the bracket multiplying the first integral over the angles. Some additional manipulation shows that neither the symmetrical nor the antisymmetrical part of this bracket is, in general, zero. Hence: *For poloidal fluid motion the coupling matrix is in general neither purely symmetrical nor purely antisymmetrical.*

Clearly, the selection rule (40) applies again. A further rule is obtained, as above, from the consideration that the integrals in (44) vanish unless the integrand is an even function of $\cos\theta$. By a similar argument as before it is found that the integrand is even when $n_\alpha+n_\beta+n_\gamma$ is even. Hence: *In the matrix elements of poloidal flow either two of the indices n are odd and one is even, or all three are even.*

Other selection rules for the indices n can be derived analogous to those given for toroidal flow. If for instance $Y(\alpha)=P_1(x)=x$, the only non-vanishing matrix elements are those where $n_\beta=n_\gamma\pm1$. Again when $Y(\alpha)=P_2(x)$ the non-vanishing matrix elements have either $n_\beta=n_\gamma$ or $n_\beta=n_\gamma\pm2$. We need not elaborate on those rules here.

Scaloidal Flow

The third type of matrix element, finally, where the velocity is a scaloidal vector is very similar to the type where the velocity is poloidal. The (θ, φ) functions of a scaloidal vector are identical with those of the corresponding poloidal vector, only the radial components are different. Hence the matrix elements of $\mathbf{U}(\alpha)$ contain the same integrals over spherical harmonics as (42) or (44); only the radial integrals G_α and G_β are replaced by somewhat different expressions. Again, the matrix elements are neither purely symmetrical nor purely antisymmetrical. The selection rules, depending only on the spherical harmonic functions, are the same as in the poloidal case.

A table of values of some of the lower matrix elements will be found in Part II.

INTEGRATION OF EQS. (34)

The discussion will be limited to the case where the fluid motion is stationary so that the coupling matrix on the right-hand side of (34) reduces to a set of constants.

Consider the system of differential equations

$$dc_\gamma/dt+\textstyle\sum_\beta M(\beta,\gamma)c_\beta=0. \tag{45}$$

The integration of this system offers no difficulties when the matrix M is either symmetrical or antisymmetrical. In this case a suitable linear transformation of the c's will make M diagonal, i.e., will lead to a system of new "normal modes" of the problem. It is furthermore known that in the Hermitian symmetry the transformation can always be assumed *unitary*. The unitary character may be expressed as a relationship between a transformation and its inverse, thus

$$g_\delta=\textstyle\sum_\gamma u_{\delta\gamma}c_\gamma,\quad c_\gamma=\sum_\delta u_{\delta\gamma}{}^*g_\delta. \tag{46}$$

If by this substitution the system (45) is transformed to normal coordinates, the eigenvalues of the matrix M being λ_δ, we have

$$dg_\delta/dt+\lambda_\delta g_\delta=0,\quad g_\delta=g_\delta(0)\exp(-\lambda_\delta t). \tag{47}$$

One must now distinguish between the cases where the matrix M is symmetrical and where it is antisymmetrical. In the Hermitian symmetry there is associated to every antisymmetrical matrix a symmetrical one by the relation (25). The eigenvalues of a symmetrical (Hermitian) matrix are real; hence the eigenvalues of an antisymmetrical matrix are purely imaginary.

It follows that in the case of an antisymmetrical matrix M in (45) the coefficients of the new normal modes are purely harmonic functions of time, so that

$$|g_\delta|^2=\text{const.}$$

It will be useful to derive a related result in a more direct way. If one multiplies (45) with $c_\gamma{}^*$ and adds to the formula obtained its conjugate complex there comes

$$d/dt|c_\gamma|^2=\textstyle\sum_\beta M(\beta,\gamma)c_\beta c_\gamma{}^*+\sum_\beta M^*(\beta,\gamma)c_\gamma c_\beta{}^*$$

and when M is antisymmetrical, i.e.,

$$M(\beta,\gamma)=-M^*(\gamma,\beta),$$

one finds, by summing over γ, that

$$d/dt[\textstyle\sum_\gamma|c_\gamma|^2]=0, \tag{48}$$

(convergence of all the bilinear forms involved being assumed). While this requirement is much weaker than the preceding one, it is also more general in that it does not presuppose a transformation to principal axes.

The equations of motion (34) are of the form (45) with

$$M(\beta, \gamma)=K(\beta, \gamma)+\Lambda_\gamma\delta(\beta, \gamma), \qquad (49)$$

where $\delta(\beta, \gamma)=1$ for $\beta=\gamma$ and $\delta(\beta, \gamma)=0$ for $\beta\neq\gamma$; and where K is the coupling matrix discussed in the last section. We have seen that when the velocity field is a toroidal vector (or a sum of toroidal vectors), K is an antisymmetrical matrix. If now the velocity is large, K becomes large and the term with Λ in (49) may be neglected in a first approximation. Then M is purely antisymmetrical and the results stated above apply. The amplitudes of the new normal modes carry out harmonic oscillations in time. If, from the complex, Hermitian representation of the problem we go to a representation where the fundamental vectors are real, the corresponding real amplitudes carry out sinusoidal oscillations.

If now we admit a finite value for the Λ's in (49), the matrix M is no longer either antisymmetrical or symmetrical. It is known that in this case the existence of a transformation of the matrix M to diagonal form is not assured. In the case of a matrix without special symmetry it can therefore not, as a rule, be assumed that normal modes exist. The integration of the field equations in a simple, closed form is then not possible and one must be content with less stringent results. One such result may be derived as follows. By the same procedure that previously led to Eq. (48) it is found that the bilinear form corresponding to the antisymmetrical matrix K vanishes, and so

$$d/dt[\textstyle\sum_\gamma|c_\gamma|^2]=-\sum_\gamma\Lambda_\gamma|c_\gamma|^2<0, \qquad (50)$$

since the Λ's are essentially positive. Hence the over-all amplitude of the field decreases continually. In physical terms the result may be stated as follows, assuming the Λ's to be small: ***For toroidal flow the induction effect consists in oscillatory changes of the field amplitudes superposed upon the slow, general decay of the field.***

We now consider the case where the coupling matrix K is symmetrical. In this case the matrix M of the Eqs. (45) is also symmetrical. The eigenvalues λ_δ of M are all real. Hence the coefficients of the new normal modes given by (47) are now real exponential functions of the time. The eigenvalues may be either positive or negative. When the coupling matrix K is small, the eigenvalues will be near the "free" ones which are Λ and the exponents (47) will then all be negative. When, however, K is large the eigenvalues are near to those of K alone. Since all elements of K are proportional to the magnitude of $\mathbf{v}$, all eigenvalues change their sign when the direction of the velocity vector is reversed. Physically, the positive exponentials represent amplification and the negative exponentials de-amplification of the corresponding normal mode. Thus, the normal modes corresponding to negative λ's (positive exponents) increase without bound while the modes corresponding to positive λ's decay to zero.

Unfortunately there is no simple type of fluid motion for which the coupling matrix is purely symmetrical. As we have seen, the matrix of poloidal flow contains both symmetrical and antisymmetrical components. One can think of eliminating the antisymmetrical part by forming a linear combination of vectors of different types, but such a velocity field would have little physical significance and be of a very complicated geometrical form.

An interesting illustration of the fact that the coupling matrix of poloidal flow is not simply symmetrical is found in a paper by Cowling.[8] This author investigates the question of whether the fundamental equations of induction, (32) or (34), can have stationary solutions. In the stationary case our problem would reduce to the set of equations

$$\textstyle\sum_\gamma M(\beta, \gamma)c_\gamma=0.$$

This is a system of homogeneous algebraic equations for the coefficients c_γ. Expressed otherwise and in more physical terms, the problem is a boundary value problem for the field Eqs. (32). Cowlings presuppose that the fluid motion is incompressible, has rotational symmetry, and that the velocity vector is contained in the $r-\theta$ plane. In our terminology this corresponds to poloidal flow where only zonal harmonics are involved ($m=0$). Cowling has

[8] T. G. Cowling, M. N. R. A. S. **94**, 39 (1934).

given a formal proof to the effect that the problem in this form does not possess characteristic solutions. If the coupling matrix were symmetrical in the poloidal case, one might indeed be led to expect that characteristic solutions are possible. This goes to show that, even apart from the fact that fluid motion on the rotating earth is intrinsically turbulent, *a stationary model of induction of simple geometrical symmetry cannot exist, on purely electromagnetic grounds.*

When the coupling matrix is neither symmetrical nor antisymmetrical, normal modes do not, as a rule, exist. The theory of integration of the differential equations then becomes far more complicated than in the case of symmetry. One would still expect that amplification or de-amplification of the field components is possible when the coupling matrix has a symmetrical part and when the velocities are large enough. If M_{sym} designates the symmetrical part of M we have, in generalization of (48) and (50),

$$d/dt[\sum_\gamma |c_\gamma|^2] = \sum_{\beta\gamma} M_{\mathrm{sym}}(\beta, \gamma) c_\beta c_\gamma^*. \qquad (51)$$

It may be noted that the choice of the coefficients c in this relation is quite arbitrary, provided only that all the bilinear forms converge. Thus Eq. (51) applies to any magnetic field. When $\mathbf{v}$ is large enough, M is proportional to $\mathbf{v}$, and by the suitable choice of a numerical factor multiplying $\mathbf{v}$ the right-hand side of (51) can be made to have either positive or negative sign and any prescribed magnitude. Also, when the form (51) is sufficiently large at a given instant, a finite time interval must pass before it can change its sign. The results may be summarized as follows: *To any given magnetic field at a given instant a fluid motion can be found which amplifies or de-amplifies this field at a prescribed rate, and continues to amplify or de-amplify it over a finite length of time.*