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Author(s): Edward Bullard and H. Gellman
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HOMOGENEOUS DYNAMOS AND TERRESTRIAL MAGNETISM

By Sir Edward Bullard, F.R.S. and H. Gellman
National Physical Laboratory and University of Toronto

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The main object of the paper is to discuss the possibility of a body of homogeneous fluid acting as a self-exciting dynamo. The discussion is for the most part confined to the solution of Maxwell's equations for a sphere of electrically conducting fluid in which there are specified velocities. Solutions are obtained by expanding the velocity and the fields in spherical harmonics to give a set of simultaneous linear differential equations which are solved by numerical methods. Solutions exist when harmonics up to degree four are included. The convergence of the solutions when more harmonics are included is discussed, but convergence has not been proved. The simultaneous solution of Maxwell's equations and the hydrodynamic equations has not been attempted, but a velocity system has been chosen that seems reasonable from a dynamical point of view. A parameter in the velocity system has been adjusted to satisfy the conservation of angular momentum in a rough way. Orders of magnitude are derived for a number of quantities connected with the dynamo theory of terrestrial magnetism. It is concluded that the dynamo theory does provide a self-consistent account of the origin of the earth's magnetic field and raises no insuperable difficulties in other directions.

1. Introduction

Many problems of geophysics and astronomy involve electromagnetic phenomena in moving conducting media, and much has been written on the subject in recent years (e.g. Cowling 1934; Elsasser 1946 *a, b*, 1947; Schwarzschild 1949; Bullard 1949 *a, b*; Alfvén 1950 *a*; Batchelor 1950; Bullard, Freedman, Gellman & Nixon 1950; Lundquist 1952; Takeuchi & Shimazu 1952 *a, b*). A central problem in this subject is to determine whether there exist motions of a simply connected, symmetrical fluid body which is homogeneous and isotropic that will cause it to act as a self-exciting dynamo and to produce a magnetic field in the absence of any sustaining field from an external source. We call such dynamos 'homogeneous dynamos' to distinguish them from the dynamos of the electrical engineer, which are multiply connected and of low symmetry. The main object of the present paper is to develop a method for deciding whether any specified motion in a fluid sphere will act as a dynamo and to apply it to motions that have been suggested as possible dynamos. It turns out that such dynamos are possible, though the treatment of the convergence of the numerical processes employed is not entirely satisfactory.

The work, which has been in progress for several years, is closely related to that of Elsasser (1946 *a*, *b*, 1947) and of Takeuchi & Shimazu (1952 *a*, *b*). We have corresponded with these authors and have derived great benefit from the exchange of unpublished results with them.

2. An example

The nature of the problem is made clearer by a consideration of the simplest 'engineering' dynamo. In a slightly idealized form, this consists of a copper disk rotating on an axle (figure 1) and surrounded by a coil whose ends are connected to brushes

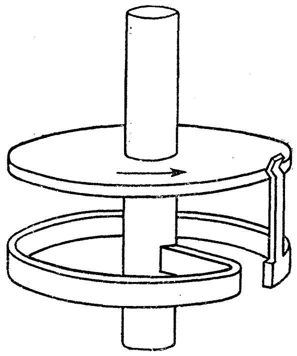


Figure 1. A simple dynamo.

rubbing one on the periphery of the disk and the other on the axle. If a uniform magnetic field H exists parallel to the axis, an electromotive force $\frac{1}{2}Hv_0 a$ will be induced in the circuit where a is the radius of the disk and v_0 the velocity of its periphery. The current I in the disk will be

$$I = \tfrac{1}{2}Hv_0 a/R,$$

where R is the resistance of the circuit. This will produce a field of approximately $2\pi I/a$ or $\pi Hv_0/R$ at the disk. If

$$v_0 = R/\pi, \tag{1}$$

this will be just the field assumed initially, and rotation at this rate will maintain an initial field of arbitrarily chosen intensity. In this argument we have not considered the radial variation of the field due to the coil. If this is taken into account, the constant π in (1) becomes $M/2\pi a$, where M is the mutual inductance of the coil and the periphery of the disk.

If the velocity is less than R/π the dynamo cannot maintain a field, and any that is initially present will collapse if not maintained by an external agency. The variation with time, t, may be shown to be $\exp[(\pi v - R)\, t/L]$, where L is the inductance of the circuit. If the velocity exceeds R/π, the state with no field and no current, though a state of equilibrium, is unstable; an indefinitely small initial field or current will grow exponentially without limit.

The exponential growth is a consequence of specifying the velocity of rotation. If the couple applied to the disk or the power dissipation are specified, there is always a limit to the growth and an equilibrium state with a definite field and velocity R/π. In this state the specified applied couple can just rotate the disk at the critical speed against the electromagnetic forces. At a higher speed the applied couple is too small and the disk decelerates.

Such a machine differs from a homogeneous dynamo in several ways:

(*a*) It is composed of solid parts with rubbing contacts at the brushes, and not of a continuous fluid. This difference is trivial, since the coil brushes and disk can, in imagination, be composed of fluid, the outer part of which can be stationary and the inner part rotate without affecting the electromagnetic problem or violating the hydrodynamic equation of continuity.

(*b*) It is multiply connected, whereas the homogeneous dynamo is singly connected. This difference is also trivial, since the coil and the disk can be joined by a thin sheet of fluid without disturbing the machine.†

(*c*) The coil has the symmetry of a clock face in which the two directions of rotation are not equivalent. It is this feature that causes the current to traverse the coil in such a direction that it produces a field which reinforces the initial field. Simple bodies, such as a sphere, do not have this property; their asymmetry can only lie in the motion. This difference is the crucial one. Does asymmetry of motion suffice for a dynamo, or is asymmetry of structure also necessary?

Alfvén (1950*b*) and Bondi & Gold (1950) have discussed the possibility of homogeneous dynamos from the point of view of the formation of loops in lines of force. The results are a useful guide to what may happen, but are inconclusive as a proof or disproof of the possibility of the process. This is essentially a mathematical question. There is agreement as to the equations governing the problem; the question is: do these equations possess solutions of the required type? The purpose of this paper is to discuss this question, as far as possible, without recourse to physical intuitions. Before entering into the rather detailed analysis necessary for this purpose, some general points will be discussed.

3. Equations to be solved

Maxwell's equations give

$$\operatorname{curl} \boldsymbol{H} = 4\pi \boldsymbol{I} = 4\pi\kappa(\boldsymbol{E} + \boldsymbol{v}_1 \times \boldsymbol{H}), \tag{2}$$

$$\operatorname{curl} \boldsymbol{E} = -\partial \boldsymbol{H}/\partial t, \tag{3}$$

$$\operatorname{div} \boldsymbol{H} = 0, \tag{4}$$

$$\operatorname{div} \boldsymbol{E} = 4\pi q c^2, \tag{5}$$

where $\boldsymbol{H}$ and $\boldsymbol{E}$ are the magnetic and electric fields, $\boldsymbol{I}$ is the current, κ the electrical conductivity and q the volume density of charge, all in electromagnetic units; $\boldsymbol{v}_1$ is the velocity of the fluid and c the velocity of light in cm/s. The displacement current, the Hall current and the magnetic field due to the motion of charge have been omitted, and the permeability and dielectric constant taken as equal to those for a vacuum. The equations are those relevant to a conducting liquid such as the core of the earth, and cannot be applied directly to astrophysical problems in which the mean free path of electrons is large compared to the radius of curvature of their orbits in the magnetic field.

The fields $\boldsymbol{E}$ and $\boldsymbol{H}$ must have no singularities either inside or outside the body. At the boundary of the body, all components of $\boldsymbol{H}$ and the tangential components of $\boldsymbol{E}$ must be continuous, and the normal component of $\boldsymbol{I}$ must vanish. The normal component of $\boldsymbol{E}$ need not be continuous; if it is not, its discontinuity will determine the surface density of

† We owe this observation to Mr T. Gold.

charge. Such a surface density and discontinuity in $\boldsymbol{E}$ may be necessary to render the body as a whole electrically neutral. Outside the body, $\boldsymbol{H}$ must be derivable from a potential and $\boldsymbol{E}$ contain a part derivable from a potential and a part connected with $\partial \boldsymbol{H}/\partial t$ by (3). In a dynamo problem all fields must vanish at infinity at least like $1/r^3$.

The equations (2), (3) and (4), with the boundary conditions, determine the fields if $\boldsymbol{v}_1$ is specified. As they are linear and homogeneous, any solution may be multiplied by a constant factor. This indeterminateness is analogous to that in the disk dynamo. It may be removed by including the hydrodynamic equations with specified forces, and using them to determine the velocities. These equations contain a force $\boldsymbol{I}\times\boldsymbol{H}$ which is quadratic in the fields. The difficulty of the problem is very greatly increased if the hydrodynamic equations are included. In this paper we specify $\boldsymbol{v}_1$ arbitrarily, subject only to the equation of continuity

$$\operatorname{div} \boldsymbol{v}_1 = 0. \tag{6}$$

Here the medium is treated as incompressible, which is unlikely to be a serious restriction. We hope later to extend the work to include the simultaneous solution of the electromagnetic and hydrodynamic equations. This is a formidable and perhaps impracticable task, and will certainly need much time and effort; the main features of this problem are discussed in an approximate way in § 9.

The present work differs fundamentally from that of Alfvén (1950*a*) and Schwarzschild (1949), who include the hydrodynamic equations, but assume the currents induced by the motion to give only a small part of the field, the main part being maintained by an unspecified process. Such arguments have given many interesting results, but they are irrelevant to the dynamo problem, the essence of which is that the whole field is produced by induction, and in which the $\boldsymbol{v}_1\times\boldsymbol{H}$ term of (2) cannot be regarded as small.

Eliminating $\boldsymbol{E}$ from (2) and (3) gives

$$\nabla^2 \boldsymbol{H} = 4\pi\kappa[\partial \boldsymbol{H}/\partial t - \operatorname{curl}(\boldsymbol{v}_1\times\boldsymbol{H})], \tag{7}$$

where, in co-ordinates other than Cartesian, ∇^2 is to be regarded as an abbreviation for grad div $-$ curl^2. Analytically the problem of the existence of homogeneous dynamos is to determine whether there are any vector fields $\boldsymbol{v}_1$ satisfying (6), for which (4) and (7) have solutions without singularities which satisfy the boundary conditions and which do not decrease to zero with the passage of time. Such solutions might be steady, with $\partial\boldsymbol{H}/\partial t = 0$ in (7), or unsteady. If $\boldsymbol{v}_1$ and $V\boldsymbol{v}_1$ are two similar steady velocity fields of different magnitude (V is a scalar constant), it is obvious from (7) that the $\boldsymbol{H}$'s corresponding to them cannot be similar. In fact it appears likely, and is proved in § 5, that (7) can have steady solutions satisfying the boundary conditions for at most a discrete set of V. The problem of the steady homogeneous dynamo is therefore that of selecting a suitable velocity field and finding a value for its magnitude such that (4) and (7) have a steady solution satisfying the boundary conditions. This is analogous to the behaviour of the disk dynamo, which gives a steady field for a single value, R/π, of the peripheral velocity.

Let lengths in (7) be measured in terms of a length a defining the size of the system; for a sphere a, will be taken as the radius. Then if times are measured in units of $4\pi\kappa a^2$ and velocities in units of $1/4\pi\kappa a$, the $4\pi\kappa$ may be removed from (7). This is only possible if the fluid is not a perfect conductor. If it is, the field is locked in the fluid and will remain constant indefinitely in a stationary fluid. If a perfectly conducting fluid is in motion, the magnetic energy may be increased indefinitely; this case has been discussed by Bondi & Gold (1950) and will not be considered further here. For finite conductivity, (7) gives, after the change in units,

$$\nabla^2 \boldsymbol{H} = \partial \boldsymbol{H}/\partial t - V \operatorname{curl}(\boldsymbol{v} \times \boldsymbol{H}). \tag{8}$$

Here $\boldsymbol{v}$ is a specified function of position giving the form of the velocity field and V is a scalar constant controlling its magnitude. V and $\boldsymbol{v}$ are dimensionless; they are related to the velocity $\boldsymbol{v}_1$ in cm/s by

$$\boldsymbol{v}_1 = V\boldsymbol{v}/4\pi\kappa a. \tag{9}$$

The critical value of V is a 'characteristic number' or 'Eigenvalue' for the equation (8) with the given boundary conditions; it depends on the form of the velocity field and the shape of the system, but not on its size or material. For the disk dynamo V may be taken as the peripheral velocity in units of $1/4\pi\kappa a$, and $|\boldsymbol{v}|$ as r/a. Neglecting the resistance of the disk, (1) gives $V = 4al/A$, where l is the length of the coil, A the cross-section of the wire and a the radius of the disk. A cannot be increased beyond a certain limit without a large part of the field produced by the coil missing the disk, which would increase V. An exact estimate would be difficult, but rough calculation suggests that a value of V of about 20 could be obtained. The disk dynamo is, for a dynamo without iron, an efficient machine, in that all the current goes round the coil and produces field and most of the field goes through the disk and produces current. A homogeneous dynamo is necessarily partly short-circuited, and part of the current travels over paths that do not help the regeneration. The critical velocity may therefore be expected to be higher than for the disk, and if V_0 is the lowest eigenvalue of (8), $V_0 \boldsymbol{v}$ will be substantially greater than unity.

This circumstance accounts for the failure to construct homogeneous dynamos in the laboratory. With $a = 10$ cm, $\kappa = 10^{-5}$ e.m.u. (10^4 ohm^{-1} cm^{-1} as for mercury), $1/4\pi\kappa a = 10^3$ cm/s, and if $V\boldsymbol{v}$ is a good deal greater than unity, the velocity required, $V\boldsymbol{v}/4\pi\kappa a$, will be higher than is easily attained. For bodies of astronomical size the velocities are much reduced; for example, with $a = 1000$ km, the critical velocity will be 10^{-7} of that in the laboratory example.

Elsasser (1946*b*) and Batchelor (1950) have pointed out that (7) possesses a simple physical interpretation. It may be written

$$\frac{\partial \boldsymbol{H}}{\partial t} = \frac{1}{4\pi\kappa} \nabla^2 \boldsymbol{H} - (\boldsymbol{v} . \nabla)\, \boldsymbol{H} + (\boldsymbol{H} . \nabla)\, \boldsymbol{v}. \tag{10}$$

If $\boldsymbol{v} = 0$ this is the vector analogue of the equation of diffusion or heat conduction, and, in the absence of poles or sources of current, always leads to a decay of the field. The second term is the gradient of the field in the direction of the velocity multiplied by the velocity; it therefore represents the convection of the field with the moving fluid. It would vanish at any point if axes were chosen moving with the fluid. The third term gives the field multiplied by the gradient of the velocity in its direction. It therefore represents the rate of stretching of lines of force. If this is sufficiently large and positive it may overcome the first term and $\boldsymbol{H}$ may increase or, as a special case, remain constant.

To make this argument more precise multiply (7) by $\boldsymbol{H}$ and integrate over the whole of the conducting body to obtain the rate of change of the magnetic energy, W_i, within it. This gives

$$4\pi\frac{\mathrm{d}W_i}{\mathrm{d}t}=\frac{1}{2}\int_i\frac{\mathrm{d}^2\boldsymbol{H}^2}{\mathrm{d}t}\mathrm{d}U=\frac{1}{4\pi\kappa}\int_i\boldsymbol{H}.\nabla^2\boldsymbol{H}\mathrm{d}U-\int_i[\boldsymbol{H}.(\boldsymbol{v}.\nabla)\,\boldsymbol{H}-\boldsymbol{H}.(\boldsymbol{H}.\nabla)\,\boldsymbol{v}]\,\mathrm{d}U.$$

Here $\mathrm{d}U$ is an element of volume and i indicates that the integral extends over the interior of the body. By evaluating the terms in Cartesian co-ordinates and using (4) it may be shown that

$$\boldsymbol{H}.\nabla^2\boldsymbol{H}=-(\mathrm{curl}\,\boldsymbol{H})^2+\nabla.(\boldsymbol{H}\times\mathrm{curl}\,\boldsymbol{H})$$

and

$$\boldsymbol{H}.(\boldsymbol{v}.\nabla)\,\boldsymbol{H}=\tfrac{1}{2}\nabla.(H^2\boldsymbol{v}).$$

Thus

$$4\pi\frac{\mathrm{d}W_i}{\mathrm{d}t}=-\frac{1}{4\pi\kappa}\int_i(\mathrm{curl}\,\boldsymbol{H})^2\,\mathrm{d}U+\frac{1}{4\pi\kappa}\int_\sigma(\boldsymbol{H}\times\mathrm{curl}\,\boldsymbol{H}).\boldsymbol{n}\,\mathrm{d}\sigma-\frac{1}{2}\int_\sigma H^2\boldsymbol{v}.\boldsymbol{n}\,\mathrm{d}\sigma+\int_i\boldsymbol{H}.(\boldsymbol{H}.\nabla)\,\boldsymbol{v}\,\mathrm{d}U,$$

where $\mathrm{d}\sigma$ is an element of the surface of the body, $\boldsymbol{n}$ is its outward normal and the integral with respect to σ is taken over the whole surface. Since the normal component of $\boldsymbol{v}$ vanishes at the surface, the third integral is zero, and the second may be written

$$\frac{1}{4\pi\kappa}\int_\sigma(\boldsymbol{H}\times\mathrm{curl}\,\boldsymbol{H}).\boldsymbol{n}\,\mathrm{d}\sigma=\int_\sigma(\boldsymbol{H}\times\boldsymbol{E}).\boldsymbol{n}\,\mathrm{d}\sigma+\int_\sigma\boldsymbol{H}\times(\boldsymbol{v}\times\boldsymbol{H}).\boldsymbol{n}\,\mathrm{d}\sigma,$$

whence

$$4\pi\frac{\mathrm{d}W_i}{\mathrm{d}t}=-\frac{1}{4\pi\kappa}\int_i(\mathrm{curl}\,\boldsymbol{H})^2\,\mathrm{d}U+\int_\sigma(\boldsymbol{H}\times\boldsymbol{E}).\boldsymbol{n}\,\mathrm{d}\sigma-\int_\sigma\boldsymbol{H}.(\boldsymbol{v}.\boldsymbol{H})\,\boldsymbol{n}\,\mathrm{d}\sigma+\int_i\boldsymbol{H}.(\boldsymbol{H}.\nabla)\,\boldsymbol{v}\,\mathrm{d}U.$$

This is Poynting's theorem for a moving conductor. Since $\mathrm{curl}\,\boldsymbol{H}=4\pi\boldsymbol{I}$, the first term is minus 4π times the Joule heat and is always negative; the second term is the flow of energy from the external field into the conductor. It is minus 4π times the rate of change of the magnetic energy in the space outside the conductor. Thus if W is the total energy, internal and external,

$$4\pi\frac{\mathrm{d}W}{\mathrm{d}t}=-\frac{4\pi}{\kappa}\int_i I^2\,\mathrm{d}U-\int_\sigma\boldsymbol{H}.(\boldsymbol{v}.\boldsymbol{H})\,\boldsymbol{n}\,\mathrm{d}\sigma+\int_i\boldsymbol{H}.(\boldsymbol{H}.\nabla)\,\boldsymbol{v}\,\mathrm{d}U; \tag{11}$$

the last term is the work done in 'stretching' the field against the tension $H^2/8\pi$ in the lines of force. The second term is the integral over the surface of $H_N H_T\,|\,\boldsymbol{v}\,|\cos\alpha$, where H_N and H_T are the normal and transverse components of the field and α is the angle between H_T and $\boldsymbol{v}$; it is the rate at which work is done against the electromagnetic stresses at the surface of the sphere (Stratton 1941, p. 102). The term vanishes if the velocity at the surface is zero. It ensures that $\mathrm{d}W/\mathrm{d}t$ is the same whether or not there is a boundary layer near the surface of the body in which the velocity increases rapidly from zero. It is easily shown that the contribution of this layer to the third integral does not vanish as the layer becomes indefinitely thin, but tends to equality with the value which would be obtained for the second integral if the velocity at the inner surface of the layer were used in it. That it is a necessary term in the energy balance may be seen by considering a solid conducting sheet in rectilinear motion through a field normal to the sheet. The last term vanishes since $\boldsymbol{v}$ is constant. The second term is the integral of the applied normal field multiplied by the product of the velocity and the tangential field due to the induced currents. This is of the right form to give the energy put into the sheet by the driving forces. The term is not of much importance for the present discussion since it can always be dropped if the velocity is zero at the surface, which is not a serious restriction. A more 'obvious' expression for the rate of change of energy may be obtained by changing the second term into a volume integral. We then get after a little transformation

$$\frac{\mathrm{d}W}{\mathrm{d}t} = -\frac{1}{\kappa}\int_i \boldsymbol{I}^2 \,\mathrm{d}U + \int \boldsymbol{v}.(\boldsymbol{H}\times\boldsymbol{I})\,\mathrm{d}U.$$

The second term here is a natural expression of the work done by the flow against the electromagnetic forces $\boldsymbol{I}\times\boldsymbol{H}$. This form, though almost self-evident, is less instructive than (11), as it does not bring out the rôle played by stretching in transferring energy from the motion to the field.

It is easy to devise systems for which (11) is initially positive and the energy increases. This, however, is not sufficient to prove the existence of a homogeneous dynamo, for the energy may not go on increasing; it may reach a maximum and then decline steadily to zero. An example of such a system has been worked out in detail in a previous paper (Bullard 1949*b*, p. 441). Consider a rotating sphere surrounded by a concentric stationary spherical shell in electrical contact with it. If we start with the simplest type of initial field with lines of force in meridian planes and a dipole external field, the motion will generate a toroidal field whose magnitude after a given time can be made as large as we please by making the rotation fast enough. The original field will decay at a rate independent of the speed of rotation, and both it and the toroidal field will ultimately decline exponentially to zero. Such a system will clearly give an initial increase in magnetic energy followed by a decline to zero. The reason for the failure of this system to maintain its initial increase in energy is that, although the field is continually pulled out and wrapped round and round the axis, nothing is done to pull out and maintain the component in the meridian planes, which steadily declines. The situation is comparable to the growth of radioactive material from a long-lived parent substance. At first as the product grows the activity increases, but finally as the parent substance decays the product must decay also.

From this discussion it seems that the energy equation (11) cannot provide a short cut to a sufficient condition for the existence of a dynamo. It will determine whether the energy will increase initially, but to find from (11) whether it will go on increasing we need the field as a function of the time. That is, we need a detailed solution of the equations (4) and (7).

Batchelor (1950) has avoided this difficulty by observing that (7), which connects the velocity and the field, is identical with the relation connecting velocity and vorticity, except that the latter has the kinematic viscosity, ν, in place of $1/4\pi\kappa$. Since we know experimentally, and in some simple cases theoretically, that turbulent motions are possible in which the vorticity does not decrease indefinitely, it follows that similar motions will maintain a magnetic field if κ is sufficiently high. A field will only be produced when $\kappa a_1 v_1$ reaches a critical value, where a_1 is the size of the eddies that do most of the stretching and v_1 is the velocity in them. As the mechanical energy fed into the system is increased the eddies get smaller in such a way that the critical value of $\kappa a_1 v_1$ is never reached for the values of κ given by actual materials, except perhaps in the attenuated gas in interstellar space. In spite of this it still follows that a material can be specified that will generate a field, and, indeed, that motions can be specified that would produce a field for any specified conductivity. Such motions would be similar to turbulent motions but with their scale or velocity increased. They are dynamically impossible without a most elaborate system of external forces, but they are consistent with (6) and do give solutions of (4) and (7) that do not decline indefinitely with time. Batchelor considers an unbounded medium, but the presence of boundaries would not affect his argument. If these solutions to our problem exist it is clear that there can be no general theorem that a solution is impossible, and it is likely that more efficient systems can be found. Turbulence is inefficient as a dynamo, firstly because the motion is split into eddies small compared to the size of the whole system, and secondly because the motion is a 'random' one in which the stretching of the lines of force depends on an unbalance between the processes of stretching and shrinking.

If (2) is integrated round a closed line of force or a closed line of zero field, the integral of $\boldsymbol{v} \times \mathbf{H}$ vanishes. The integral of $\mathbf{E}$ round a closed circuit is equal to the integral of curl $\mathbf{E}$ over a surface bounded by the circuit; by (3) this vanishes if the field is steady. Thus the integral of curl $\mathbf{H}$ round the circuit must vanish, and in a steady dynamo the lines of force cannot run like a spiral spring with its ends joined, nor like a set of rings strung on a closed string. Cowling's theorem, that an axially symmetric motion cannot act as a dynamo, follows from this by noting than an axially symmetric field necessarily has a closed line of zero meridional field running along a circle of latitude. For more general types of motion this theorem can be avoided either by having no closed lines of force, or by having curl $\mathbf{H}$ zero on them, or by having curl $\mathbf{H}$ in opposite directions on different parts of them.

Equation (8) shows that, as with the disk dynamo, the existence of a homogeneous dynamo with the field in one direction implies the possibility of one with the same velocity and the field in the reverse direction. The velocity, however, cannot in general be reversed. Reversal is of course possible for dynamos in which it is equivalent to a symmetry operation. Such dynamos must have a field that is not exactly reversed by the symmetry operation. An example is given in § 7. If a motion in which reversal is not equivalent to a symmetry operation will work as a dynamo in either direction, then the fields produced by the direct and reverse motion must be quite different.

The foregoing discussion defines the problem, but does little towards solving it. A direct general proof of the existence of the required solutions of (4) and (8) is hardly to be expected, though Batchelor's argument strongly suggests that solutions exist for some types of motion. It seems most likely that progress will be made by a discussion of particular examples. Considerable advances in this direction have been made by Elsasser (1946 *a*, *b*, 1947) and by Takeuchi & Shimazu (1952 *a*, *b*). In particular, the latter authors have shown that the equations for a dynamo system suggested by Bullard (1949 *a*) have solutions, at any rate up to the point where spherical harmonics of the first and second degree are included in the argument. We now undertake a detailed discussion of the solutions of (4) and (8) for a spherical body.

4. Expansion of the solutions for a sphere

It is impracticable to solve (4) and (8) by a direct arithmetical method. This is partly due to the complexity of (8), which is a vector equation and is equivalent to three simultaneous scalar partial differential equations in four independent variables, and partly to the difficulty of satisfying the boundary conditions, which require that at the surface of the body $\boldsymbol{E}$ and $\boldsymbol{H}$ should match an external field that tends to zero at a great distance.

The only practicable method is to expand the solutions in spherical harmonics. The appropriate functions for the expansion of a vector field are well known, and have been used by many authors (e.g. Lamb 1881; Stratton 1941; Elsasser 1946 *a*, *b*, 1947; Bullard 1949 *b*). We take them in the form

$$T_r = 0, \quad T_\theta = \frac{T(r)}{r\sin\theta}\frac{\partial Y}{\partial\phi}, \quad T_\phi = -\frac{T(r)}{r}\frac{\partial Y}{\partial\theta}, \tag{12}$$

$$S_r = \frac{n(n+1)}{r^2}S(r)\,Y, \quad S_\theta = \frac{1}{r}\frac{\partial S}{\partial r}\frac{\partial Y}{\partial\theta}, \quad S_\phi = \frac{1}{r\sin\theta}\frac{\partial S}{\partial r}\frac{\partial Y}{\partial\phi}, \tag{13}$$

where Y is the surface harmonic $P_n^m(\cos\theta)\,{\sin \atop \cos}\,m\phi$. T_r, T_θ and T_ϕ are the components in spherical polar co-ordinates $(r\theta\phi)$ of a vector $\boldsymbol{T}_n^{mc}$ or $\boldsymbol{T}_n^{ms}$, which is derived from the surface harmonic $P_n^m \cos m\phi$ or $P_n^m \sin m\phi$. $T(r)$ is a scalar function of r and will be denoted by T_n^{mc} or T_n^{ms} when it is necessary to distinguish the functions belonging to different harmonics. S_r, S_θ and S_ϕ are similarly components of a vector $\boldsymbol{S}_n^{mc}$ or $\boldsymbol{S}_n^{ms}$. T_n^c and T_n^s will be written for T_n^{1c} and T_n^{1s}, and T_n for T_n^{0c} (T_n^{0s} is zero), and similarly for S; the radial functions S_n^m and T_n^m associated with a spherical harmonic of degree n will themselves be said to be 'of degree n'. We use unnormalized spherical harmonics as given by Stratton (1941, p. 608). The functions $\boldsymbol{T}$ are called 'toroidal' and have no radial component; the functions $\boldsymbol{S}$ are called 'poloidal'. The 'lines of force' or 'stream lines' for the $\boldsymbol{T}_n^{mc}$ of degrees up to four are shown in figure 2; they are contour maps of the surface harmonics. The lines are spaced so that an equal flux passes between each pair of neighbouring lines. The $\boldsymbol{T}_n^{ms}$ can be obtained by rotating the $\boldsymbol{T}_n^{mc}$ through an angle $\pi/2m$ about the axis $\theta = 0$. The S's are difficult to illustrate satisfactorily, but can be visualized by imagining figure 2 to give lines of current flow; the S's will then be the associated magnetic fields.

The following properties are easily established:

(*a*) $\boldsymbol{S}$ and $\boldsymbol{T}$ satisfy (4).

(*b*) The $\boldsymbol{S}$'s and $\boldsymbol{T}$'s are orthogonal when integrated over the surface of a sphere; that is

$$\iint \boldsymbol{S}.\boldsymbol{S}'\,\mathrm{d}\sigma = \iint \boldsymbol{T}.\boldsymbol{T}'\,\mathrm{d}\sigma = 0,$$

unless the $\mathbf{S}$ and $\mathbf{S}'$ or $\mathbf{T}$ and $\mathbf{T}'$ are derived from the same surface harmonic;

$$\iint \boldsymbol{S}.\boldsymbol{T}\,\mathrm{d}\sigma = 0$$

for all $\boldsymbol{S}$ and $\boldsymbol{T}$.

(*c*) If $\boldsymbol{T}$, $\boldsymbol{T}'$, $\boldsymbol{S}$ and $\boldsymbol{S}'$ are derived from the same surface harmonic

$$\iint \boldsymbol{T}.\boldsymbol{T}'\,\mathrm{d}\sigma = N_n TT',$$

$$\iint \boldsymbol{S}.\boldsymbol{S}'\,\mathrm{d}\sigma = N_n\left[n(n+1)\,SS'/r^2 + \frac{\mathrm{d}S}{\mathrm{d}r}\frac{\mathrm{d}S'}{\mathrm{d}r}\right],$$

where

$$\left.\begin{aligned} N_n &= \frac{2\pi n(n+1)}{2n+1}\frac{(n+m)!}{(n-m)!} \quad \text{if} \quad m \neq 0, \\ N_n &= \frac{4\pi n(n+1)}{2n+1} \quad \text{if} \quad m = 0. \end{aligned}\right\} \tag{14}$$

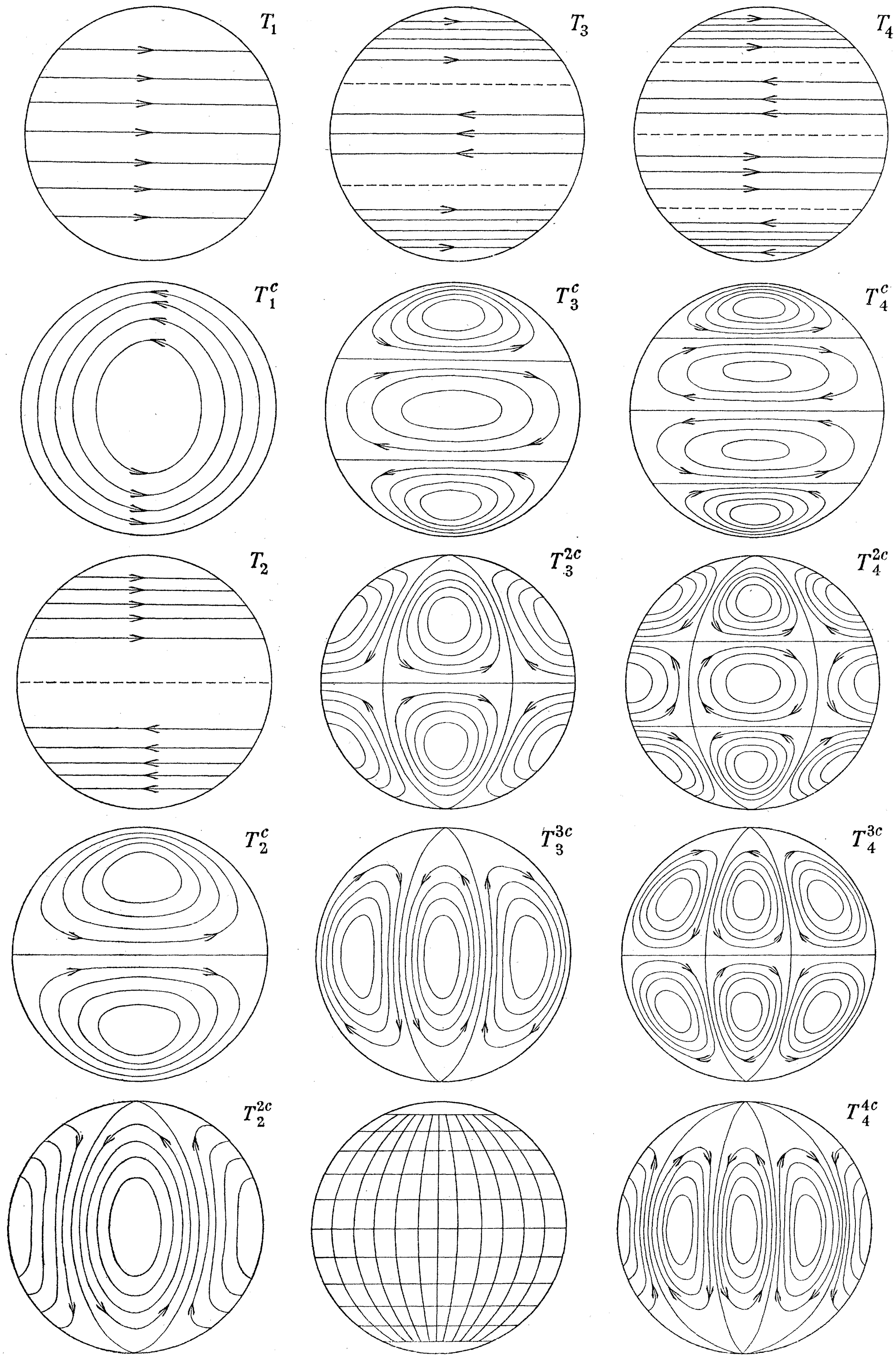


FIGURE 2. T_n^{mc} with $\theta = 90°$, $\phi = 0°$ in the centre.

(*d*) Any vector whose components are a continuous function of position and which satisfies (4) can be expressed as the sum of a series of $\boldsymbol{S}$'s and $\boldsymbol{T}$'s.

(*e*) If $\boldsymbol{S}$ and $\boldsymbol{T}$ and all their derivatives are continuous at the origin, S and T must behave there like r^{n+1} multiplied by an even function of r.

(*f*)
$$\left.\begin{aligned} \operatorname{curl} \boldsymbol{T} &= \overline{\boldsymbol{S}}, \\ \operatorname{curl}^2 \boldsymbol{T} &= \operatorname{curl} \overline{\boldsymbol{S}} = \boldsymbol{T}^*, \\ \operatorname{curl}^2 \boldsymbol{S} &= \boldsymbol{S}^*, \end{aligned}\right\} \tag{15}$$

where $\overline{\boldsymbol{S}}$ is the $\boldsymbol{S}$ of (13) with T written in place of S and $\boldsymbol{T}^*$ and $\boldsymbol{S}^*$ are given by (12) and (13) with $T(r)$ and $S(r)$ replaced by

$$\left.\begin{aligned} T^*(r) &= -\frac{\mathrm{d}^2 T}{\mathrm{d}r^2} + \frac{n(n+1)}{r^2}\, T, \\ S^*(r) &= -\frac{\mathrm{d}^2 S}{\mathrm{d}r^2} + \frac{n(n+1)}{r^2}\, S. \end{aligned}\right\} \tag{16}$$

In most previous applications the radial functions have been expanded in a series of Bessel functions of half-integer order. For the present problem it seems better to keep them as arbitrary functions of r.

Suppose the magnetic field to be expanded in a series of $\boldsymbol{S}$'s and $\boldsymbol{T}$'s,

$$\boldsymbol{H} = \sum_{\beta} (\boldsymbol{S}_\beta + \boldsymbol{T}_\beta), \tag{17}$$

where the suffix β indicates the surface harmonic from which $\boldsymbol{S}_\beta$ and $\boldsymbol{T}_\beta$ are derived, that is, it specifies the degree and order of the harmonic and whether it contains $\cos m\phi$ or $\sin m\phi$. We also use β, when it is not a suffix, to indicate the degree of the harmonic. This double use of the same symbol for a number and as a label considerably simplifies the typography; a similar convention is used below for α and γ. The radial functions T_β and S_β are functions of time as well as of r.

Our procedure differs from Elsasser's (1946*a*) since he expands not the field but a vector potential, $\boldsymbol{A}$, defined by

$$\boldsymbol{H} = \operatorname{curl} \boldsymbol{A}, \quad \boldsymbol{E} = -\partial \boldsymbol{A}/\partial t, \quad \operatorname{div} \boldsymbol{A} = 0. \tag{18}$$

In fact these relations are not mutually compatible, for taking the divergence of (2)

$$\operatorname{div} \boldsymbol{E} + \operatorname{div} (\boldsymbol{v}_1 \times \boldsymbol{H}) = 0,$$

and (18) therefore implies $$\operatorname{div} (\boldsymbol{v} \times \boldsymbol{H}) = 0,$$

which is untrue is most problems of induction in moving media (e.g. Bullard 1949*b*, p. 425). The currents associated with the electric field produced by the charge distribution cannot be neglected, though those produced by convection of charge, that is, terms of the form $q\boldsymbol{v}$, can. In spite of the unsuitable choice of vector potential and the resulting incorrectness of the final equations, a large part of Elsasser's results is correct. In particular, his selection rules (1946*a*, p. 113) are unaffected. It is not known whether his equations have solutions satisfying the boundary conditions.

28-2

Since (17) already satisfies (4), the problem is reduced to showing that there exist $S_\beta(r,t)$ and $T_\beta(r,t)$ such that (8) and the boundary conditions are also satisfied. Substituting the expansion (17) in (8) gives

$$\sum_\beta \left[\operatorname{curl}^2(\boldsymbol{S}_\beta+\boldsymbol{T}_\beta)+\frac{\partial \boldsymbol{S}_\beta}{\partial t}+\frac{\partial \boldsymbol{T}_\beta}{\partial t}-V\operatorname{curl}(\boldsymbol{\nu}\times\boldsymbol{S}_\beta)-V\operatorname{curl}(\boldsymbol{\nu}\times\boldsymbol{T}_\beta)\right]=0.$$

Multiplying this by $\boldsymbol{S}'_\gamma$ and $\boldsymbol{T}'_\gamma$, whose radial functions are unity for all r, using (15) and integrating over a sphere of radius r, gives

$$\left.\begin{aligned}\frac{\partial S_\gamma}{\partial t}&=\frac{\partial^2 S_\gamma}{\partial r^2}-\gamma(\gamma+1)\,r^{-2}S_\gamma+\frac{Vr^4}{\gamma(\gamma+1)\,N_\gamma}\sum_\beta\iint \boldsymbol{S}'_\gamma\cdot\operatorname{curl}(\boldsymbol{\nu}\times\boldsymbol{S}_\beta+\boldsymbol{\nu}\times\boldsymbol{T}_\beta)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi,\\ \frac{\partial T_\gamma}{\partial t}&=\frac{\partial^2 T_\gamma}{\partial r^2}-\gamma(\gamma+1)\,r^{-2}T_\gamma+\frac{Vr^2}{N_\gamma}\sum_\beta\iint \boldsymbol{T}'_\gamma\cdot\operatorname{curl}(\boldsymbol{\nu}\times\boldsymbol{S}_\beta+\boldsymbol{\nu}\times\boldsymbol{T}_\beta)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi.\end{aligned}\right\}\quad(19)$$

Let $\boldsymbol{\nu}$ be also expanded in a series of $\boldsymbol{S}$'s and $\boldsymbol{T}$'s,

$$\boldsymbol{\nu}=\sum_\alpha(\boldsymbol{S}_\alpha+\boldsymbol{T}_\alpha),\quad(20)$$

where the suffix α has been used to avoid confusion with the functions used in the expansion of the magnetic field. In arithmetical work this will be taken as a terminating series; that is, the velocity fields will be restricted to those that can be represented by a finite sum of spherical harmonic terms. Since v_r must vanish at $r=1$, the S_α must also vanish there. For a viscous fluid in a rigid envelope, v_θ and v_ϕ must also vanish, but in the application to the earth's core it is possible that viscosity will only have an appreciable effect in a thin boundary layer, and it is likely to be a good approximation to neglect this and to assume that v_θ and v_ϕ do not vanish at $r=1$. This is probably nearer to the truth than taking simple forms for v_θ and v_ϕ that vanish at $r=1$.

Equation (19) will contain eight types of integral on its right-hand side. One of these always vanishes,

$$\iint \boldsymbol{S}'_\gamma\cdot\operatorname{curl}(\boldsymbol{T}_\alpha\times\boldsymbol{T}_\beta)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi=0.$$

The rest may be made to depend on the two integrals

$$\left.\begin{aligned}K_{\alpha\beta\gamma}&=\iint Y_\alpha Y_\beta Y_\gamma\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi,\\ L_{\alpha\beta\gamma}&=\iint Y_\alpha\left(\frac{\partial Y_\beta}{\partial\theta}\frac{\partial Y_\gamma}{\partial\phi}-\frac{\partial Y_\beta}{\partial\phi}\frac{\partial Y_\gamma}{\partial\theta}\right)\mathrm{d}\theta\,\mathrm{d}\phi,\end{aligned}\right\}\quad(21)$$

where, as in (12) and (13), the Y's are surface harmonics. By integration by parts it may be shown that $L_{\alpha\beta\gamma}$ is not changed by an even permutation of the indices, but has its sign reversed by an odd permutation. K and L without suffixes will be taken to mean $K_{\alpha\beta\gamma}$ and $L_{\alpha\beta\gamma}$. With this convention, (19) becomes

$$r^2\frac{\partial S_\gamma}{\partial t}=r^2\frac{\partial^2 S_\gamma}{\partial r^2}-\gamma(\gamma+1)\,S_\gamma-V\sum_{\alpha\beta}[(S_\alpha S_\beta S_\gamma)+(T_\alpha S_\beta S_\gamma)+(S_\alpha T_\beta S_\gamma)],\quad(22)$$

$$r^2\frac{\partial T_\gamma}{\partial t}=r^2\frac{\partial^2 T_\gamma}{\partial r^2}-\gamma(\gamma+1)\,T_\gamma-V\sum_{\alpha\beta}[(S_\alpha S_\beta T_\gamma)+(T_\alpha S_\beta T_\gamma)+(S_\alpha T_\beta T_\gamma)+(T_\alpha T_\beta T_\gamma)],\quad(23)$$

where

$$\left.\begin{aligned}
(S_\alpha S_\beta S_\gamma) &= -\frac{K}{2N_\gamma}\Big[\alpha(\alpha+1)\{\alpha(\alpha+1)-\beta(\beta+1)-\gamma(\gamma+1)\}S_\alpha\frac{\partial S_\beta}{\partial r}\\
&\qquad +\beta(\beta+1)\{\alpha(\alpha+1)-\beta(\beta+1)+\gamma(\gamma+1)\}\frac{\partial S_\alpha}{\partial r}S_\beta\Big],\\
(T_\alpha S_\beta S_\gamma) &= -L\beta(\beta+1)\,T_\alpha S_\beta/N_\gamma,\\
(S_\alpha T_\beta S_\gamma) &= -L\alpha(\alpha+1)\,S_\alpha T_\beta/N_\gamma,\\
(S_\alpha S_\beta T_\gamma) &= \frac{L}{N_\gamma}\Big[\alpha(\alpha+1)\,S_\alpha\frac{\partial^2 S_\beta}{\partial r^2}+\Big\{[\alpha(\alpha+1)+\beta(\beta+1)-\gamma(\gamma+1)]\frac{\partial S_\alpha}{\partial r}\\
&\qquad -\alpha(\alpha+1)\frac{2S_\alpha}{r}\Big\}\frac{\partial S_\beta}{\partial r}+\beta(\beta+1)\left(\frac{\partial^2 S_\alpha}{\partial r^2}-\frac{2}{r}\frac{\partial S_\alpha}{\partial r}\right)S_\beta\Big],\\
(T_\alpha S_\beta T_\gamma) &= -\frac{K}{2N_\gamma}\Big[\{\beta(\beta+1)\,[\alpha(\alpha+1)-\beta(\beta+1)+\gamma(\gamma+1)]\\
&\qquad +\gamma(\gamma+1)\,[\alpha(\alpha+1)+\beta(\beta+1)-\gamma(\gamma+1)]\}\,T_\alpha\frac{\partial S_\beta}{\partial r}\\
&\qquad +\beta(\beta+1)\{\alpha(\alpha+1)-\beta(\beta+1)+\gamma(\gamma+1)\}\left(\frac{\partial T_\alpha}{\partial r}-\frac{2T_\alpha}{r}\right)S_\beta\Big],\\
(S_\alpha T_\beta T_\gamma) &= \frac{K}{2N_\gamma}\Big[\alpha(\alpha+1)\{-\alpha(\alpha+1)+\beta(\beta+1)+\gamma(\gamma+1)\}S_\alpha\frac{\partial T_\beta}{\partial r}\\
&\qquad +\Big\{\alpha(\alpha+1)\,[-\alpha(\alpha+1)+\beta(\beta+1)+\gamma(\gamma+1)]\left(\frac{\partial S_\alpha}{\partial r}-\frac{2S_\alpha}{r}\right)\\
&\qquad +\gamma(\gamma+1)\,[\alpha(\alpha+1)+\beta(\beta+1)-\gamma(\gamma+1)]\frac{\partial S_\alpha}{\partial r}\Big\}T_\beta\Big],\\
(T_\alpha T_\beta T_\gamma) &= -L\gamma(\gamma+1)\,T_\alpha T_\beta/N_\gamma.
\end{aligned}\right\} \tag{24}$$

The equations (22) and (23) have a simple physical interpretation. The left-hand side is the rate of increase of the part of the field derived from a particular spherical harmonic. The first two terms on the right-hand side give the decrease of this part by diffusion, which is independent of the motion. The remaining terms give the rate at which the component S_α or T_α of the motion interacts with the component S_β or T_β of the field to produce, by induction, the field component S_γ or T_γ. Thus in any equation the suffix β represents the inducing field and γ the induced field. If the units of distance and time are changed from those adopted for (8) to centimetres and seconds, the diffusive terms are divided by $4\pi\kappa$; for infinite conductivity they vanish as they should.

For a dynamo producing a steady field the terms in V must exactly balance the diffusive terms and

$$\left.\begin{aligned}
r^2\frac{\mathrm{d}^2S_\gamma}{\mathrm{d}r^2}-\gamma(\gamma+1)\,S_\gamma &= V\sum_{\alpha\beta}[(S_\alpha S_\beta S_\gamma)+(T_\alpha S_\beta S_\gamma)+(S_\alpha T_\beta S_\gamma)],\\
r^2\frac{\mathrm{d}^2T_\gamma}{\mathrm{d}r^2}-\gamma(\gamma+1)\,T_\gamma &= V\sum_{\alpha\beta}[(S_\alpha S_\beta T_\gamma)+(T_\alpha S_\beta T_\gamma)+(S_\alpha T_\beta T_\gamma)+(T_\alpha T_\beta T_\gamma)],
\end{aligned}\right\} \tag{25}$$

where the terms on the right are given by (24) with $\partial/\partial r$ replaced by $\mathrm{d}/\mathrm{d}r$.

The boundary conditions require $\boldsymbol{H}$ to be continuous at the surface of the sphere and to join an external field that vanishes at least like $1/r^3$ at infinity. Outside the sphere the magnetic field $\boldsymbol{H}_e$ must be derivable from a potential

$$\boldsymbol{H}_e = \sum_\beta c_\beta(t)\,\mathrm{grad}\,(Y_\beta/r^{\beta+1}), \tag{26}$$

where the c_β's are functions of the time only. (17) and (26) show that the field will be continuous at $r=1$ if

$$H_r = \sum_\beta \beta(\beta+1)\,S_\beta Y_\beta = -\sum_\beta (\beta+1)\,c_\beta Y_\beta,$$

$$H_\theta = \sum_\beta \left(\frac{\partial S_\beta}{\partial r}\frac{\partial Y_\beta}{\partial \theta} + \frac{T_\beta}{\sin\theta}\frac{\partial Y_\beta}{\partial \phi}\right) = \sum_\beta c_\beta \frac{\partial Y_\beta}{\partial \theta},$$

$$H_\phi = \sum_\beta \left(\frac{1}{\sin\theta}\frac{\partial S_\beta}{\partial r}\frac{\partial Y_\beta}{\partial \phi} - T_\beta\frac{\partial Y_\beta}{\partial \theta}\right) = \sum_\beta \frac{c_\beta}{\sin\theta}\frac{\partial Y_\beta}{\partial \phi},$$

which are satisfied if

$$\left.\begin{aligned} T_\beta &= 0 \\ \partial S_\beta/\partial r + \beta S_\beta &= 0 \\ c_\beta &= -\beta S_\beta \end{aligned}\right\} \quad \text{at} \quad r=1; \tag{27}$$

the first two of these are the boundary conditions for (22), (23) and (25), the third determines the external magnetic field (26).

It remains to show that the boundary conditions for the current and the electric field can also be satisfied. The normal component of the current must vanish at the boundary; the current is equal to curl $\boldsymbol{H}/4\pi$, and this implies the vanishing of the normal component of curl $\boldsymbol{H}$. Since the normal component of curl $\boldsymbol{S}$ always vanishes and, by (27), all the $\boldsymbol{T}$ are zero at the boundary, this is automatically satisfied.

The tangential components of the electric field must be continuous at the boundary and must join an external field $\boldsymbol{E}_e$ that vanishes at infinity at least as $1/r^3$. The latter condition usually implies a discontinuity in the normal component and a charge on the surface. It may be shown that it is always possible to find a suitable external field. The derivation of the expression for the field is somewhat complicated and will be omitted.

The internal electric field can be found from (2) when the magnetic field is known. The volume distribution of charge can then be found from (5), and the surface charge density from the discontinuity in the radial electric field.

The problem of finding a steady self-exciting dynamo is now reduced to finding S_α and T_α as functions of r, and a value of V for which the solutions of the set of ordinary simultaneous equations (25) satisfy the conditions

$$\left.\begin{aligned} S_\beta = T_\beta &= O(r^{\beta+1}) \quad \text{at} \quad r=0, \\ \mathrm{d}S_\beta/\mathrm{d}r + \gamma S_\beta = T_\beta &= 0 \quad \text{at} \quad r=1, \end{aligned}\right\} \tag{28}$$

5. General properties of equations (25)

The origin is the only singular point of (25). Near it there are solutions that behave like $r^{\gamma+1}$ and solutions that behave like $r^{-\gamma}$. The latter give $\boldsymbol{S}$'s and $\boldsymbol{T}$'s with singularities at the origin and must be excluded in the present problem. If the S_α and T_α have been chosen so that the velocity is continuous at $r = 0$, the $\boldsymbol{S}$'s and $\boldsymbol{T}$'s derived from the $r^{\gamma+1}$ solutions are also continuous. N equations will give N such solutions containing, in all, N constants of integration. As both the equations and the boundary conditions are homogeneous, only $N-1$ of these are available to satisfy the N boundary conditions; the remaining constant is a factor fixing the magnitude of the field. The remaining condition must be satisfied by a correct choice of V. The problem is therefore one in which a characteristic value must be found for a parameter and boundary conditions satisfied at two points. It differs from most problems of this kind hitherto discussed in that the parameter whose characteristic value is to be found, is multiplied by the first and second derivatives of some of the dependent variables as well as by the variables themselves.

The difficulties are peculiar to the self-exciting dynamo. If the problem considered had been that of induction in a field maintained by some external agency, (26) would have contained terms $a_\beta(t)\,\mathrm{grad}\,(r^\beta Y_\beta)$, where the a_β are functions of the time only, and (27) would be replaced by

$$\left.\begin{aligned} T_\beta &= 0 \\ \partial S_\beta/\partial r + \beta S_\beta - (\beta+1)\,a_\beta &= 0 \\ c_\beta &= -\beta(S_\beta - a_\beta) \end{aligned}\right\} \quad \text{at} \quad r = 1.$$

If any of the a_β differ from zero, the boundary conditions are no longer homogeneous, all N constants are available to satisfy them, and a solution can be obtained for any V. As would be expected from physical arguments, it is only when external currents and poles are excluded by putting all the a_β equal to zero that there is any problem concerning the existence of solutions.

If the S's and T's can be divided into two classes such that no one of the equations (25) contains members of the same class on both the left and right sides, then a change of sign of one class and of V leaves the equations unaltered. Thus if such equations have a positive characteristic number they also have an equal negative one associated with the same characteristic functions, except that half of them are reversed in sign. This condition is satisfied by the equations considered in § 7, but is not always satisfied. As stated in § 3, it must always be satisfied if reversal of the velocity is equivalent to a symmetry operation.

It will be shown in § 6 that, except for a purely rotational motion, the equations (25) are always infinite in number, but that each contains only a finite number of terms on its right-hand side. The convergence of the solutions as the number of equations is increased is considered in § 8; in this section we discuss the solutions of the finite set of N equations derived from (25) by omitting all terms derived from harmonics above a certain degree and ignoring all equations containing such terms on the left. The velocity radial functions, S_α and T_α, will be taken to be continuous and to have continuous differential coefficients of all orders; near $r = 0$ they vanish at least like $r^{\alpha+1}$; in arithmetical work they will be taken as powers of r or as polynomials containing only two or three terms. The restrictions on S_α and T_α could probably be relaxed somewhat, but for the present problem there is no object and some inconvenience in doing so. With these restrictions, the radial functions for the magnetic field are continuous and have a convergent Taylor series in $0 \leqslant r \leqslant 1$.

Some simple examples of one and two equations are treated analytically in another paper (Bullard 1955); the results are useful as a general indication of the behaviour of the equations, but to establish the possibility of homogeneous dynamos it is necessary to treat larger systems of equations, which can only be handled numerically. This is done in § 7 of the present paper.

A well-known argument (Picard 1930) shows that, since the range of r is finite and all the N solutions have a Taylor series in $0 \leqslant r \leqslant 1$, the characteristic numbers are discrete. For a Sturm-Liouville equation it can further be shown that the characteristic numbers are real, infinitely numerous and without upper bound. These theorems cannot be extended to (25). Many examples of complex characteristic numbers are known, and in the cases that are simple enough to be investigated there is an upper bound to the real characteristic numbers. The Sturmian theorems on the interlacing of the zeros corresponding to successive characteristic numbers also do not apply to (25); in fact, examples are known in which a single equation has two characteristic functions, both without a zero in $0 < r < 1$.

There are no solutions if the motion is purely toroidal. To prove this put $S_\alpha = 0$, multiply each of the equations (22) by $N_\gamma \gamma(\gamma+1) S_\gamma$ and add. This gives

$$\sum_\gamma N_\gamma \gamma(\gamma+1)\left[-r^2 S_\gamma \frac{\partial S_\gamma}{\partial t} + r^2 S_\gamma \frac{\partial^2 S_\gamma^2}{\partial r^2} - \gamma(\gamma+1) S_\gamma^2\right] = -V \sum_{\alpha\beta\gamma} \gamma(\gamma+1) L\beta(\beta+1) T_\alpha S_\beta S_\gamma.$$

Since $L_{\alpha\beta\gamma} = -L_{\alpha\gamma\beta}$, the right-hand side of this is zero. Integrate the left-hand side from $r = 0$ to 1:

$$\sum_\gamma N_\gamma \gamma(\gamma+1) \int_0^1 \left[-\tfrac{1}{2} r^2 \frac{\partial S_\gamma^2}{\partial t} + r^2 S_\gamma \frac{\partial^2 S_\gamma}{\partial r^2} - \gamma(\gamma+1) S_\gamma^2\right] \mathrm{d}r = 0.$$

Integrate the second term twice by parts:

$$\sum_\gamma N_\gamma \gamma(\gamma+1)\left[\left| r^2 S_\gamma \frac{\partial S_\gamma}{\partial r} - r S_\gamma^2 \right|_0^1 - \int_0^1 \left\{\tfrac{1}{2} r^2 \frac{\partial S_\gamma^2}{\partial t} + (\gamma^2+\gamma-1) S_\gamma^2 + r^2\left(\frac{\partial S_\gamma}{\partial r}\right)^2\right\} \mathrm{d}r\right] = 0.$$

With the boundary conditions (27) this gives

$$\frac{\partial}{\partial t} \sum_\gamma N_\gamma \gamma(\gamma+1) \int_0^1 r^2 S_\gamma^2 \,\mathrm{d}r = -2 \sum_\gamma N_\gamma \gamma(\gamma+1)\left[(\gamma+1) S_\gamma^2(1) + \int_0^1 \left\{(\gamma^2+\gamma-1) S_\gamma^2 + r^2\left(\frac{\partial S_\gamma}{\partial r}\right)^2\right\} \mathrm{d}r\right],$$

whence
$$\frac{\partial}{\partial t} \sum_\gamma N_\gamma \gamma(\gamma+1) \int_0^1 r^2 S_\gamma^2 \,\mathrm{d}r < -\sum_\gamma N_\gamma \gamma(\gamma+1) \int_0^1 r^2 S_\gamma^2 \,\mathrm{d}r.$$

This can be true only if all the S_γ decrease below any assigned limit after a sufficient time. In (23) put S_α and $S_\beta = 0$, multiply by $T_\gamma/\gamma(\gamma+1)$ and add, then

$$\sum_\gamma \frac{N_\gamma}{\gamma(\gamma+1)}\left[-r^2 T_\gamma \frac{\partial T_\gamma}{\partial t} + r^2 T_\gamma \frac{\partial^2 T_\gamma}{\partial r^2} - \gamma(\gamma+1)\, T_\gamma^2\right] = \sum_{\alpha\beta\gamma} VLT_\alpha T_\beta T_\gamma,$$

whence, by an argument similar to that for S_γ, all the T_γ decrease, and the only solution compatible with the boundary conditions gives a field that decreases to zero everywhere. Elsasser has shown that this also follows from his equations.

If all the S_β and T_β and their first derivatives vanish at $r = 1$, then by (25) the second and all higher derivatives will vanish also ($\mathrm{d}^2 S_\beta/\mathrm{d}r^2$ on the right of (24) is multiplied by S_α which is zero at $r = 1$). The field will therefore be zero everywhere and, with a material of finite conductivity, no steady dynamo can exist in which there is no external field and the toroidal field has zero gradient at the surface. Similarly, it can be shown that with material of finite conductivity no steady dynamo can exist in which the region of fluid containing the field is surrounded by field-free fluid. This result is to be expected, since the lines of force repel each other and, if given time, will reach the boundaries of the fluid.

These are the only general theorems that have been deduced from (22) and (23) without a detailed consideration of (24). We have not been able to prove that a dynamo with no external magnetic field is impossible, though it would doubtless require a very special velocity field.

6. Selection rules

The behaviour of the solutions of (25) depends on which terms are present in (24). This in turn depends on which of the velocity radial functions, S_α and T_α, are assumed to be present, and on the vanishing of certain of the K's and L's. Important results follow from the mere enumeration of the non-vanishing terms; this does not depend on the form of S_α and T_α as functions of r, nor on the terms in $\Big[\quad\Big]$ in (24), but on the properties of the K's and L's. The rules for the vanishing of the K's and L's are called 'selection rules' by analogy with the closely similar rules in quantum mechanics. Most of the selection rules have been given by Elsasser and are unaffected by his restrictive choice of vector potential. They have been examined in detail by Bird (1949), to whom we are indebted for most of the material on which this section is based.

Let the three harmonics occurring in $K_{\alpha\beta\gamma}$ and $L_{\alpha\beta\gamma}$ be $P_\alpha^{m_\alpha}(\cos\theta)\,{}^{\cos}_{\sin}\,m_\alpha\phi$, etc. Then the rules, which may be deduced from (21) and (24), are:

(1) $(S_\alpha S_\beta S_\gamma)$, $(S_\alpha T_\beta T_\gamma)$ and $(T_\alpha S_\beta T_\gamma)$ depend on K and are zero unless the conditions (*a*) to (*d*) are all satisfied,

- (*a*) $\alpha+\beta+\gamma$ is even,
- (*b*) α, β and γ can form the sides of a triangle (the degenerate cases $\alpha=\beta+\gamma$, etc., count as triangles),
- (*c*) one or more of the four expressions $m_\alpha \pm m_\beta \pm m_\gamma$ vanishes,
- (*d*) three of the harmonics have $\cos m\phi$ or one has ($m=0$ counts as a cosine).

(2) $(S_\alpha T_\beta S_\gamma)$, $(S_\alpha S_\beta T_\gamma)$, $(T_\alpha S_\beta S_\gamma)$, $(T_\alpha T_\beta T_\gamma)$ depend on L and are zero unless the conditions (*a*) to (*e*) are all satisfied,

- (*a*) $\alpha+\beta+\gamma$ is odd,
- (*b*) α, β and γ can form the sides of a triangle (the degenerate cases are excluded by (*a*)),
- (*c*) one or more of the four expressions $m_\alpha \pm m_\beta \pm m_\gamma$ vanishes,
- (*d*) two of the harmonics have $\cos m\phi$ or none has ($m=0$ counts as a cosine),
- (*e*) no two harmonics are identical.

(3) $(T_\alpha T_\beta S_\gamma)$ does not occur in (24) as it is always zero.

Rules (1*b*) and (2*b*) requires that $|\alpha-\gamma| \leqslant \beta \leqslant \alpha+\gamma$, and thus that only a finite number of terms appear on the right of (25).

The computation of the K's and L's has been considered by Bird (1949) and by Infeld & Hull (1951) and of the K's by Gaunt (1929). They show that the integrals can be expressed in closed form, but it is usually easier to compute particular examples directly from (21). Bird has computed all K's and L's for harmonics up to the fourth degree. He finds that in addition to those shown to be zero by the selection rules, the terms containing the K derived from $P_3^2 P_3^2 P_2$ also vanish.

The relations given by the selection rules, and thus the structure of (25), can be conveniently illustrated by diagrams in which each S_γ occurring on the left-hand side of (25) is represented by a labelled point. If the equation with S_γ on the left has a non-zero term containing S_β on the right, an arrow is drawn from S_β to S_γ and labelled with the symbol for the S_α involved. Thus the occurrence of an arrow in the diagram means that the velocity component attached to it interacts with the S_β or T_β from which it points to produce the S_γ or T_γ towards which it points. For example, consider a T_1 motion (a rotation about an axis) interacting with an S_1 field (a dipole field). This can give terms $(T_1 S_1 S_\gamma^m)$ and $(T_1 S_1 T_\gamma^m)$. From rule (2c) m must be zero, but then (2 a, b and e) cannot all be satisfied, and $(T_1 S_1 S_\gamma^m) = 0$.

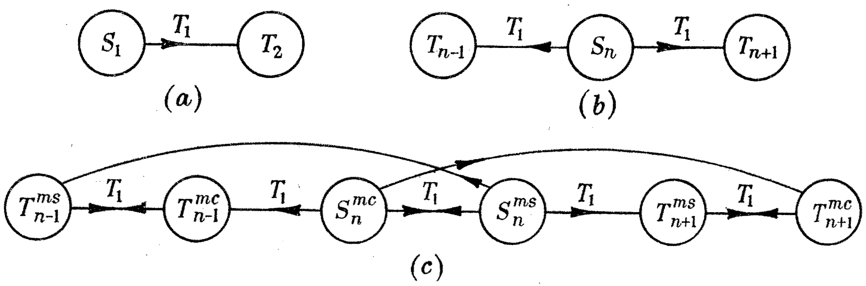


FIGURE 3. Interactions with a T_1 motion.

From rules (1a) and (1b), T_γ^m must be T_2. The diagram representing this connexion is then as in figure 3a. The reverse coupling $(T_1 T_2 S_1)$ and connexions to other S fields are excluded by rule (3), and connexions of T_2 to other T fields by rule (2). The diagram is therefore complete. Other connexions possible with a T_1 motion are shown in figures 3b and 3c. These and the corresponding ones for T_1^c and T_1^s motions are the only diagrams with a finite number of elements; it is this simplification that makes it possible to discuss them comparatively easily (Bullard 1949b). These systems cannot form dynamos, as the motion is purely toroidal. In any case those of figures 3a and 3b would be excluded, as there is no arrow pointing towards the S component and therefore no term on the right of (25) to maintain the S field, which will decay as described in § 2.

There are sixteen possible components of the velocity field derived from spherical harmonics of degrees one and two, eight S's and eight T's. The interactions of these taken one and two at a time with all possible fields will now be considered.

(i) *Single-velocity component*

The velocity components T_1, T_1^c and T_1^s are incapable of forming dynamos, since they are toroidal and also because each is symmetrical about an axis. The five second-degree toroidal components are also excluded.

The first-degree poloidal velocity components S_1, S_1^c and S_1^s are incapable of forming dynamos, since they have an axis of symmetry. The connexions for S_1 are shown in figure 4. There are an infinite number of such diagrams, two for each m, each of which extends indefinitely in the direction of increasing n. This multiplicity is a common feature of the diagrams for the simpler motions; those for the more complex ones form a single network.

Where the network consists of discrete parts it is necessary to examine each separately to see if it will give a dynamo. If more than one does, that with the lowest characteristic number will be the one that occurs in practice.

The second degree poloidal velocity component S_2 is symmetrical about an axis and therefore cannot give a dynamo. The other four second degree harmonics are all geometrically similar and can be converted into each other by suitable rotations. It is therefore only necessary to consider one, and S_2^{2c} will be taken. There are eight possible diagrams, of which six are not convertible into each other by rotations. They are shown in figure 5. The dynamo with S_2^{2c} motion is considered further in § 7.

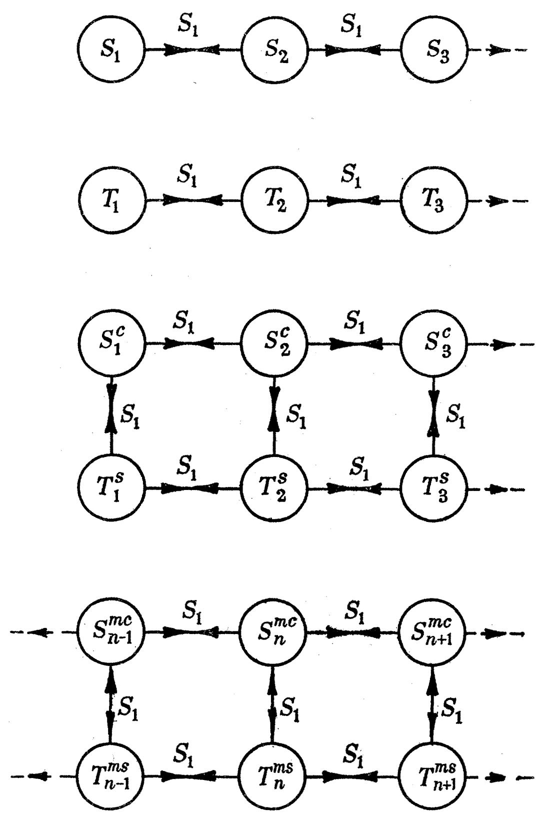


FIGURE 4. Interactions with an S_1 motion.

(ii) *Two-velocity components*

There are 136 ways of choosing a pair of components from the 16 S's and T's of degrees one and two. Of these, 47 are excluded as being either purely toroidal or symmetrical about an axis. Of the remaining 89, only 29 are physically distinct, the rest being derivable from them by rotations. These facts are summarized in table 1, where C means that the system is known from Cowling's theorem not to be a dynamo, and T means that it is purely toroidal and therefore not a dynamo; CT means that it is excluded on both grounds. The figures indicate which systems are physically distinct. For example, the pairs $S_2S_1^c$ and $S_2S_1^s$ are both marked 7 because they are merely different views of the same system and can be superposed if rotated through a right angle about the axis.

If S_1 and S_1^c have the same radial functions, the pair $S_1S_1^c$ is similar to S_1 and has an axis of symmetry; but if the radial functions are different this is not so and the pairs 2 of table 1

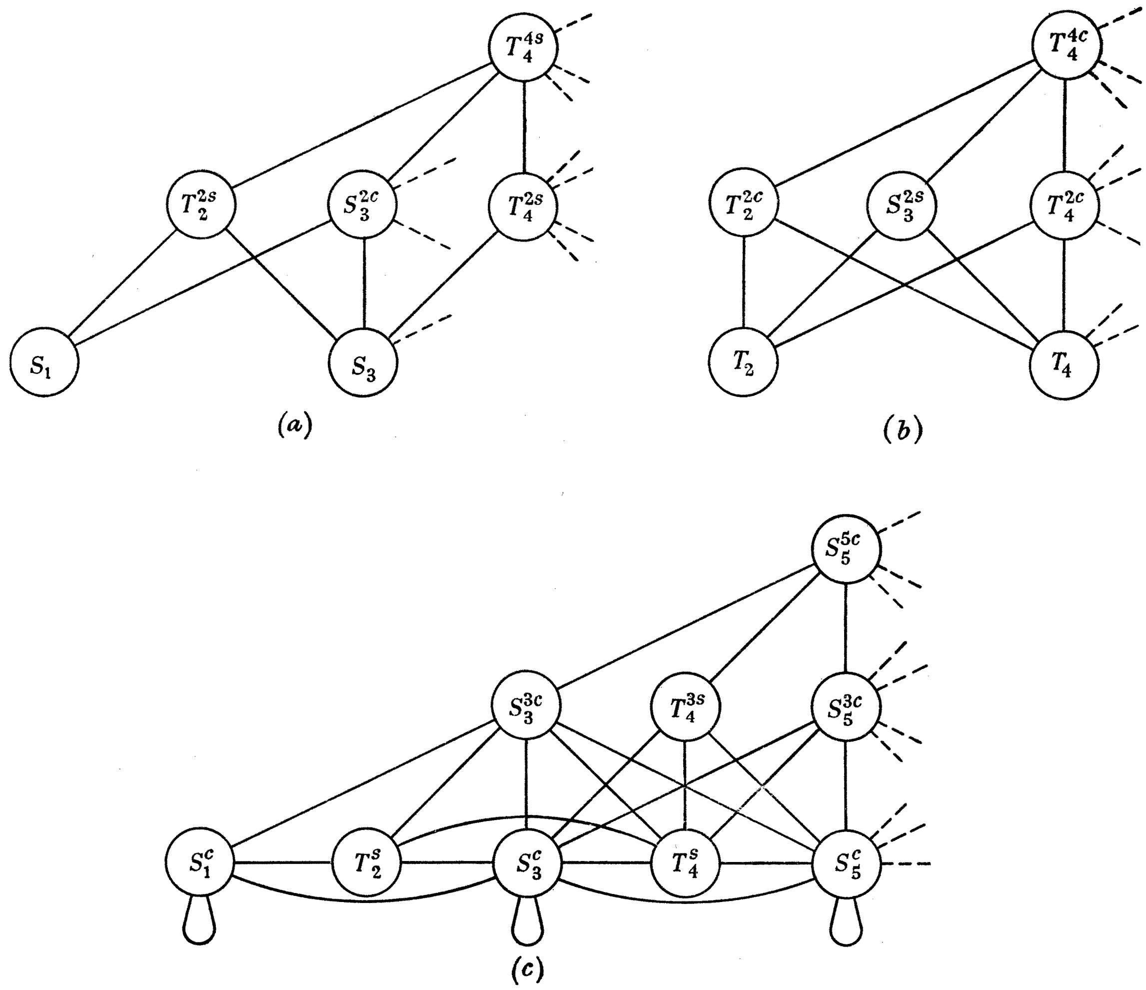


FIGURE 5. Interactions with an S_2^{2c} motion. A fourth diagram can be obtained by changing the suffix c to s in (c), and four more by changing S to T and T to S everywhere. The loop below S_1^c means that S_1^c is coupled to itself, that is, it occurs on both sides of the equation (25). Arrows have been omitted, as all interactions are reversible.

TABLE 1. COMBINATIONS OF MOTIONS DERIVED FROM PAIRS OF SPHERICAL HARMONICS OF NOT ABOVE THE SECOND DEGREE

S_1	S_1^c	S_1^s	T_1	T_1^c	T_1^s	S_2	S_2^c	S_2^s	S_2^{2c}	S_2^{2s}	T_2	T_2^c	T_2^s	T_2^{2c}	T_2^{2s}	
C	2	2	C	3	3	C	4	4	5	5	C	8	8	9	9	S_1
	C	2	3	C	3	7	4	5	6	4	10	8	9	11	8	S_1^c
		C	3	3	C	7	5	4	6	4	10	9	8	11	8	S_1^s
			TC	T	T	C	12	12	13	13	TC	T	T	T	T	T_1
				T	T	14	12	13	15	12	T	T	T	T	T	T_1^c
					T	14	13	12	15	12	T	T	T	T	T	T_1^s
						C	16	16	17	17	C	18	18	19	19	S_2
							1	20	21	20	23	24	25	26	25	S_2^c
								1	21	20	23	25	24	26	25	S_2^s
									1	22	27	28	28	24	29	S_2^{2c}
										1	27	25	25	29	24	S_2^{2s}
											TC	T	T	T	T	T_2
												T	T	T	T	T_2^c
													T	T	T	T_2^s
														T	T	T_2^{2c}
															T	T_2^{2s}

must be considered separately from the system with a single first harmonic. Similar remarks apply to other pairs such as S_2^c and S_2^s.

It is clearly impracticable to examine 29 possible dynamos in detail, particularly as most of them will have several possible types of field, as occurs for the S_2^{2c} motion in figure 5. The choice of which to investigate further has been made on dynamical grounds. The main interest of the problem is in connexion with the magnetic fields of the earth, sun and stars, which are all rotating. A rigid body rotation will have no effect, since (8) is unchanged by it (Bullard 1949*b*, p. 415). However, in a rotating body an S motion will produce a rotation

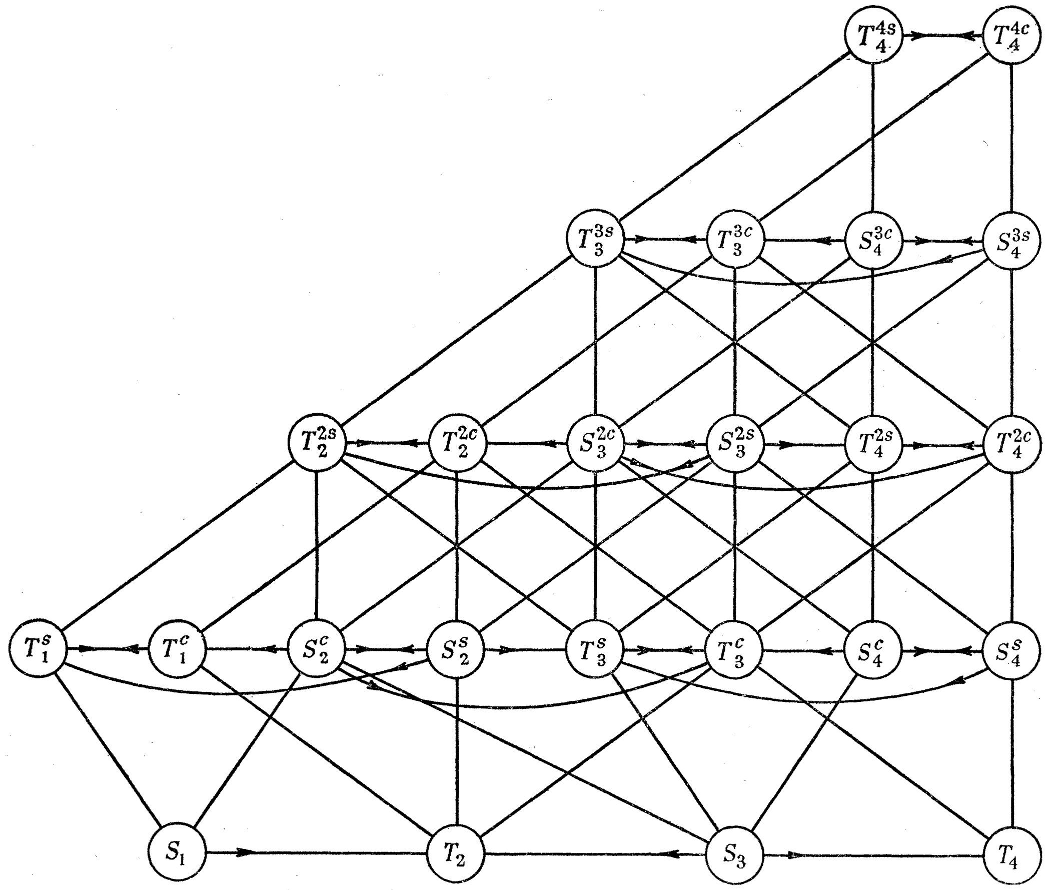


FIGURE 6. Interactions with T_1 and S_1^c motions. The coupling between rows is by the S_1^c motion, that along rows by the T_1 motion. All the S_1^c interactions are reversible and no arrows are shown for them. The connexions to terms of degree above four are not shown, owing to their complexity.

which varies with the radius. It has been estimated that the velocity associated with this will be much greater than the S velocity (Bullard 1949*a*, p. 444, and § 10 below). It may therefore be expected that T_1 motions will play a prominent part in terrestrial and astronomical dynamos. Only three of the physically distinct pairs in table 1 contain a T_1 motion; they are $T_1S_1^c$, $T_1S_2^c$ and $T_1S_2^{2c}$. The first two of these give two diagrams each. One of each is shown in figures 6 and 7; the others are obtained by writing T for S and S for T in these figures and reversing the one-way arrows, the indices being left where they are. The pair $T_1S_2^{2c}$ gives four distinct diagrams, two of which are shown in figures 8 and 9; the other two can be obtained from them as before. Only one diagram for each of the three velocity fields

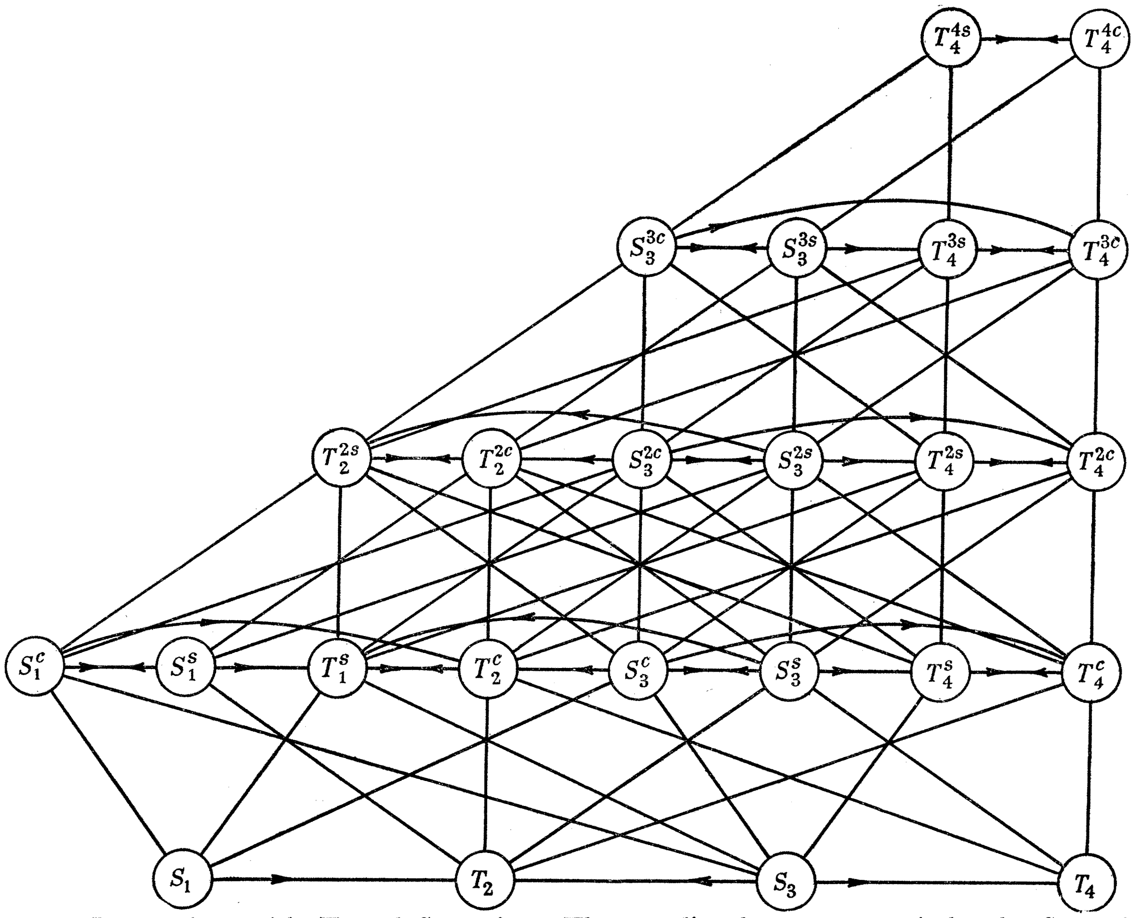


FIGURE 7. Interactions with T_1 and S_2^c motions. The coupling between rows is by the S_2^c motion, that along rows by the T_1 motion. All the S_2^c interactions are reversible and no arrows are shown for them. The connexions to terms of degree above four are not shown, owing to their complexity.

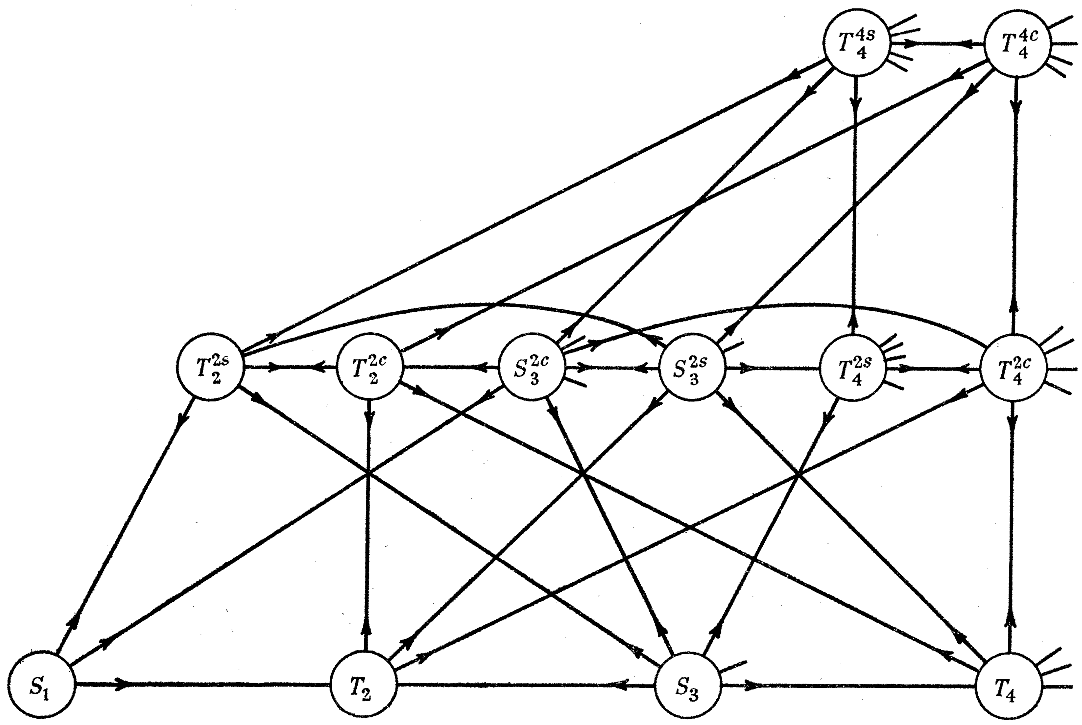


FIGURE 8. Interactions with T_1 and S_2^{2c} motions. The coupling between rows is by the S_2^{2c} motion, that along rows by the T_1 motion.

contains an S_1 field. Since a dipole field approximately parallel to the axis of rotation is the main feature of the earth's field, it is natural to investigate first the modes that give it. We thus arrive at the result previously stated without proof (Bullard 1949*a*), that of all the simple motions three stand out as likely to give dynamos relevant to the explanation of the earth's field. Of these, we have chosen to investigate the pair $T_1 S_2^{2c}$, mainly because figure 8 is simpler than figures 6 or 7. Also the pair $T_1 S_2^c$ seems less likely than the other two on dynamical grounds (see § 10). In a complete investigation, all four diagrams belonging to $T_1 S_2^{2c}$ should be investigated to discover which gives the lowest characteristic number; this has not been done.

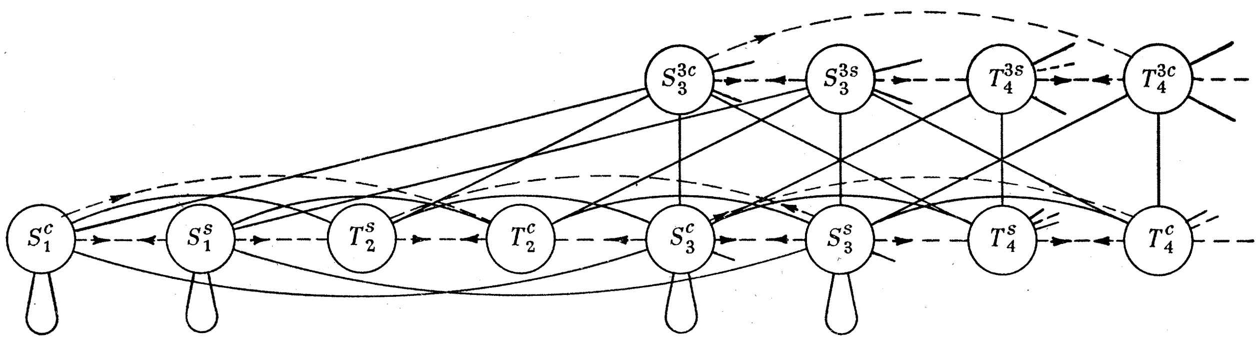


FIGURE 9. Interactions with T_1 and S_2^{2c} motions. The full lines represent coupling by the S_2^{2c} motion and are all reversible; no arrows are shown for these. The dotted lines represent coupling by the T_1 motion.

The condition given in § 5 for the occurrence of equal positive and negative values of V is equivalent to the occurrence of only even-sided figures in the diagrams of this section. This is true of figure 8 of the $T_1 S_2^{2c}$ velocity system.

7. The solution of equations (25)

The arguments of § 6 classify the sets of equations (25) that can occur, but they do nothing to show whether or not they have solutions satisfying the boundary conditions. It is impracticable to obtain numerical solutions of an infinite set of equations; we therefore work with the equations for a finite number of S's and T's and later consider the order of magnitude of the neglected terms.

It is natural to suppose that if the velocity distribution contains only harmonics of low degree, the main part of the field can also be expressed by the first few harmonics. For the reasons given in § 6 a motion compounded from S_2^{2c} and T_1 has been chosen for detailed investigation. The equations for harmonics of degree one and two in the field will first be considered, then those for degrees one to four and finally the effect of the higher harmonics.

(a) *Harmonics of degrees one and two*

Figure 10 gives the part of figure 8 referring to harmonics of degrees one and two. It contains one harmonic of the first degree, the field S_1 having the symmetry of a dipole but not its singularity, and three second harmonics T_2, T_2^{2c} and T_2^{2s}. The corresponding equations are, from (25),

$$\left.\begin{aligned} r^2\ddot{S}_1 - 2S_1 &= -\tfrac{216}{5} V Q_S T_2^{2s}, \\ r^2\ddot{T}_2 - 6T_2 &= -\tfrac{2}{3} V(\dot{Q}_T - 2Q_T/r)\, S_1 - \tfrac{72}{7} V[Q_S \dot{T}_2^{2c} + 2(\dot{Q}_S - Q_S/r)\, T_2^{2c}], \\ r^2\ddot{T}_2^{2c} - 6T_2^{2c} &= -\tfrac{6}{7} V[Q_S \dot{T}_2 + 2(\dot{Q}_S - Q_S/r)\, T_2] + 2VQ_T T_2^{2s}, \\ r^2\ddot{T}_2^{2s} - 6T_2^{2s} &= -2VQ_T T_2^{2c} - \tfrac{2}{3} V[3Q_S \dddot{S}_1 + (\dot{Q}_S - 6Q_S/r)\, \dot{S}_1 + (\ddot{Q}_S - 2\dot{Q}_S/r)\, S_1]. \end{aligned}\right\} \quad (29)$$

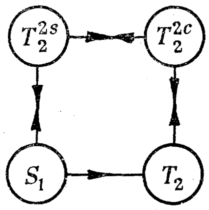


FIGURE 10. First and second degree harmonics in a dynamo with S_2^{2c} and T_1 motion. The horizontal connexions refer to the T_1 motion, the vertical ones to the S_2^{2c}.

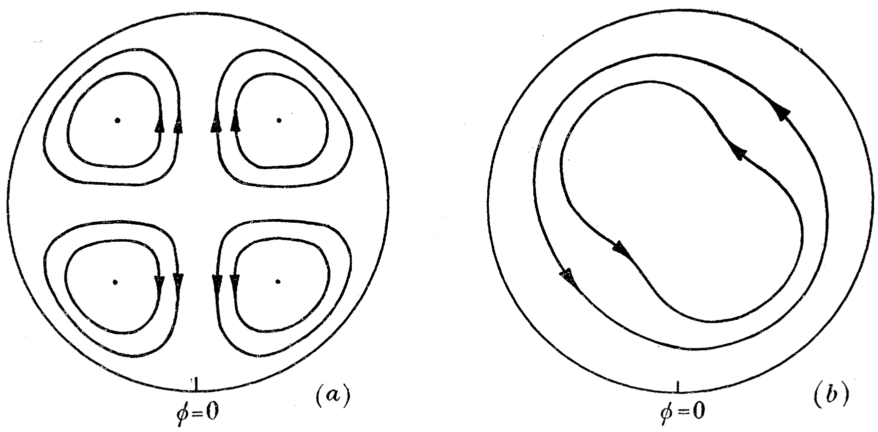


FIGURE 11. Stream lines in the equatorial section, (a) S_2^{2c} motion with $Q_S = r^3(1-r)^2$, (b) S_2^{2c} and T_1 motion with $Q_S = r^3(1-r)^2$, $Q_T = 10r^3$, ϵ has been taken to be 10 rather than a larger value, since the characteristics of the motion then show more plainly in a diagram.

Here the dots signify differentiation with respect to r and Q_S and Q_T have been written for the velocity radial functions in place of S_α and T_α so as to show more clearly which symbols refer to the field and which to the velocity. In the numerical work, Q_S and Q_T will usually be taken as†

$$Q_S = r^3(1-r)^2, \quad Q_T = \epsilon r^3, \quad (30)$$

or

$$Q_S = r^3(1-r)^2, \quad Q_T = \epsilon r^2(1-r), \quad (31)$$

where ϵ is a constant which determines the ratio of the T_1 and S_2^{2c} motions. (29) and (31) are equivalent to the equations studied by Takeuchi & Shimazu (1952 a, b). Equatorial sections of the stream lines of the motion (30) with $\epsilon = 0$ and 10 are given in figure 11. The

† The functions (30) and (31) give some discontinuous second derivatives for the velocity components. It would have been better to take $Q_S = r^3(1-r^2)^2$, $Q_T = \epsilon r^4$ and $\epsilon r^2(1-r^2)$, but the error was not noticed till the work was complete. It is considered unlikely that the solutions are greatly affected by the unsuitable choice of Q_S and Q_T, and in view of the great labour and expense involved, the calculations have not been repeated.

S_2^{2c} velocity vanishes at the centre and at the surface of the sphere. The T_1 velocity is a rotation with angular velocity proportional to the radius in (30) and proportional to $(1-r)$ in (31). When applied to the earth the whole system rotates with the earth, and the velocities calculated from (30) and (31) are to be thought of as superposed on the diurnal rotation, which itself has no effect on the field (an r^2 term in Q_T corresponds to a rigid-body rotation, but this has an effect as it leaves the S_2^{2c} velocity system fixed and is thus not a rotation of the whole system). Bullard (1949*a*) has estimated, from very crude arguments concerning the conservation of angular momentum, that the maximum of the T_1 velocity will be about 160 times the mean of the radial S_2^{2c} velocity at $r = 0{\cdot}8$ (see § 9 below). This would require ϵ to be about 20. The estimate is very rough, but it does suggest that the main practical interest is likely to be in values of ϵ of the order of 10 to 100.

As was shown in § 5, V can be taken as positive, since the latent roots always occur in pairs, $\pm V$; ϵ can also be taken as positive since its reversal merely changes the signs of T_2 and T_2^{2c}. These are general properties of the $S_2^{2c}T_1$ velocity system and are not due to the inclusion of only four equations in (29); they follow from the equivalence of a reversal of the velocities to rotation of the whole system through 180° about an axis in longitude 45° in the equatorial plane.

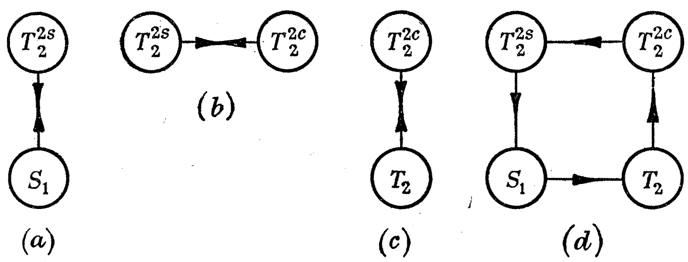


FIGURE 12. Closed loops from figure 10.

Figure 10 is made up of the four closed loops shown in figure 12, to which correspond four sets of terms from equations (29). Although (29) is later solved in its complete form, it is useful, for a physical understanding of the problem, to consider which of the loops 'really' drives the dynamo, that is, which are the terms in (29) that make the system act as a dynamo. The arguments of § 5 show that the connexions of figure 12*b* cannot give the required solution, since the motion is toroidal. The connexions of figure 12*a* and 12*c* are discussed below and in Bullard (1955); they have large characteristic values and very peculiar characteristic functions, and play little part in maintaining the dynamo for large ϵ. The square of figure 12*d* gives the essential features of the dynamo and contains the interactions discussed in Bullard (1949*a*). It is not difficult to satisfy oneself that the equations containing only the terms shown in figure 12*d* have solutions, and the main point at issue is whether the other terms, particularly the back coupling of T_2^{2s} to T_2^{2c} (the term $2VQ_TT_2^{2s}$ in (29)) will destroy the possibility of solutions.

A considerable variety of methods is available for the solution of equations such as (29). We have tried most of them and have found unexpected difficulties in getting reliable solutions. The essential point is that the method should not give spurious solutions when none exist.

(i) *Step-by-step integration*

If a trial value of V is assumed N independent solutions of N equations (25) may be found by integrating out from the origin. These are conveniently taken as the solutions in which one of the S_γ or T_γ behaves like $ar^{\gamma+1}$ at the origin and the rest have the corresponding term zero (a is a constant of integration). The boundary conditions at $r=1$ give N homogeneous linear equations connecting the N a's. If the trial V is the right one, the determinant of the coefficients of the a's will vanish. If it does not, a new V must be tried. The principal difficulty of this method is that although the N solutions are formally independent they are found to be very nearly proportional to each other for large V. The characteristic functions are then given as a linear combination of terms whose sum is small compared to the individual terms. Another practical disadvantage is the amount of rather specialized coding that is necessary if the work is to be done on an electronic computer. The computation of the series, which are necessary to start the integration, is also troublesome. In view of these difficulties the method has only been used for single equations and for pairs, and even then has not always been successful. Takeuchi & Shimazu have used this method for $\epsilon = \infty$.

(ii) *Iteration*

The equation
$$r^2\ddot{S}_\gamma - \gamma(\gamma+1)\,S_\gamma = VR$$
may be written
$$S_\gamma/r^{\gamma+1} = a_\gamma + V\int_0^r r^{-2\gamma-2}\int_0^r r^{\gamma-1}R\,\mathrm{d}r\,\mathrm{d}r. \tag{32}$$

Equations (25) will give a set of simultaneous equations of this form, where R represents the terms (24) and contains S's and T's which also occur on the left. It might be thought that if approximations to the characteristic functions are used on the right, and the a_γ and V chosen so that the boundary conditions are satisfied, improved functions would be obtained. It may be shown that such a process only converges to the required solution if the characteristic value V is that of smallest modulus; in fact, the smallest real V is usually not that of smallest modulus, and the process cannot be used. If V is left as a symbol, repeated application of (32) gives a polynomial in V whose coefficients are functions of r and whose roots for $r=1$ are the characteristic values.

The latter method has been used successfully for $\epsilon = 0$ (see below), but it is often difficult to judge whether it is converging. In one instance, in which a solution was known to exist, twenty iterations were performed only to find that the successive polynomials in V alternately had and had not a real root.

(iii) *Taylor series*

If the iteration of (32) is started by putting $r^{\beta+1}$ for the S_β and T_β in R, it provides a rapid method of computing the Taylor series for the S's and T's. We have expended much effort on these expansions, with very little success. The main difficulty is that in the series, which is a power series in V and r, the polynomial in r which multiplies a given power of V has a sum which is much less than its individual terms. Thus a large number of figures must be kept to give a significant result.

Takeuchi & Shimazu have expanded the T's and S's in series which in our notation would be $\Sigma a_n r^n(1-r)$ and $\Sigma a_n r^n[1-(n+\gamma)\,r/(1+n+\gamma)]$. Here each term satisfies the boundary conditions and the a's are determined by an adaptation of Rayleigh's principle. This method gives good characteristic values for V, though it is often difficult to be sure that the series are converging, particularly as the method becomes unmanageable if more than three terms are taken. The characteristic functions are naturally not well represented by so few terms. Rayleigh's principle applied to (25) will not give quadratic convergence to a characteristic value of V, though in practice the convergence seems satisfactory.

(iv) *Expansion in orthogonal functions*

Elsasser has suggested expanding the S's and T's in series of Bessel functions of half-integral order and solving the resulting linear algebraic equations. The coefficients are troublesome to compute as they contain integrals of triple products of Bessel functions and their derivatives. There seems no reason to suppose that the results obtained would be any better than those given by method (v) below.

(v) *Methods using matrices*

If the range from $r = 0$ to 1 is divided into M equal parts, the N differential equations and their boundary conditions may be replaced by linear homogeneous algebraic equations connecting the $N(M+1)$ ordinates at the points of division and at the end-points. If y_ν be the value of a dependent variable y at the νth point of division, the algebraic equations are obtained by substituting in the differential equations and the boundary conditions

$$\begin{aligned} y &= y_\nu, \\ \dot{y} &= M\delta' = \tfrac{1}{2}M(y_{\nu+1}-y_{\nu-1}), \\ \ddot{y} &= M^2\delta'' = M^2(y_{\nu+1}-2y_\nu-y_{\nu-1}), \end{aligned}$$

where δ' and δ'' are the central differences at the point y_ν. The equations corresponding to a single differential equation are got by giving ν the values $1, 2, \ldots, M$. For a T component of the field the values at $r = 0$ and 1 may be put equal to zero and there are $(M-1)$ ordinates to determine. For an S component the value at $r = 0$ is zero, but at $r = 1$, $\dot{S}_\gamma + \gamma S_\gamma = 0$. Let S_0 be the value of S_γ at $r = 1$, S_- that at $r = 1-1/M$, and S_+ that at $r = 1+1/M$. The latter has no physical existence, but can be used for computing $\dot{S}$ and $\ddot{S}$ at $r = 1$. At $r = 1$ the boundary condition gives

$$\tfrac{1}{2}M(S_+ - S_-) + \gamma S_0 = 0,$$

and thus

$$S_+ = S_- - 2\gamma S_0/M.$$

This value of S_+ may now be used to find $\dot{S}$ and $\ddot{S}$ when setting up the finite difference equation at $r = 1$. The right-hand side of the equation (29) for S_1 vanishes at $r = 1$; thus the equation derived from it at $r = 1$ can be used to express S_0 in terms of S_-. When this happens

$$S_0 = M^2S_-/[M^2 + \gamma M + \tfrac{1}{2}\gamma(\gamma+1)],$$

and the total number of equations and of unknowns will be $N(M-1)$.

The problem now consists in finding a real value of V such that the $N(M-1)$ equations have a solution, that is, in finding the latent roots and columns of a square matrix of order $N(M-1)$. The matrix can be partitioned into square submatrices† of order $(M-1)$ each

† If the equation for the point $r = 1$ has to be included for some, but not all, of the radial functions, some of the submatrices will not be square, but this causes no difficulty.

containing terms derived from a single S or T component of the field in a single equation of (29). If a term on the right of (29) contains S or T but not their differential coefficients, the corresponding submatrix will be diagonal; if it contains first or second derivatives the submatrix will have non-zero terms only on the diagonals and the adjacent lines parallel to the diagonals. Such matrices are called 'continuants'. If diagonal submatrices are represented by D_1, D_2, etc., continuants derived from the right-hand side of (29) by B_1, B_2, etc., and those from the left by F_1, F_2, etc., the matrix equation corresponding to (29) may be written

$$\begin{bmatrix} \mathsf{F}_1\lambda & 0 & 0 & \mathsf{D}_1 \\ 0 & \mathsf{F}_2\lambda & \mathsf{B}_2 & \epsilon\mathsf{D}_2 \\ \epsilon\mathsf{D}_3 & 12\mathsf{B}_2 & \mathsf{F}_2\lambda & 0 \\ \mathsf{B}_3 & -\epsilon\mathsf{D}_2 & 0 & \mathsf{F}_2\lambda \end{bmatrix} \begin{bmatrix} \mathsf{S}_1 \\ \mathsf{T}_2^{2c} \\ \mathsf{T}_2 \\ \mathsf{T}_2^{2s} \end{bmatrix} = 0, \tag{33}$$

where $\lambda = 1/V$ and the B's and D's do not contain V or ϵ. S_1 is the column matrix whose elements are the $(M-1)$ values of S_1 (here and elsewhere in this paper matrices will be

Table 2. Latent roots, $10^5\lambda^2$, from solutions of four equations with the range $0 \leqslant r \leqslant 1$ divided into 10 parts, $V = 1/\lambda$

$Q_S = r^3(1-r)^2,\ Q_T = \epsilon r^3$				$Q_S = r^3(1-r)^2,\ Q_T = \epsilon r^2(1-r)$		
$\epsilon = 10$	$\epsilon = 20$	$\epsilon = 100$	$\epsilon = \infty$	$\epsilon = 10$	$\epsilon = 100$	$\epsilon = \infty$
−18447·2	−67959·95	−1655470	+224·81	−8741·114	−558441	+56·553
− 3756·7	−10299·81	− 221917	+ 25·96	−2186·017	−113303	and 8
− 1453·3	− 3436·60	− 66699	+ 12·98	− 986·668	− 39394	smaller
− 799·4	− 1675·56	− 30164	and 6	− 590·962	− 18884	roots
− 548·7	− 1026·27	− 17198	smaller	− 464·599	− 11710	
− 387·3	− 728·02	− 9906	roots	− 308·205	− 7278	
− 341·3	− 583·05	− 4791		− 99·144	− 4765	
− 232·2	− 375·02	− 1497		− 85·218	− 3188	
− 43·0	+ 91·74	+ 205·5		+ 44·312	− 669	
+21·58 ± 15·90i	− 44·52	− 103		and 9	+ 56·4	
+ 1·789 ± 13·941i	and 8	+ 18		smaller	and 8	
+ 2·775 ± 2·145i	smaller	+ 15		roots	smaller	
+ 2·136	roots	+ 8			roots	
+ 0·167		+ 6				
0·000		+ 2				
		+ 1				
		and 2				
		smaller				
		roots				
$V = 216{\cdot}4$†	$V = 33{\cdot}02$	$V = 22{\cdot}06$	$V = 21{\cdot}09$	$V = 47{\cdot}50$	$V = 42{\cdot}1$	$V = 42{\cdot}05$

† Spurious root.

distinguished from vectors by the use of 'sans serif' type). By operations on rows, F_1 and F_2 can be made diagonal. This will turn the B's and D's into matrices without zero elements. Call the result

$$\begin{bmatrix} \mathsf{I}\lambda & \mathsf{A} \\ \mathsf{B} & \mathsf{I}\lambda \end{bmatrix} \begin{bmatrix} \mathsf{X}_1 \\ \mathsf{X}_2 \end{bmatrix} = 0, \tag{34}$$

where I is the unit matrix, X_1 is the column whose elements are the values of S_1 and T_2^{2c} and X_2 is similarly related to T_2 and T_2^{2s}. The squares of the λ's satisfying this are the latent roots of AB which is of order $2(M-1)$ instead of the $4(M-1)$ of (33) and (34). The corresponding latent column is X_1 (i.e. the values of S_1 and T_2^{2c} at the $(M-1)$ points). The column X_2 can be found from

$$\mathsf{X}_2 = -\lambda^{-1}\mathsf{B}\mathsf{X}_1,$$

or more symmetrically by finding the latent column of BA; the latter method gives λ^2 also and thus provides a check.

This process of halving the order is always possible when there are equal positive and negative roots for V. In § 6 it is shown that this occurs only if the diagram of the system consists entirely of even-sided figures. This is true of figure 8 and of some other simple systems, but is not true in general; for example, if an S_2^{2s} motion was added to figure 8, T_2 would be coupled to T_2^{2s} and a triangle would be formed. The top left and bottom right quarters of (33) would then contain terms without λ and the reduction to (34) would not be possible.

The matrix (33) was set up and reduced to the form (34) on a desk calculating machine. The elements were then punched as nine digit decimal numbers on Hollerith cards and converted to thirty-two digit binary numbers, twelve on a card, by the electronic computer, A.C.E. The A.C.E. then inserted a particular value of ϵ, formed the product AB and computed the latent roots, λ^2, by repeated multiplication by AB starting with an arbitrary column. The root required is the largest positive one, as this gives the smallest real value of V. The results are given in tables 2, 3 and 4 and in figures 15 and 16. The work involved is substantial. If the range is divided into ten parts the matrix (33) will be 36×36 and AB will be 18×18. The work is considerably increased by the existence of a number of negative values of λ^2 whose moduli are much larger than the required positive root and which have to be removed one by one before finding the positive root. These negative values of λ^2 are associated with the interaction shown in figures 12*b* and give imaginary values of V. The terms concerned in this in (29) with the Q_T from (30) give

$$\left.\begin{aligned} r^2\ddot{T}_2^{2c} - 6T_2 &= 2Vr^3T_2^{2s}, \\ r^2\ddot{T}_2^{2s} - 6T_2^{2s} &= -2Vr^3T_2^{2c}, \end{aligned}\right\} \tag{35}$$

which has solutions $\qquad T_2^{2c} = -\mathrm{i}\,T_2^{2s} = rJ_{\frac{5}{3}}(x),$

where $J_{\frac{5}{3}}$ is a Bessel function and $\qquad x^2 = \frac{2}{9}\mathrm{i}\epsilon Vr^3.$

The characteristic values of λ, derived from (35) are therefore

$$\lambda = 1/V = 2\epsilon\mathrm{i}/9x_s^2, \tag{36}$$

where the x_s are the zeros of $J_{\frac{5}{3}}(x)$. For $\epsilon = 100$ the first few of these give

iV from (36)	0·2495	0·7124	1·397	2·304	3·433	4·784	6·36	8·15	10·17
iV from (29)	0·2458	0·6713	1·224	1·821	2·411	3·177	4·57	8·17	22·1i†

† Real V.

The agreement is good for the lower roots, but as the roots increase the neglected terms in (29) get bigger and the approximation (35) poorer till the connexions of figure 12*d* take charge and a positive λ^2 and a real V are produced.

In order to get a rough idea of the nature of the solutions and the importance of the various terms, several solutions were first found with the range divided into only three parts.

TABLE 3. RADIAL FUNCTIONS FOR FOUR EQUATIONS, $Q_S = r^3(1-r)^2$

$Q_T = \epsilon r^3$

	$\epsilon=20,\ V=33\cdot02$				$\epsilon=100,\ V=22\cdot06$				$\epsilon\to\infty,\ V=21\cdot09$			
r	S_1	T_2	T_2^{2c}	T_2^{2s}	S_1	T_2	T_2^{2c}	T_2^{2s}	S_1	T_2/ϵ	ϵT_2^{2c}	T_2^{2s}
0·0	0·0000	0·000	0·0000	0·0000	0·00000	0·000	0·00000	0·0000	0·00	0·000	—	0·00
0·1	0·0230	0·255	0·0677	0·0064	0·00749	0·066	0·00768	0·0044	0·99	0·058	263·9	1·29
0·2	0·0847	0·962	0·2313	0·0339	0·02657	0·349	0·01823	0·0177	2·99	0·329	208·9	2·20
0·3	0·1269	1·000	0·1739	0·0346	0·03740	0·679	0·01117	0·0161	3·94	0·673	107·5	1·69
0·4	0·1247	0·897	0·0662	0·0223	0·03401	0·911	0·00250	0·0074	3·49	0·911	20·9	0·72
0·5	0·0922	0·896	0·0007	0·0067	0·02470	1·000	−0·00074	0·0005	2·54	1·000	−8·1	0·03
0·6	0·0552	0·883	−0·0179	−0·0084	0·01680	0·971	−0·00081	−0·0027	1·75	0·972	−7·9	−0·27
0·7	0·0327	0·772	−0·0094	−0·0143	0·01229	0·855	−0·00032	−0·0031	1·30	0·859	−3·2	−0·31
0·8	0·0243	0·579	−0·0019	−0·0099	0·01015	0·663	−0·00009	−0·0021	1·08	0·668	−0·9	−0·21
0·9	0·0213	0·324	−0·0003	−0·0032	0·00898	0·383	−0·00003	−0·0007	0·96	0·387	−0·3	−0·07
1·0	0·0192	0·000	0·0000	0·0000	0·00809	0·000	0·00000	0·0000	0·86	0·000	—	0·00

$Q_T = \epsilon r^2(1-r)$

	$\epsilon=10,\ V=47\cdot50$				$\epsilon=100,\ V=42\cdot1$				$\epsilon\to\infty,\ V=42\cdot05$			
r	S_1	T_2	T_2^{2c}	T_2^{2s}	S_1	T_2	T_2^{2c}	T_2^{2s}	S_1	T_2/ϵ	ϵT_2^{2c}	T_2^{2s}
0·0	0·0000	0·000	0·0000	0·0000	0·00000	0·000	0·00000	0·0000	0·000	0·000	0·00	0·000
0·1	0·0065	0·032	0·0072	0·0033	0·00020	−0·002	0·00000	−0·0001	0·019	−0·002	0·00	−0·006
0·2	0·0204	0·120	0·0287	0·0068	0·00088	−0·013	−0·00002	−0·0002	0·084	−0·014	−0·31	−0·021
0·3	0·0266	0·117	0·0285	0·0052	0·00244	−0·020	0·00012	0·0001	0·239	−0·022	0·97	0·001
0·4	0·0230	0·122	0·0284	0·0055	0·00441	0·026	0·00079	0·0012	0·444	0·022	7·93	0·121
0·5	0·0060	0·182	0·0275	0·0076	0·00376	0·200	0·00128	0·0027	0·388	0·199	13·35	0·273
0·6	−0·0299	0·385	0·0021	0·0069	−0·00262	0·530	0·00020	0·0017	−0·257	0·532	2·03	0·169
0·7	−0·0811	0·769	−0·0612	−0·0130	−0·01205	0·880	−0·00149	−0·0038	−1·211	0·882	−15·24	−0·394
0·8	−0·1188	1·000	−0·0978	−0·0638	−0·01755	1·000	−0·00153	−0·0108	−1·764	1·000	−15·32	−1·085
0·9	−0·1183	0·723	−0·0664	−0·0766	−0·01732	0·717	−0·00108	−0·0117	−1·741	0·717	−10·98	−1·177
1·0	−0·1066	0·000	0·0000	0·0000	−0·01560	0·000	0·00000	0·0000	−1·568	0·000	—	0·000

Such a coarse division cannot be expected to give accurate results, but it does enable a number of possibilities to be studied with little labour. The systems indicated in figure 13 were studied in turn and gave the following values of V for $\epsilon = 100$ and Q_S and Q_T as in (30):

figure 13	*a*	*b*	*c*	*d*
V	5·04	18·8	18·8	19·0

This confirms the conjecture (Bullard 1949*a*) that the important terms are those represented by the arrows running counter-clockwise round figure 13*a* and the back-coupling of T_2^{2s} to T_2^{2c}, and that the couplings from S_1 to T_2^{2s} and T_2^{2c} to T_2 are of secondary importance.

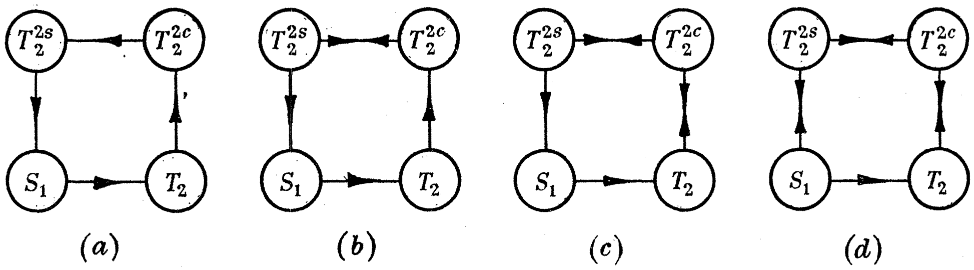


FIGURE 13. Stages in the preliminary calculations for the dynamo with S_2^{2c} and T_1 motions (cf. figure 10).

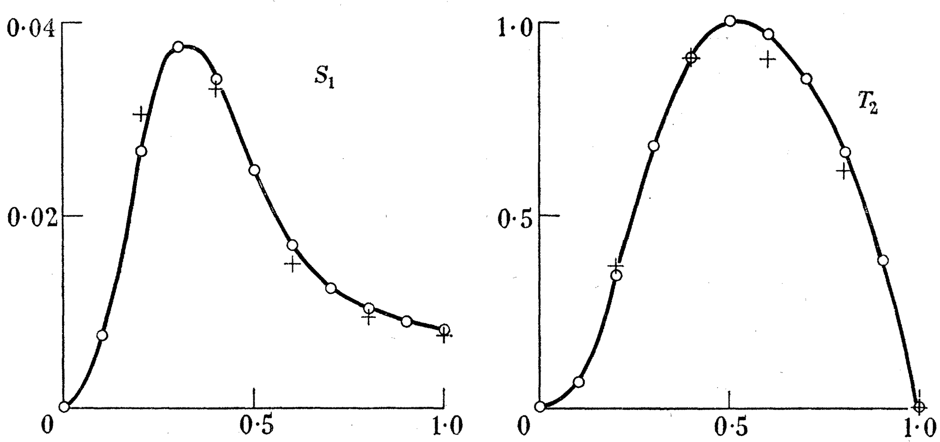


FIGURE 14. Radial functions S_1 and T_2 for solutions of four equations with the range divided into five parts (+) and into ten parts (○). $Q_S = r^3(1-r)^2$, $Q_T = \epsilon r^3$, $\epsilon = 100$.

Solutions were then obtained for $\epsilon = 100$, $Q_T = r^3$ with the range divided into five and ten parts. The comparison of the values obtained for V with increasing fineness of division provides an important check on the adequacy of the numerical methods. The results are

M	3	5	10
V	19·0	22·0	22·06

It appears that increasing the fineness of the division will change V by not more than a few parts in a thousand. The radial functions S_1 and T_2 for $\epsilon = 100$, $M = 5$ and 10 are compared in figure 14; the scale has been chosen so that the T_2 functions agree at $r = 0·4$. The accuracy may be estimated by Richardson's method of the 'deferred approach to the limit'. In this the value x_M of a quantity, estimated by dividing the range into M parts is assumed to be

$$x_M = x + a/M^2,$$

where x is the true value and a is independent of M. From this it follows that the correction required by x_{10} is $\frac{1}{3}(x_{10}-x_5)$. The results for the four radial functions with $Q_T = \epsilon r^3$, $\epsilon = 100$ are:

	S_1			T_2^{2c}			T_2			T_2^{2s}		
r	$M=10$	$M=5$	%	$M=10$	$M=5$	%	$M=10$	$M=5$	%	$M=10$	$M=5$	%
0·2	0·02657	0·03060	−3·6	0·01823	0·01459	+6·6	0·349	0·370	−0·7	+0·0177	0·0183	+1·0
0·4	0·03401	0·03323	+0·7	0·00250	0·00259	−0·2	0·911	0·911	(0·0)	+0·0074	0·0068	+1·0
0·6	0·01680	0·01515	+1·5	−0·00081	−0·00076	−0·1	0·971	0·906	+2·2	−0·0027	−0·0025	+0·6
0·8	0·01015	0·00966	+0·4	−0·00009	−0·00008	0·0	0·663	0·621	+1·4	−0·0021	−0·0020	+0·0
1·0	0·00809	0·00779	+0·3	—	—	—	—	—	—	—	—	—

The columns headed % give the value of $\frac{1}{3}(x_{10}-x_5)$ expressed as a percentage of the largest of the ten values of the radial function obtained when the range is divided into ten parts. It appears that the radial functions are reliable to about 1 % of their maxima except near the origin, where some of them may be in error by 5 or 10 %.

This estimate was confirmed and the calculations checked by substituting the results in the differential equations (29). If only second differences are retained the calculated values should satisfy (29) exactly. If third and fourth differences are included δ' will be replaced by $(\delta'-\frac{1}{6}\delta''')$ and δ'' by $(\delta''-\frac{1}{12}\delta^{\text{iv}})$; V calculated from the ratio of the right- and left-hand sides of (29) will now not agree exactly with the previous value, and the difference will give an indication of the uncertainty of the result. The procedure may be illustrated by the results for the second of equations (29) for $Q_T = r^3$, $\epsilon = 100$ and $M = 10$. The finite difference form of this equation is

$$[100r^2(\delta''-\tfrac{1}{12}\delta^{\text{iv}})-6]\,T_2 = V[-\tfrac{200}{3}r^2S_1-\tfrac{72}{7}\{10r^3(1-r)^2\,(\delta'-\tfrac{1}{6}\delta''')+4r^2(1-r)\,(1-2r)\}\,T_2^{2c}],$$

where δ', etc., represent central differences. The values of T_2, S_1 and T_2^{2c} from table 3 give:

	l.h.s.		r.h.s. ÷ V		V	
r	without δ^{iv}	with δ^{iv}	without δ'''	with δ'''	without δ''' and δ^{iv}	with δ''' and δ^{iv}
0·2	−1·906	−1·914	−0·0867	−0·0870	22·0	22·0
0·3	−4·956	−5·028	−0·2253	−0·2227	22·0	22·3
0·4	−7·754	−7·850	−0·3506	−0·3469	22·1	22·6
0·5	−8·95	−8·95	−0·4064	−0·4077	22·0	22·0
0·6	−8·96	−8·89	−0·4049	−0·4059	22·1	22·1
0·7	−8·85	−8·76	−0·4034	−0·4037	21·9	21·9
0·8	−9·61	−9·61	−0·4337	−0·4337	22·2	22·2
				mean	22·04	22·16

The value of V obtained without the δ''' and δ^{iv} terms agrees with that calculated from the matrix. This check was performed on every entry in table 3 and provides a complete check on the setting up and solution of the matrices. The change in V on including the δ''' and δ^{iv} terms is 0·5 % and hardly exceeds the uncertainty due to rounding off T_2 to three decimals.

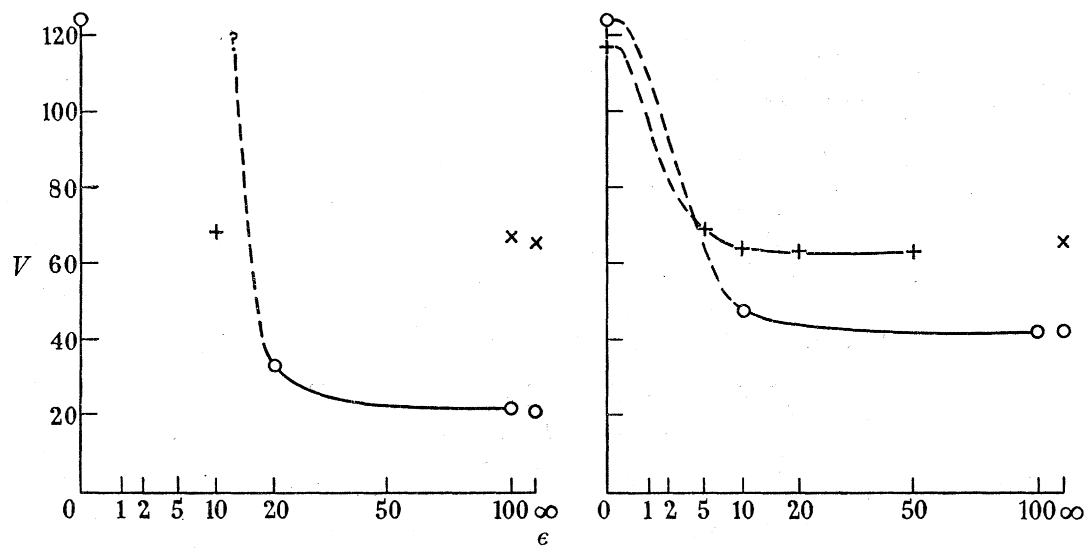
An $N\times N$ matrix such as (33) possesses N latent roots and N latent columns; most of these are approximations to the characteristic numbers and characteristic functions of the differential equations (29), but a few of the smallest may not be. These small roots correspond to functions that alternate in sign many times in the range $0<r<1$; their value depends critically on the fineness of the division of the range. If a latent column derived from the matrix (33) gives four smoothly varying radial functions each with not more than two zeros, and if the functions and the latent root do not change much with the fineness of division of the range, the results can be accepted as approximations to the radial functions and characteristic number of (29), and the accuracy estimated as above. If the sign of the terms in the latent column alternates many times they do not give an approximation to the solution of (29); one example of this has occurred in the present work. Intermediate cases where the results are of doubtful significance can arise. These can be resolved only by repeating the calculation at a finer interval, but fortunately none has occurred in the present work.

The values of λ^2 obtained with the range divided into ten parts are given in table 2. Each matrix has 18 roots except for $\epsilon = \infty$ when there are only 9 (see below); when a smaller number is given in table 2 the remainder have not been computed. The variation of V with ϵ is shown in table 4 and figure 15; in drawing this figure $\mathrm{d}V/\mathrm{d}\epsilon$ has been made zero at $\epsilon = 0$, this can be proved to be so by showing that V is an even function of ϵ, which follows from

TABLE 4. CRITICAL VALUES OF V FOR $Q_S = r^3(1-r)^2$, RANGE OF r DIVIDED INTO TEN PARTS

Q_T	four equations		seven equations		twelve equations	
ϵ	ϵr^3	$\epsilon r^2(1-r)$	ϵr^3	$\epsilon r^2(1-r)$	ϵr^3	$\epsilon r^2(1-r)$
0	124	124	117·6	117·6	—	—
5	—	—	—	68·8	—	—
10	none	47·50	69·0	63·9	—	—
20	33·02	—	—	63·3	—	—
50	—	—	—	63·3	—	—
100	22·06	42·1	—	—	67·4	—
∞	21·09	42·05	—	—	65·8	65·5

A — indicates that no solution has been attempted; it is known that the results for seven equations are considerably increased by further subdivision of the range for r (see text).



(a) $Q_T = \epsilon r^3$ (b) $Q_T = \epsilon r^2(1-r)$

FIGURE 15. Variation of V with ϵ; $Q_S = r^3(1-r)^2$; range of r divided into ten parts (except for $\epsilon = 0$); $\circ$, from four equations; $+$, from seven equations; $\times$, from twelve equations.

Q_T	ϵ	V
ϵr^3	20	33·02
ϵr^3	100	22·06
ϵr^3	$\rightarrow\infty$	21·09
$\epsilon r^2(1-r)$	10	47·50
$\epsilon r^2(1-r)$	100	42·1
$\epsilon r^2(1-r)$	$\rightarrow\infty$	42·05

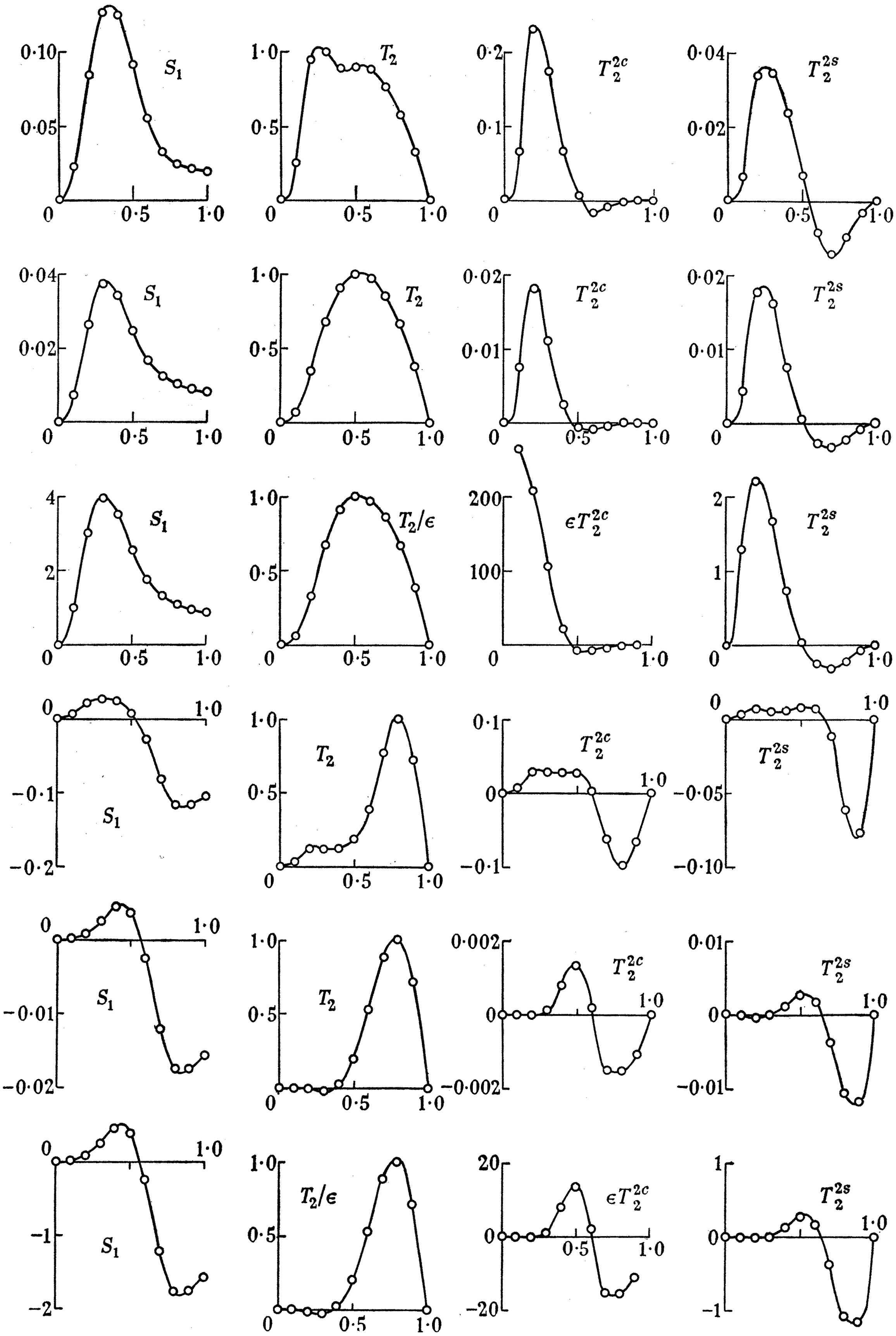


FIGURE 16. Radial functions from four equations with the range divided into ten parts; $Q_S = r^3(1-r)^2$.

(40) below. The largest positive root for $\epsilon = 10$, $Q_T = r^3$ was $V = 216$; this gave six or seven zeros in the interval $0<r<1$ for all four vectors and is clearly a spurious root of the kind discussed above. It seems likely that with this Q_T the four equations have no solution when ϵ is in a range running from a small value to somewhere between 10 and 20. The radial functions for the largest positive λ^2 are given in table 3 and figure 16; they have been normalized so that the largest value of T_2 is unity.

The special cases $\epsilon = 0$ and $\epsilon \to \infty$ will now be considered.

No T_1 motion, $\epsilon = 0$

If ϵ, and therefore Q_T, is zero, the equations (29) separate into two unconnected pairs. One of these pairs is

$$r^2\ddot{T}_2 - 6T_2 = -\tfrac{72}{7}V[Q_S\dot{T}_2^{2c} + 2(\dot{Q}_S - Q_S/r)\,T_2^{2c}], \tag{37}$$

$$r^2\ddot{T}_2^{2c} - 6T_2^{2c} = -\tfrac{6}{7}V[Q_S\dot{T}_2 + 2(\dot{Q}_S - Q_S/r)\,T_2]. \tag{38}$$

If T_2 and V are replaced by T_2' and V' given by

$$T_2 = 2\sqrt{3}\,T_2', \quad V = 7V'/12\sqrt{3},$$

the constants $\frac{72}{7}$ and $\frac{6}{7}$ disappear from (37) and (38). The solutions of the resulting pair of equations that vanish at $r = 0$ may be written

$$T_2' = af_1(r, V') + bf_2(r, V'),$$

$$T_2^{2c} = bf_1(r, V') + af_2(r, V'),$$

where a and b are constants and f_1 and f_2 are independent solutions which could be found by integrating outwards from the origin. If T_2 and T_2^{2c} are to vanish at $r = 1$, a and b must be chosen so that a/b is equal both to $-f_1(1, V')/f_2(1, V')$ and to $-f_2(1, V')/f_1(1, V')$. This requires V' and a/b to be chosen so that either

$$a = b, \quad f_1(1, V') = -f_2(1, V')$$

or

$$a = -b, \quad f_1(1, V') = f_2(1, V').$$

The former gives $T_2' = T_2^{2c}$ and the latter $T_2' = -T_2^{2c}$. The characteristic values of V corresponding to these two solutions will be equal and of opposite sign, for (37) and (38) are unaffected if the sign of both V and T_2/T_2^{2c} are changed. It is thus only necessary to consider the solutions in which $T_2' = T_2^{2c}$. For these (37) and (38) are identical and we have the single equation

$$r^2\ddot{T} - 6T = -V'[Q_S\dot{T} + 2(\dot{Q}_S - Q_S/r)\,T], \tag{39}$$

where T stands for either T_2' or T_2^{2c}. For every solution of this with a characteristic value V' there will be two solutions of (37) and (38) with characteristic values $\pm V$, but in (39) the sign of V' cannot be reversed.

It is known that (39) possesses solutions satisfying the boundary conditions if Q_S has no zeros in $0<r<1$ and if suitable restrictions are placed on its behaviour near $r = 0$ and $r = 1$. The solutions are of two kinds. One kind depends on the form of Q_S near $r = 0$. If Q_S behaves like $a\,r^\sigma$ for small r, where σ is a positive integer which for continuity at $r = 0$ must not be less than three, $2(\dot{Q}_S - Q_S/r)$ will behave like $2a(\sigma-1)\,r^{\sigma-1}$, and a sufficient condition for the existence of a solution with positive V is (Bullard 1955)

$$\Gamma\left(\frac{3-\sigma}{1-\sigma}\right) < 0.$$

Since for all σ exceeding three, $0 > (3-\sigma)/(1-\sigma) > -1$, there exist solutions for all Q_S which vanish at the origin at least as rapidly as r^4. If $\sigma = 3$, there may or may not be a solution of this kind; the necessary conditions are not known, but there is believed to be none for $Q_S = r^3(1-r)^2$. The solution for $Q_S = r^4(1-r)^2$ has been found by step-by-step integration for a series of trial values of V'. For convenience in the numerical work, which was done on the A.C.E., the term in $\dot{T}$ in (39) was removed before integration by the substitution

$$T = \Phi \exp\left(-\tfrac{1}{2}V' \int Q_S \,\mathrm{d}r\right).$$

Some values of T are given in table 5 and figure 17; V was found to be 137.

Table 5. Solutions for $\epsilon=0$ normalized so that $T/r^3=1$ at $r=0$

	$Q_S=r^3(1-r)^2$		$Q_S=r^4(1-r)^2$		$Q_S=r^{17}(1-r)^2$
V	124		137	239	1307
r	ψ/r^3	T/r^3	T/r^3	T/r^3	T/r^3
0·00	1·0000	1·000	1·000	1·000	1·00000
0·05	0·9965	1·538	0·983	1·031	1·00000
0·10	0·9760	4·930	0·879	1·293	1·00000
0·15	0·9344	$2·776\times10$	0·671	1·990	1·00000
0·20	0·8761	$2·366\times10^2$	0·422	4·374	1·00000
0·25	0·8073	$2·672\times10^3$	0·214	$1·346\times10$	1·00000
0·30	0·7328	$3·542\times10^4$	$8·85\times10^{-2}$	$5·685\times10$	1·00000
0·35	0·6561	$4·947\times10^5$	$3·02\times10^{-2}$	$3·170\times10^2$	1·00000
0·40	0·5795	$6·622\times10^6$	$8·81\times10^{-3}$	$2·200\times10^3$	0·99997
0·45	0·5047	$7·827\times10^7$	$2·27\times10^{-3}$	$1·766\times10^4$	0·99968
0·50	0·4328	$7·626\times10^8$	$5·44\times10^{-4}$	$1·510\times10^5$	0·99834
0·55	0·3647	$5·796\times10^9$	$1·26\times10^{-4}$	$1·259\times10^6$	0·99355
0·60	0·3012	$3·294\times10^{10}$	$2·97\times10^{-5}$	$9·343\times10^6$	0·97891
0·65	0·2428	$1·358\times10^{11}$	$7·4\times10^{-6}$	$5·670\times10^7$	0·94175
0·70	0·1899	$3·996\times10^{11}$	$2·0\times10^{-6}$	$2·605\times10^8$	0·85644
0·75	0·1429	$8·333\times10^{11}$	$6·1\times10^{-7}$	$8·493\times10^8$	0·70352
0·80	0·1020	$1·234\times10^{12}$	$2·1\times10^{-7}$	$1·872\times10^9$	0·48771
0·85	0·0674	$1·307\times10^{12}$	$8·3\times10^{-8}$	$2·697\times10^9$	0·26627
0·90	0·0391	$0·979\times10^{12}$	$3·5\times10^{-8}$	$2·463\times10^9$	0·11012
0·95	0·0168	$0·466\times10^{12}$	$1·3\times10^{-8}$	$1·280\times10^9$	0·03374
1·00	0·0000	0·000	0	0	0·00000

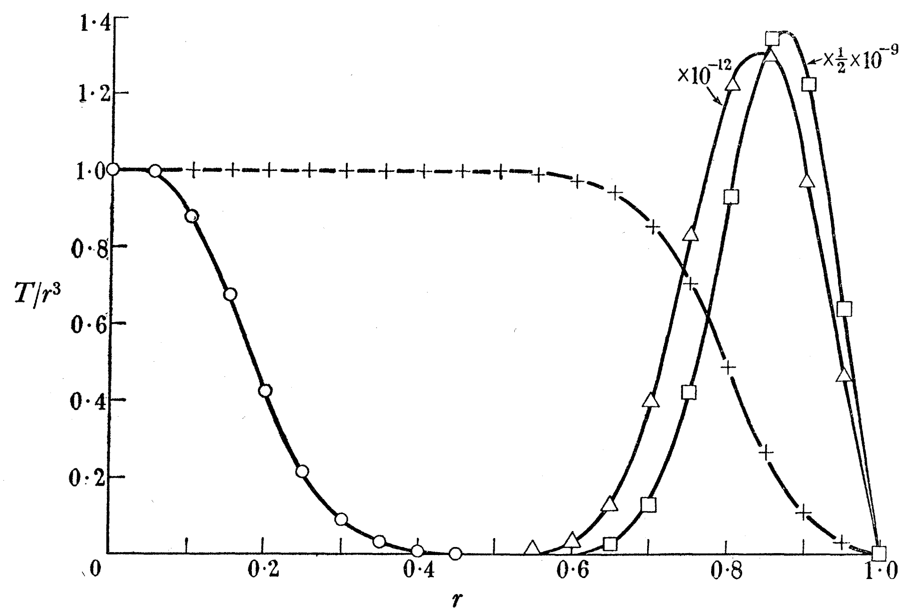


Figure 17. Solutions with $\epsilon=0$, normalized so that $T/r^3=1$ at $r=1$. ○, $Q_S=r^4(1-r)^2$, $V=137$; □, $Q_S=r^4(1-r)^2$, $V=239$; △, $Q_S=r^3(1-r)^2$, $V=124$; +, $Q_S=r^{17}(1-r)^2$, $V=1307$.

A solution was also found for $Q_S = r^{17}(1-r)^2$ by repeated use of (32); V was 1307 and settled to 7 figures after 10 iterations. T is given in table 5 and figure 17.

Solutions having V' negative and depending on the form of Q_S near $r = 1$ are also possible. A sufficient condition for these to exist when Q_S behaves like $(1-r)^{\sigma_1}$ near $r = 1$ is that

$$\Gamma\left(\frac{1-\sigma_1}{1+\sigma_1}\right) < 0.$$

This is satisfied for all σ_1 greater than one. Solutions can exist with $\sigma_1 = 1$, but it is not known if they always do; none are believed to exist for $Q_S = r^3(1-r)$. The solutions for $Q_S = r^3(1-r)^2$ and $r^4(1-r)^2$ have been obtained by step-by-step integration. The values found for V were 124 and 239. The results for T/r^3 are given in table 5 and figure 17. The integration is more difficult than in the preceding cases because T is very small over the lower half of the range and then increases with extreme rapidity so that the ratio of its values at $r = 0{\cdot}2$ and $0{\cdot}8$ is 3×10^{11}. To make the numerical work easier the substitution

$$T = \psi \exp\left(-V\int r^{-2} Q_S \,\mathrm{d}r\right)$$

was made before integrating. This removes the leading term in the asymptotic solution for large V obtained by the W.B.K. method. ψ/r^3 is given in table 5 and is seen to be a slowly varying function.

All the solutions for $\epsilon = 0$ are somewhat pathological functions, in that they have an extensive region where they are very small compared to their values over the rest of the range. The main part of the changes in the functions are included in a small fraction of the range of r and a fine division is necessary to get accurate results. For example, dividing the range into ten parts gave $V = 140$ for $Q_S = r^3(1-r)^2$ compared to the value of 124 obtained by a series of step-by-step integrations with trial values of V, using an interval of $0{\cdot}005$ in r. Small additional terms on the right-hand side of the equation in the region where the function is small might have a large effect throughout the range, and it would be rash to take the existence of these solutions of (29) as implying the existence of solutions of the infinite set of equations including terms of all degrees when $\epsilon = 0$.

The other pair of equations for $\epsilon = 0$ is

$$r^2\ddot{S}_1 - 2S_1 = -\tfrac{216}{5} V Q_S T_2^{2s},$$
$$r^2\ddot{T}_2^{2s} - 6T_2^{2s} = -2V[Q_S\ddot{S}_1 + \tfrac{1}{3}(\dot{Q}_S - 6Q_S/r)\,\dot{S}_1 + \tfrac{1}{3}(\ddot{Q}_S - 2\dot{Q}_S/r)\,S_1].$$

It is known (Bullard 1955) that this pair of equations possesses a solution satisfying the boundary conditions if the behaviour of Q_S near $r = 0$ is suitably restricted, but all attempts to find it by the matrix and step-by-step methods have failed. It may be that the characteristic value is very large.

T_1 *motion much faster than* S_2^{2c}, $\epsilon \gg 1$

There can be no solution of (29) satisfying the boundary conditions if $Q_S = 0$, no matter how large Q_T may be, for the latter is purely toroidal. In this section the possibility of solutions for indefinitely large ϵ and Q_T, and finite V and Q_S is examined. We try to find solutions for which as $\epsilon \to \infty$ the ratio of T_2, T_2^{2c} and T_2^{2s} to S_1 for any r is $O(\epsilon^s)$, where s is a small integer which is the same for all r but may be different for the three T's. The argument is most easily presented, in a form that can be generalized for more equations, by regarding (33) an a condensed version of the differential equations (29) and not as a numerical matrix. The S's and T's are then the continuous radial functions S and T and not column matrices. The F's and B's are, for this purpose, regarded as differential operators and the D's as functions of r (e.g. $\mathsf{F}_1 = (r^2\,\mathrm{d}^2/\mathrm{d}r^2 - 2)$). Consider the orders of magnitude of the terms as functions of ϵ. Since the first row does not contain ϵ and relates S_1 and T_2^{2s}, these two functions must be of the same order in ϵ, say order one. The fourth row contains a term $-\epsilon \mathsf{D}_2 T_2^{2c}$, which must be balanced by the terms $\mathsf{B}_3 S_1$ and $\mathsf{F}_2 T_2^{2s}$ of order one. Thus T_2^{2c} must be $O(\epsilon^{-1})$ except near the zeros of D_2. The other two equations then show T_2 to be of order ϵ except near zeros of $\mathsf{D}_2 T_2^{2s}$ and $\mathsf{D}_3 S_1$. This suggests putting

$$T_2^{2c} = \overline{T}_2^{2c}/\epsilon, \quad T_2 = \epsilon \overline{T}_2.$$

With this substitution (33) becomes

$$\begin{bmatrix} \mathsf{F}_1\lambda & 0 & 0 & \mathsf{D}_1 \\ 0 & \mathsf{F}_2\lambda/\epsilon^2 & \mathsf{B}_2 & \mathsf{D}_2 \\ \mathsf{D}_3 & 12\mathsf{B}_2/\epsilon^2 & \mathsf{F}_2\lambda & 0 \\ \mathsf{B}_3 & -\mathsf{D}_2 & 0 & \mathsf{F}_2\lambda \end{bmatrix} \begin{bmatrix} S_1 \\ \overline{T}_2^{2c} \\ \overline{T}_2 \\ T_2^{2s} \end{bmatrix} = 0. \tag{40}$$

This expression can be manipulated like a set of algebraic equations provided it is remembered that multiplication by the F's and B's is non-commutative. If the terms in $1/\epsilon^2$ are neglected the fourth row is the only one to contain $\overline{T}_2^{2c}$ and gives

$$\overline{T}_2^{2c} = \mathsf{D}_2^{-1}(\mathsf{B}_3 S_1 + \mathsf{F}_2 T_2^{2s}\lambda), \tag{41}$$

which fixes T_2^{2c} when λ, S_1 and T_2^{2s} have been found. Eliminating D_1 from the first row gives

$$\begin{bmatrix} \mathsf{F}_1\lambda & -\mathsf{D}_1\mathsf{D}_2^{-1}\mathsf{B}_2 & 0 \\ 0 & \mathsf{B}_2 & \mathsf{D}_2 \\ \mathsf{D}_3 & \mathsf{F}_2\lambda & 0 \end{bmatrix} \begin{bmatrix} S_1 \\ \overline{T}_2 \\ T_2^{2s} \end{bmatrix} = 0.$$

The second row of this gives
$$T_2^{2s} = -\mathsf{D}_2^{-1}\mathsf{B}_2\overline{T}_2, \tag{42}$$

and the other two give
$$\begin{bmatrix} \mathsf{F}_1\lambda & -\mathsf{D}_1\mathsf{D}_2^{-1}\mathsf{B}_2 \\ \mathsf{D}_3 & \mathsf{F}_2\lambda \end{bmatrix} \begin{bmatrix} S_1 \\ \overline{T}_2 \end{bmatrix} = 0. \tag{43}$$

This represents a pair of second-order differential equations which can be solved by regarding (43) as a numerical matrix analogous to (33). Suppose the terms in λ^2 in (43) to be converted into unit matrices I by operations on rows which transform $-\mathsf{D}_1\mathsf{D}_2^{-1}\mathsf{B}_2$ and D_3 into $\overline{\mathsf{B}}_2$ and $\overline{\mathsf{D}}_3$; (43) then becomes

$$\begin{bmatrix} \mathsf{I}\lambda & \overline{\mathsf{B}}_2 \\ \overline{\mathsf{D}}_3 & \mathsf{I}\lambda \end{bmatrix} \begin{bmatrix} \mathsf{S}_1 \\ \overline{\mathsf{T}}_2 \end{bmatrix} = 0. \tag{44}$$

This shows that λ^2 is the latent root of $\overline{\mathsf{B}}_2\overline{\mathsf{D}}_3$, the corresponding latent column being S_1. $\overline{\mathsf{T}}_2$ is the latent column of $\overline{\mathsf{D}}_3\overline{\mathsf{B}}_2$, and T_2^{2c} and T_2^{2s} can be found from (41) and (42). All these operations can be carried out numerically and involve only multiplications and latent-root extractions on matrices of order $(M-1)$. As the matrix (33) is of order $4(M-1)$, this is a very substantial reduction in the arithmetical work.

If $\overline{\mathsf{B}}_2\overline{\mathsf{D}}_3$ possesses a real positive latent root, all the quantities will be determinable and will be of the required orders in ϵ; S_1 and $\overline{T}_2$ will satisfy the boundary conditions, but T_2^{2c} and T_2^{2s} may not. To find the conditions for a solution to exist, it is convenient to consider the differential equations corresponding to (41), (42) and (43). They are

$$\left.\begin{aligned} r^2\frac{d^2S_1}{dr^2}-2S_1 &= -\frac{648V}{35}\frac{Q_S}{Q_T}\left[Q_S\frac{dT_2}{dr}+2\left(\frac{dQ_S}{dr}-\frac{Q_S}{r}\right)T_2\right],\\ r^2\frac{d^2T_2}{dr^2}-6T_2 &= -\frac{2V}{3}\left(\frac{dQ_T}{dr}-\frac{2Q_T}{r}\right)S_1, \end{aligned}\right\} \tag{45}$$

$$T_2^{2s} = \frac{3}{7Q_T}\left[Q_S\frac{dT_2}{dr}+2\left(\frac{dQ_S}{dr}-\frac{Q_S}{r}\right)T_2\right], \tag{46}$$

$$T_2^{2c} = -\frac{1}{2VQ_T}\left(r^2\frac{d^2T_2^{2s}}{dr^2}-6T_2^{2s}\right)-\frac{1}{3Q_T}\left[3Q_S\frac{d^2S_1}{dr^2}+\left(\frac{dQ_S}{dr}-\frac{6Q_S}{r}\right)\frac{dS_1}{dr}+\left(\frac{d^2Q_S}{dr^2}-\frac{2Q_S}{r}\right)S_1\right]. \tag{47}$$

Solutions of (45) are known to exist for a wide variety of Q_S and Q_T (Bullard 1955).

The S_1 and T_2 calculated from (45) satisfy the boundary conditions, but the T_2^{2s} and T_2^{2c} obtained from (46) and (47) in general do not. This is a consequence of the omission of the term in $\dot{T}_2^{2c}$ in getting (46) from (29); this reduces the order of the equations and leaves insufficient constants to satisfy the boundary conditions. An examination of the behaviour of the functions near $r = 0$ and 1 shows that if Q_S and Q_T behave like r^σ and r^τ near $r = 0$, and like $(1-r)^{\sigma_1}$ and $(1-r)^{\tau_1}$ near $r = 1$, the boundary conditions will be satisfied if $\sigma > 2\tau$ and $\sigma_1 > 2(\tau_1+1)$; (30) and (31) do not satisfy these conditions and will therefore give functions that do not satisfy the boundary conditions as $\epsilon \to \infty$.

Takeuchi & Shimazu (1952 *a*, *b*) have treated the case $\epsilon \to \infty$ by putting $T_2^{2c} = 0$ without detailed discussion of the reasons or consequences. In fact, none of their examples satisfies the above conditions (they take $\sigma_1 = 2$, $\tau_1 = 1$). They do not give the form of T_2^{2c}, but if it were worked out it would be infinite at either $r = 0$ or 1 or at both, their other radial functions for $Q_T = r^2(1-r)$ agree closely with ours.

We have solved (45) to (47) by the matrix method for the Q_S and Q_T given by (30) and (31). The solutions are shown in figure 16. As would be expected they give real values of V, but do not satisfy the boundary conditions for T_2^{2c}.

The difficulty about the boundary conditions arises from the assumption that T_2^{2c} is $O(\epsilon^{-1})$ over the whole range of r which may not be true near $r = 0$ and 1. In fact, the numerical work shows that there are solutions for the four equations for large finite ϵ which closely resemble the limiting forms for $\epsilon \to \infty$ except near the ends of the range (compare, for example, T_2^{2c} in figure 16 for $\epsilon = 100$ and $\epsilon \to \infty$). The value obtained for V for $\epsilon \to \infty$ is also close to that for finite ϵ.

The interest of this limiting case is reduced by the proof, which will be published elsewhere by Takeuchi & Bullard, that the V obtained when $\epsilon \to \infty$ increases without limit as the number of the equations included in the calculation is increased (see § 8 below).

Summary of results for four equations

In this section the solution of (29) has been discussed. From the analytical theorems of Bullard (1955) it has been shown that solutions exist for $\epsilon = 0$ for a wide variety of Q_S and Q_T. When ϵ is finite, no analytical existence theorems are known, but numerical solutions have been obtained for Q_S and Q_T given by (30) and (31) for a wide range of ϵ. Except for small ϵ the results are most satisfactorily stable, V varies only slowly with ϵ and is of the same order of magnitude for the two very different forms taken for Q_T. There seems little doubt that solutions exist for almost any simple form of the velocity radial functions. It remains to show that the solutions still exist when harmonics of higher degree are included.

(b) *Harmonics of degrees one to four*

In the last section, solutions have been found to the equations (29), which are obtained by omitting all radial functions of degree higher than the second. This is a very crude approximation, and it is desirable to include some further terms. The natural next step is to add those of degree three and then those of degree four, that is to include all those shown in figure 8. When this is done the equations for the first and second degree radial functions are complete, for with T_1 and S_2^{2c} motions they can contain no terms of degree above four.

There are three radial functions of degree three, S_3, S_3^{2c} and S_3^{2s}, and five of degree four, T_4, T_4^{2c}, T_4^{2s}, T_4^{4c} and T_4^{4s}. With the one first-degree function, S_1, and the three second-degree ones, T_2, T_2^{2c} and T_2^{2s}, this gives seven radial functions and seven equations up to degree three and twelve up to degree four. Unfortunately, the equations are of great complexity. They are:

$$\left.\begin{aligned}
r^2\ddot{S}_1-2S_1 &= V\{-\tfrac{216}{5}Q_S T_2^{2s}+\tfrac{432}{7}(Q_S\dot{S}_3^{2c}+\dot{Q}_S S_3^{2c})\},\\
r^2\ddot{T}_2-6T_2 &= V\{-\tfrac{2}{3}(\dot{Q}_T-2Q_T/r)\,S_1+\tfrac{12}{7}(\dot{Q}_T-2Q_T/r)\,S_3-\tfrac{72}{7}[Q_S\dot{T}_2^{2c}+2(\dot{Q}_S-Q_S/r)\,T_2^{2c}]\\
&\quad+\tfrac{360}{7}[Q_S\ddot{S}_3^{2s}+2(\dot{Q}_S-Q_S/r)\,\dot{S}_3^{2s}+2(\ddot{Q}_S-2\dot{Q}_S/r)\,S_3^{2s}]\\
&\quad+\tfrac{600}{7}[Q_S\dot{T}_4^{2c}+2(\dot{Q}_S-Q_S/r)\,T_4^{2c}]\},\\
r^2\ddot{S}_3-12S_3 &= V\{\tfrac{36}{5}Q_S T_2^{2s}-12(3Q_S\dot{S}_3^{2c}-2\dot{Q}_S S_3^{2c})-60Q_S T_4^{2s}\},\\
r^2\ddot{T}_4-20T_4 &= V\{-\tfrac{12}{7}(\dot{Q}_T-2Q_T/r)\,S_3+\tfrac{36}{35}[3Q_S\dot{T}_2^{2c}-(\dot{Q}_S+6Q_S/r)\,T_2^{2c}]\\
&\quad-\tfrac{36}{7}[3Q_S\ddot{S}_3^{2s}-(\dot{Q}_S+6Q_S/r)\,\dot{S}_3^{2s}+6(\ddot{Q}_S-2\dot{Q}_S/r)\,S_3^{2s}]\\
&\quad-\tfrac{324}{77}[17Q_S\dot{T}_4^{2c}+(27\dot{Q}_S-34Q_S/r)\,T_4^{2c}]\},\\
r^2\ddot{T}_2^{2s}-6T_2^{2s} &= V\{-\tfrac{2}{3}[3Q_S\ddot{S}_1+(\dot{Q}_S-6Q_S/r)\,\dot{S}_1+(\ddot{Q}_S-2\dot{Q}_S/r)\,S_1]\\
&\quad+\tfrac{6}{7}[Q_S\ddot{S}_3+2(\dot{Q}_S-Q_S/r)\,\dot{S}_3+2(\ddot{Q}_S-2\dot{Q}_S/r)\,S_3]\\
&\quad-2Q_T T_2^{2c}+\tfrac{20}{7}(\dot{Q}_T-2Q_T/r)\,S_3^{2s}+400[Q_S\dot{T}_4^{4s}+2(\dot{Q}_S-Q_S/r)\,T_4^{4s}]\},\\
r^2\ddot{T}_2^{2c}-6T_2^{2c} &= V\{-\tfrac{6}{7}[Q_S\dot{T}_2+2(\dot{Q}_S-Q_S/r)\,T_2]+\tfrac{10}{21}[Q_S\dot{T}_4+2(\dot{Q}_S-Q_S/r)\,T_4]\\
&\quad+2Q_T T_2^{2s}+\tfrac{20}{7}(\dot{Q}_T-2Q_T/r)\,S_3^{2c}+400[Q_S\dot{T}_4^{4c}+2(\dot{Q}_S-Q_S/r)\,T_4^{4c}]\},\\
r^2\ddot{S}_3^{2c}-12S_3^{2c} &= V\{\tfrac{2}{15}(3Q_S\dot{S}_1-2\dot{Q}_S S_1)-\tfrac{1}{5}(3Q_S\dot{S}_3-2\dot{Q}_S S_3)+2Q_T S_3^{2s}-56Q_S T_4^{4s}\},\\
r^2\ddot{S}_3^{2s}-12S_3^{2s} &= V\{\tfrac{3}{5}Q_S T_2-\tfrac{1}{3}Q_S T_4-2Q_T S_3^{2c}+56Q_S T_4^{4c}\},\\
r^2\ddot{T}_4^{2s}-20T_4^{2s} &= V\{-\tfrac{3}{35}[3Q_S\ddot{S}_3-(\dot{Q}_S+6Q_S/r)\,\dot{S}_3+6(\ddot{Q}_S-2\dot{Q}_S/r)\,S_3]-\tfrac{6}{7}(\dot{Q}_T-2Q_T/r)\,S_3^{2s}\\
&\quad-2Q_T T_4^{2c}-\tfrac{72}{55}[17Q_S\dot{T}_4^{4s}+(27\dot{Q}_S-34Q_S/r)\,T_4^{4s}]\},\\
r^2\ddot{T}_4^{2c}-20T_4^{2c} &= V\{\tfrac{3}{35}[3Q_S\dot{T}_2-(\dot{Q}_S+6Q_S/r)\,T_2]-\tfrac{18}{770}[17Q_S\dot{T}_4+(27\dot{Q}_S-34Q_S/r)\,T_4]\\
&\quad-\tfrac{6}{7}(\dot{Q}_T-2Q_T/r)\,S_3^{2c}+2Q_T T_4^{2s}-\tfrac{72}{55}[17Q_S\dot{T}_4^{4c}+(27\dot{Q}_S-34Q_S/r)\,T_4^{4c}]\},\\
r^2\ddot{T}_4^{4s}-20T_4^{4s} &= V\{\tfrac{3}{70}[3Q_S\dot{T}_2^{2s}-(\dot{Q}_S+6Q_S/r)\,T_2^{2s}]\\
&\quad-\tfrac{3}{70}[3Q_S\ddot{S}_3^{2c}-(\dot{Q}_S+6Q_S/r)\,\dot{S}_3^{2c}+6(\ddot{Q}_S-2\dot{Q}_S/r)\,S_3^{2c}]\\
&\quad-\tfrac{9}{770}[17Q_S\dot{T}_4^{2s}+(27\dot{Q}_S-34Q_S/r)\,T_4^{2s}]-4Q_T T_4^{4c}\},\\
r^2\ddot{T}_4^{4c}-20T_4^{4c} &= V\{\tfrac{3}{70}[3Q_S\dot{T}_2^{2c}-(\dot{Q}_S+6Q_S/r)\,T_2^{2c}]\\
&\quad+\tfrac{3}{70}[3Q_S\ddot{S}_3^{2s}-(\dot{Q}_S+6Q_S/r)\,\dot{S}_3^{2s}+6(\ddot{Q}_S-2\dot{Q}_S/r)\,S_3^{2s}]\\
&\quad-\tfrac{9}{770}[17Q_S\dot{T}_4^{2c}+(27\dot{Q}_S-34Q_S/r)\,T_4^{2c}]+4Q_T T_4^{4s}\}.
\end{aligned}\right\} \tag{48}$$

Considerable trouble has been taken to ensure their correctness; the coefficients have been calculated independently by the two authors and the final result checked by Dr Takeuchi, to whom the authors are much indebted. As a further check, the coefficients obtained from (24) were compared with those obtained from (19) and from table 7 below.

The seven or the twelve equations may be turned into a set of algebraic equations analogous to (33). As the right-hand sides of the S_3^{2c} and S_3^{2s} equations do not vanish at $r = 1$ unless Q_T does, the values of these functions at $r = 1$ must be retained as unknowns. If the range is divided into 10 parts the resulting matrix is of order 65 for seven equations and 110 for twelve.

The matrix for twelve equations may be written

$$\begin{bmatrix}
\mathsf{F}_1\lambda & 0 & 0 & 0 & 0 & 0 & 0 & \mathsf{D}_1 & \mathsf{B}_1 & 0 & 0 & 0 \\
0 & \mathsf{F}_2\lambda & 0 & 0 & 0 & 0 & \mathsf{B}_2 & \epsilon\mathsf{D}_2 & \epsilon\mathsf{D}_3 & 0 & \mathsf{B}_3 & \mathsf{B}_4 \\
0 & 0 & \mathsf{F}_3\lambda & 0 & 0 & 0 & 0 & \mathsf{D}_4 & \mathsf{B}_5 & \mathsf{D}_5 & 0 & 0 \\
0 & 0 & 0 & \mathsf{F}_4\lambda & 0 & 0 & \mathsf{D}_6 & 0 & \epsilon\mathsf{D}_7 & 0 & \mathsf{D}_8 & \mathsf{D}_9 \\
0 & 0 & 0 & 0 & \mathsf{F}_5\lambda & 0 & \mathsf{B}_6 & 0 & \epsilon\mathsf{D}_{10} & \epsilon\mathsf{D}_{11} & \mathsf{B}_7 & \mathsf{B}_8 \\
0 & 0 & 0 & 0 & 0 & \mathsf{F}_6\lambda & 0 & \mathsf{B}_9 & \mathsf{B}_{10} & \mathsf{B}_{11} & \epsilon\mathsf{D}_{12} & 0 \\
\epsilon\mathsf{D}_{13} & \mathsf{B}_{12} & \epsilon\mathsf{D}_{14} & \mathsf{B}_{13} & \mathsf{B}_{14} & 0 & \mathsf{F}_7\lambda & 0 & 0 & 0 & 0 & 0 \\
\mathsf{B}_{15} & \epsilon\mathsf{D}_{15} & \mathsf{B}_{16} & \epsilon\mathsf{D}_{16} & 0 & \mathsf{B}_{17} & 0 & \mathsf{F}_8\lambda & 0 & 0 & 0 & 0 \\
\mathsf{B}_{18} & 0 & \mathsf{B}_{19} & \epsilon\mathsf{D}_{17} & 0 & \mathsf{D}_{18} & 0 & 0 & \mathsf{F}_9\lambda & 0 & 0 & 0 \\
0 & 0 & \mathsf{B}_{20} & \epsilon\mathsf{D}_{19} & \epsilon\mathsf{D}_{20} & \mathsf{B}_{21} & 0 & 0 & 0 & \mathsf{F}_{10}\lambda & 0 & 0 \\
0 & \mathsf{B}_{22} & 0 & \mathsf{B}_{23} & \mathsf{B}_{24} & \epsilon\mathsf{D}_{21} & 0 & 0 & 0 & 0 & \mathsf{F}_{11}\lambda & 0 \\
0 & \mathsf{B}_{25} & \epsilon\mathsf{D}_{22} & \mathsf{B}_{26} & \mathsf{B}_{27} & 0 & 0 & 0 & 0 & 0 & 0 & \mathsf{F}_{12}\lambda
\end{bmatrix}
\begin{bmatrix}
\mathsf{S}_1 \\ \mathsf{T}_2^{2c} \\ \mathsf{S}_3 \\ \mathsf{S}_3^{2s} \\ \mathsf{T}_4^{2c} \\ \mathsf{T}_4^{4s} \\ \mathsf{T}_2 \\ \mathsf{T}_2^{2s} \\ \mathsf{S}_3^{2c} \\ \mathsf{T}_4^{2s} \\ \mathsf{T}_4^{4c} \\ \mathsf{T}_4
\end{bmatrix} = 0, \quad (49)$$

where, as before, $\lambda = 1/V$, the D's are diagonal matrices and the F's and B's are continuants (the numbering of the submatrices is not the same as in (33)). The order may be reduced by the process used to get (34) from (33); this gives a 28×28 matrix for seven equations and a 55×55 matrix for twelve equations. The successive removal of negative and complex roots from these large matrices is inconvenient; the required positive root was therefore found directly. The matrix of order 65 or 110 was set up and reduced to the form (34) using double-length arithmetic (18 decimal digits), and scaling factors introduced to make the largest term in each column lie between a half and one. Call this matrix (cf. (34))

$$\begin{bmatrix} \mathsf{I}\lambda & \mathsf{A} \\ \mathsf{B} & \mathsf{I}\lambda \end{bmatrix}.$$

The product AB was then formed using single-length arithmetic and double-length products and an approximate latent root of AB found by evaluating the determinant $|\mathsf{AB} - \lambda^2\mathsf{I}|$ for trial values of λ.

It is difficult by this method to get a λ sufficiently accurate to give reliable radial functions. A more accurate λ was found from

$$[\mathsf{AB} - (1+\mu)\,\lambda_0^2\mathsf{I}], \qquad (50)$$

where λ_0 is the approximation to λ previously found and μ is a suitable constant (usually about $\frac{1}{3}$). If the required root is the only one in a circle of radius $\mu\lambda_0^2$ centred at λ_0^2 in the Argand diagram of λ^2, then the required root will be that of largest modulus for the matrix inverse to (50). (50) was formed with scaling factors in each column to make the largest term lie between a half and one. It was inverted using single-length arithmetic and the latent root found by iteration. This method was devised by Mr J. H. Wilkinson, who will

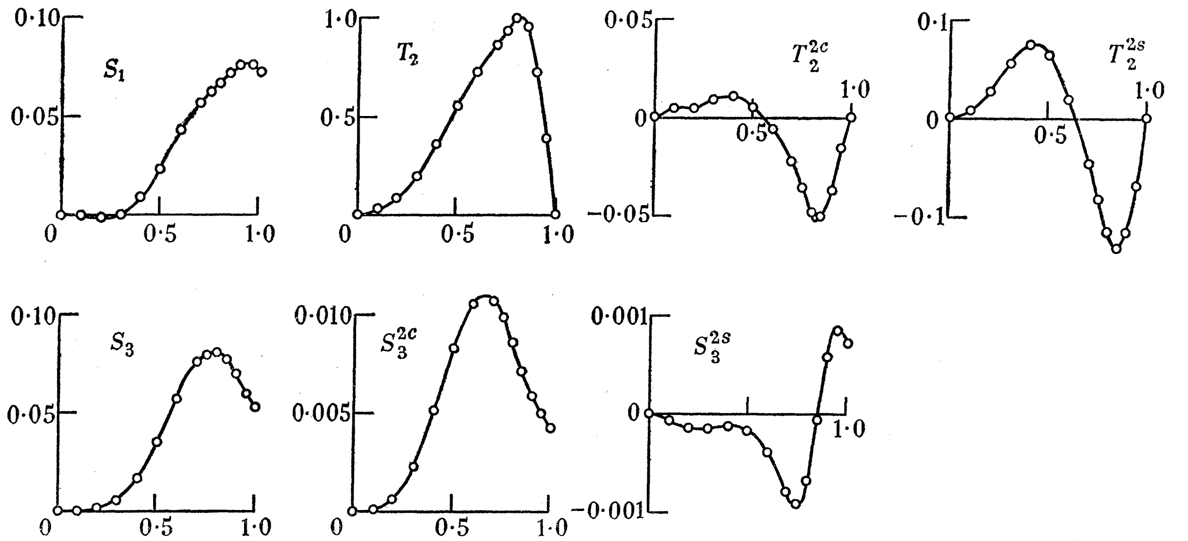


FIGURE 18. Radial functions for $Q_T = \epsilon r^2(1-r)$, $\epsilon = 5$ from seven equations with the radius divided at an interval of 0·1 from 0 to 0·7 and at an interval of 0·05 from 0·7 to 1. $V = 83 \cdot 9$.

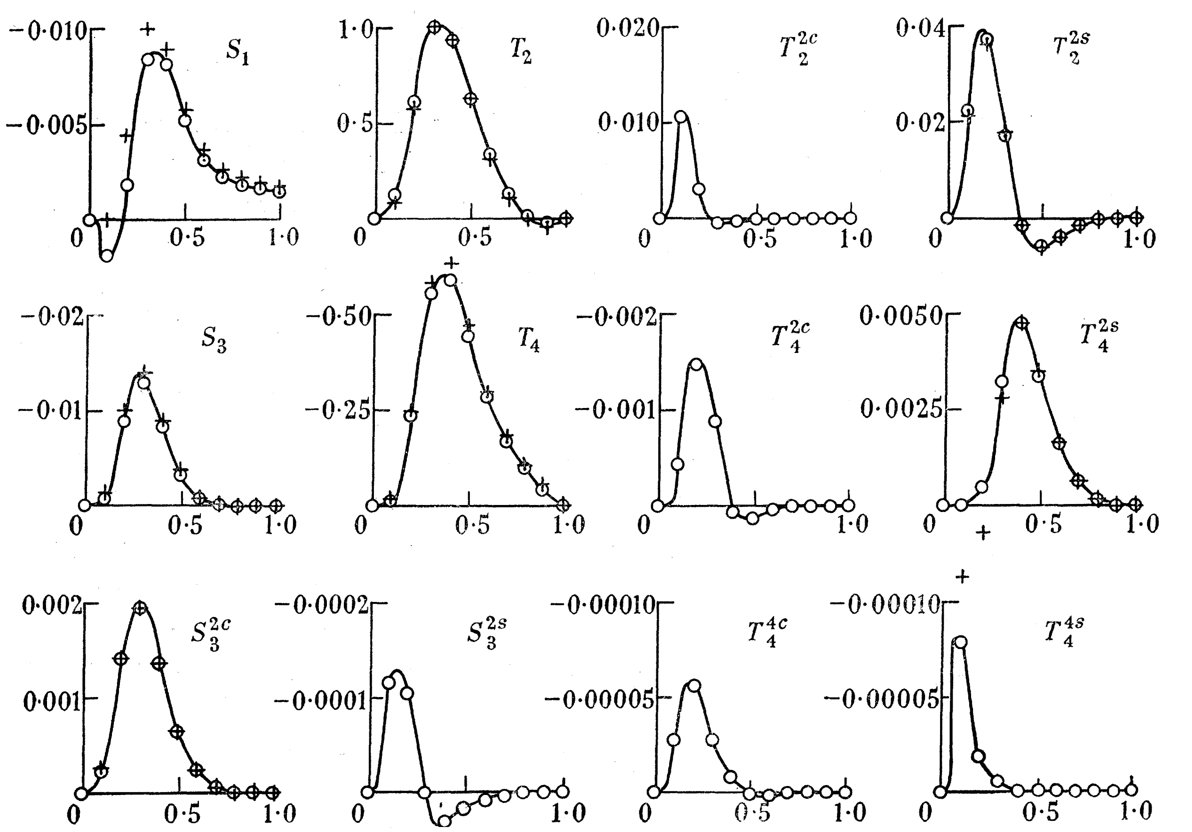


FIGURE 19. Radial functions for $Q_T = \epsilon r^3$ from twelve equations with the radius divided into ten parts. ○, $\epsilon = 100$; +, $\epsilon \to \infty$.

publish a more detailed account of it elsewhere. This gives one-half, say X_1, of the latent column of (49); the other half, X_2, can be found from

$$\mathsf{X}_2 = -\lambda^{-1}\mathsf{B}\mathsf{X}_1,$$

but the result is sometimes sensitive to small errors in X_1 and it is then preferable to repeat the whole calculation starting from BA instead of AB. The work of forming the matrix and finding a solution for twelve equations for $\epsilon = 100$ took 16 hours on the A.C.E. in 1953; it could now be done substantially faster.

The values found for V are given in table 4 and figure 15. In view of the great amount of work involved, the latent columns were calculated for only two cases, one for seven equations and one for twelve; the results, together with those for $\epsilon \to \infty$, are given in tables 6 *a* and *b* and figures 18 and 19. With seven equations, the S_1 obtained for $\epsilon = 5$ did not difference smoothly near $r = 1$; the calculations were therefore repeated with the range divided at intervals of 0·05 between $r = 0·7$ and 1·0. The radial functions obtained are believed to be solutions of (49) with an uncertainty of a few units in the last place given. How closely they represent the solutions of the differential equations (48) it is difficult to say. The

TABLE 6*a*. RADIAL FUNCTIONS FOR SEVEN EQUATIONS,

$Q_S = r^3(1-r)^2,\ Q_T = \epsilon r^2(1-r),\ \epsilon = 5,\ V = 83·90$

r	S_1	T_2^{2c}	S_3	S_3^{2s}	T_2	T_2^{2s}	S_3^{2c}
0·0	0·000000	0·000000	0·000000	0·000000	0·000000	0·000000	0·000000
0·1	−0·000762	0·004686	0·000178	−0·000060	0·031822	0·007473	0·000077
0·2	−0·001556	0·004399	0·001258	−0·000123	0·078023	0·027798	0·000687
0·3	0·000710	0·008147	0·005510	−0·000141	0·190770	0·055147	0·002366
0·4	0·009014	0·009937	0·016324	−0·000125	0·360390	0·072485	0·005183
0·5	0·024209	0·005148	0·034918	−0·000168	0·555170	0·062075	0·008350
0·6	0·042777	−0·005695	0·057334	−0·000385	0·733306	0·018581	0·010577
0·7	0·057941	−0·022834	0·075292	−0·000789	0·872913	−0·047006	0·010719
0·75	0·062887	−0·035481	0·079558	−0·000902	0·943820	−0·084049	0·009891
0·8	0·067186	−0·048650	0·080307	−0·000681	1·000000	−0·115927	0·008593
0·85	0·072059	−0·051618	0·077169	−0·000070	0·953888	−0·132398	0·007159
0·9	0·076397	−0·037563	0·070507	0·000582	0·733282	−0·117909	0·005925
0·95	0·076797	−0·015310	0·061849	0·000843	0·385804	−0·069945	0·004996
1·0	0·072966	0·000000	0·053089	0·000724	0·000000	0·000000	0·004288

reduction of the interval from 0·1 to 0·05 for the seven equations with $\epsilon = 5$ changed V from 68·8 to 83·9; this is a substantial change, but the functions are now smooth enough to suggest that further subdivision would not produce much additional change. If the other cases for seven and twelve equations given in table 4 were recomputed with a smaller interval in r the values of V would presumably also be increased.

When $\epsilon = 0$ the seven equations break into two sets, one containing the four functions S_1, T_2^{2s}, S_3 and S_3^{2c} and the other the three functions T_2, T_2^{2c} and S_3^{2s}. No real root has been found for the second of these, but the first, with the range subdivided as for $\epsilon = 5$, gives $V = 117·6$. This is not very different from the result, 124, obtained from four equations when $\epsilon = 0$; the agreement is, however, fortuitous, since the result for four equations was from the pair containing T_2 and T_2^{2c} and not from that containing S_1 and T_2^{2s}. The solutions for $\epsilon = 0$ are being further considered by Dr Takeuchi.

It frequently happens that the terms on the right-hand side of one or more of the differential equations approximately cancel. The radial function occurring on the left will then

TABLE 6b. RADIAL FUNCTIONS FOR TWELVE EQUATIONS, $Q_S = r^3(1-r)^2$

$Q_T = \epsilon r^3$, $\epsilon = 100$, $V = 67\cdot4127$

r	S_1	T_2^{2c}	S_3	S_3^{2s}	T_4^{2c}	T_4^{4s}	T_2	T_2^{2s}	S_3^{2c}	T_4^{2s}	T_4^{4c}	T_4
0·0	0·000000	0·000000	0·000000	0·000000	0·000000	0·000000	0·000000	0·000000	0·000000	0·000000	0·000000	0·000000
0·1	0·001919	0·010862	−0·000915	−0·000117	−0·000450	0·000079	0·137513	0·023409	0·000235	0·000036	−0·000027	−0·014788
0·2	−0·001879	0·003079	−0·009035	−0·000105	−0·001498	0·000019	0·613624	0·037673	0·001409	0·000490	−0·000056	−0·239565
0·3	−0·008380	−0·000437	−0·013115	0·000000	−0·000888	0·000006	1·000000	0·017249	0·001921	0·003254	−0·000027	−0·554429
0·4	−0·008085	−0·000215	−0·008268	0·000029	0·000057	0·000001	0·930782	−0·001758	0·001363	0·004744	−0·000008	−0·594784
0·5	−0·005162	0·000121	−0·003234	0·000014	0·000128	0·000000	0·631740	−0·006005	0·000661	0·003412	0·000002	−0·448318
0·6	−0·003127	0·000106	−0·000935	0·000004	0·000035	0·000000	0·339618	−0·003934	0·000239	0·001657	0·000002	−0·285881
0·7	−0·002217	0·000041	−0·000243	0·000002	0·000002	0·000000	0·135487	−0·001619	0·000062	0·000626	0·000001	−0·171812
0·8	−0·001833	0·000017	−0·000094	0·000001	−0·000002	0·000000	0·019793	−0·000403	0·000009	0·000194	0·000000	−0·099454
0·9	−0·001621	0·000010	−0·000062	0·000000	−0·000001	0·000000	−0·022182	−0·000023	0·000000	0·000039	0·000000	−0·047439
1·0	−0·001460	0·000000	−0·000046	0·000000	0·000000	0·000000	0·000000	0·000000	0·000000	0·000000	0·000000	0·000000

	$Q_T = \epsilon r^3$, $\epsilon \to \infty$, $V = 65\cdot8$							$Q_T = \epsilon r^2(1-r)$, $\epsilon \to \infty$, $V = 65\cdot5$						
r	S_1	S_3	T_2/ϵ	T_2^{2s}	S_3^{2c}	T_4^{2s}	T_4/ϵ	S_1	S_3	T_2/ϵ	T_2^{2s}	S_3^{2c}	T_4^{2s}	T_4/ϵ
0·0	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000
0·1	−0·005	−0·142	0·092	2·158	0·025	−0·185	−0·019	0·000	0·000	0·001	0·002	0·000	0·000	0·000
0·2	−0·440	−1·005	0·581	3·645	0·138	−0·070	−0·248	0·003	0·000	0·005	0·011	0·000	0·000	0·001
0·3	−0·986	−1·404	1·000	1·785	0·195	0·286	−0·583	0·008	−0·006	0·019	0·034	0·001	0·000	0·004
0·4	−0·893	−0·876	0·935	−0·173	0·139	0·479	−0·629	0·020	−0·038	0·056	0·108	0·004	−0·018	0·009
0·5	−0·571	−0·347	0·623	−0·616	0·066	0·349	−0·472	0·036	−0·089	0·165	0·287	0·013	−0·087	−0·017
0·6	−0·358	−0·104	0·320	−0·398	0·023	0·169	−0·303	0·020	−0·030	0·431	0·402	0·037	−0·166	−0·157
0·7	−0·260	−0·029	0·110	−0·158	0·006	0·064	−0·184	−0·068	0·231	0·810	−0·038	0·066	−0·100	−0·418
0·8	−0·217	−0·012	−0·005	−0·035	0·001	0·020	−0·108	−0·152	0·435	1·000	−0·930	0·064	0·107	−0·590
0·9	−0·192	−0·008	−0·038	0·000	0·000	0·004	−0·052	−0·166	0·392	0·729	−1·226	0·027	0·204	−0·458
1·0	−0·159	−0·004	0·000	0·000	0·000	0·000	0·000	−0·138	0·236	0·000	0·000	0·000	0·000	0·000

be small but badly determined. In such circumstances it is extremely difficult to estimate the error incurred by a finite difference approximation; owing to the substitution of finite differences for derivatives, the functions all have errors which are fairly smooth functions of r. The proportional error produced in a right-hand side may be greatly magnified when the right hand side almost vanishes. These difficulties are well shown by the solutions obtained when $\epsilon \to \infty$, which we now consider.

(c) *Large* ϵ

When ϵ is large it is possible to treat the seven and twelve equations by a method similar to that used for the four equations. The first step is to determine the orders of magnitude of the radial functions as functions of ϵ when $\epsilon \to \infty$. From the first row of (49) it may be seen that two of the three radial functions S_1, T_2^{2s} and S_3^{2c} must be of the same order in ϵ; call this order one. The remaining one of the three must be of the same or of a lower order. With the usual convention that $x = O(1)$ means that as $\epsilon \to \infty$ x is less than some constant *or tends to zero*, we may take all three functions as $O(1)$. If one of them is really of lower order, this will appear later in the calculation by the approximate cancelling of two of the terms, leaving a remainder of lower order to be balanced by the third term. As before, these order-of-magnitude estimates may not apply near zeros of the D's and B's. By somewhat elaborate arguments of this kind the orders of magnitude of all the radial functions can be determined. The results are:

T_2	T_4	S_1	S_3	S_3^{2c}	T_2^{2s}	T_4^{2s}	T_2^{2c}	S_3^{2s}	T_4^{2c}	T_4^{4c}	T_4^{4s}
ϵ	ϵ	1	1	1	1	1	ϵ^{-1}	ϵ^{-1}	ϵ^{-1}	ϵ^{-1}	ϵ^{-2}

If these orders of magnitude are inserted in (49) and terms which are $O(\epsilon^{-2})$ relative to the largest terms in each equation are neglected, all but four of the functions can be eliminated and (49) is reduced to

$$\begin{bmatrix} \mathsf{D}_2\mathsf{F}_1\lambda & 0 & -(\mathsf{D}_1\mathsf{B}_2+\mathsf{G}_1\mathsf{D}_7^{-1}\mathsf{D}_6) & -(\mathsf{D}_1\mathsf{B}_4+\mathsf{G}_1\mathsf{D}_7^{-1}\mathsf{D}_9) \\ 0 & \mathsf{D}_2\mathsf{D}_{11}\mathsf{F}_3\lambda & -(\mathsf{D}_2\mathsf{D}_5\mathsf{B}_6+\mathsf{D}_{11}\mathsf{D}_4\mathsf{B}_2+\mathsf{G}_2\mathsf{D}_7^{-1}\mathsf{D}_6) & -(\mathsf{D}_2\mathsf{D}_5\mathsf{B}_8+\mathsf{D}_{11}\mathsf{D}_4\mathsf{B}_4+\mathsf{G}_2\mathsf{D}_7^{-1}\mathsf{D}_9) \\ \mathsf{D}_{13} & \mathsf{D}_{14} & \mathsf{F}_7\lambda & 0 \\ 0 & \mathsf{D}_{22} & 0 & \mathsf{F}_{12}\lambda \end{bmatrix} \begin{bmatrix} \mathsf{S}_1 \\ \mathsf{S}_3 \\ \mathsf{T}_2/\epsilon \\ \mathsf{T}_4/\epsilon \end{bmatrix} = 0, \qquad (51)$$

where

$$\mathsf{G}_1 = \mathsf{D}_2\mathsf{B}_1 - \mathsf{D}_1\mathsf{D}_3,$$
$$\mathsf{G}_2 = \mathsf{D}_2(\mathsf{D}_{11}\mathsf{B}_5 - \mathsf{D}_5\mathsf{D}_{10}) - \mathsf{D}_{11}\mathsf{D}_4\mathsf{D}_3.$$

The order of this may again be halved by the method used for (43). The latent root and the functions T_2 and T_4 have been found from the resulting matrix by treating the B's and D's as numerical matrices and using an interval of 0·1 in r. The other radial functions can then be found by substitution in (51) and (49); in view of the large amount of work involved those that are $O(\epsilon^{-1})$ have not been calculated. The results are given in table 6*b*. V was found to be 65·8 if $Q_T = r^3$ and 65·5 if $Q_T = r^2(1-r)$, taking $Q_S = r^3(1-r)^2$ in both cases.

Dr Takeuchi has pointed out to us that if a finite difference approximation is not made (that is, if the B's in (51) are regarded as differential operators), the third and fourth terms in the first row and the third term in the second row vanish identically. From this it is easily seen that S_1 must be zero and (51) may be reduced to a single fourth-order equation for T_2 or T_4. S_1 will not be exactly zero when (51) is solved, as it has been above, by finite difference methods, treating the B's and D's as numerical matrices. This illustrates a general difficulty in the numerical solution of complicated sets of differential equations. If one of the variables is found to be small compared to the rest it cannot be assumed to be an approximation to a solution of the differential equation even if it is a smooth function of r; its true form may be entirely obscured by small errors in the finite-difference approximation to the other functions. In principle, such difficulties can be resolved by decreasing the interval of the independent variable till the solutions become unaffected by further subdivision. For four equations an interval of 0·1 in r was satisfactory, but for seven and twelve a finer division seems necessary if the smaller radial functions are to be determined. Further subdivision for the twelve equations is impracticable, and the significance of the smaller radial functions in table 6b is doubtful. In particular, the similarity of S_1 for $\epsilon = 100$ to that for $\epsilon \to \infty$, where it should be zero, strongly suggests that S_1 for $\epsilon = 100$ cannot be determined with an interval of 0·1 in r. It has been verified by retaining terms of order $1/\epsilon^2$ in (51) that S_1 does not vanish identically for finite ϵ. Difficulties with the higher harmonics might have been anticipated; the spherical harmonics of high degree are rapidly varying functions of θ and ϕ, and it is not surprising that a fine division must be used in the r direction if their radial functions are to be determined.

(*d*) *Summary of results*

From this discussion it is clear that the A.C.E. is not capable of giving entirely satisfactory solutions of (48) in a reasonable time. To get radial functions with an accuracy of a few parts in a hundred would require a finer division of r than we have been able to use. Nevertheless, the calculations do establish that solutions exist when third and fourth harmonics are retained in the calculations.

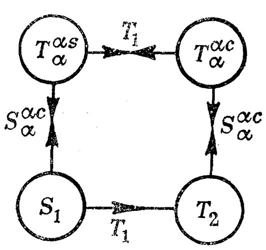


FIGURE 20. Interactions with T_1 and $S_\alpha^{\alpha c}$ motions.

On comparing figures 16 and 18, the radial functions for T_2, T_2^{2c} and T_2^{2s} are seen not to be greatly affected by the introduction of the harmonics of degree 3. The dipole field, S_1, seems, however, to be much more sensitive both to the form of Q_T and to the number of harmonics included. On comparing figures 16 and 18, the sign of the external field is seen to be reversed by including the third harmonics. This instability of the S_1 field is due to the existence of two simple opposing mechanisms by which it may be formed from T_2; one of these is by the chain T_2—T_2^{2c}—T_2^{2s}—S_1 and the other by T_2—S_3^{2s}—S_3^{2c}—S_1.

The two rising and two falling motions of the S_2^{2c} motion constitute a somewhat arbitrary choice. $S_\alpha^{\alpha c}$ gives a more general motion in which there are α rising and α falling currents. If this is combined with a T_1 motion, it gives a diagram analogous to figure 8, whose terms of lowest degree are shown in figure 20. The equations of this system are closely similar to (20) but have different numerical coefficients. For a large T_1 motion they reduce to a pair of equations

$$r^2\ddot{S}_1 - 2S_1 = -\frac{9\alpha^2(\alpha+1)\,(2\alpha)!\,V_\alpha\,Q_S}{4(2\alpha+1)\,(2\alpha+3)\,Q_T}[Q_S\dot{T}_2 + 2(\dot{Q}_S - Q_S/r)\,T_2],$$
$$r^2\ddot{T}_2 - 6T_2 = -\tfrac{2}{3}V_\alpha(\dot{Q}_T - 2Q_T/r)\,S_1.$$

If S_1 and V_α are replaced by S_1' and V_2 given by

$$V_2 = \mu V_\alpha, \quad S_1' = S_1/\mu, \quad \mu^2 = \frac{35\alpha^2(\alpha+1)\,(2\alpha)!}{288(2\alpha+1)\,(2\alpha+3)},$$

these equations become identical with (45) and the critical V_α can at once be deduced from that, V_2, for (45)

$$V_\alpha = \frac{12}{\alpha}\left\{\frac{2(2\alpha+1)\,(2\alpha+3)}{35(\alpha+1)\,(2\alpha)!}\right\}^{\frac{1}{2}} V_2.$$

From this it appears that for large α, $V_\alpha \ll V_2$; the quantity of physical interest is, however, not V_α but the actual velocity whose radial component is from (9) and (13)

$$\frac{\alpha(\alpha+1)\,V_\alpha}{4\pi\kappa a r^2}\,Q_S P_\alpha^\alpha(\cos\theta)\cos\alpha\phi.$$

The r.m.s. value of this is $\surd[(\alpha+1)\,(2\alpha+3)/21]$ times that for $\alpha = 2$; this increases with α as expected. The S_2^{2c} velocity, though a somethat arbitrary choice, thus appears to be a typical representative of the class of motions with rising and falling currents spaced around the equator.

(e) *Lines of force*

The process by which the field is regenerated may be illustrated by considering the distortion of the lines of force by the motion. The lines of force of any field, such as a dipole field, that has a radial component will cross regions of differing angular velocity (the T_1 motion). The lines of force will tend to move with the fluid and will be sheared and wrapped around the axis of rotation. This mechanism has been discussed in a previous paper (Bullard 1949*b*); it gives the connexions shown in the bottom line of figure 8. Starting with an S_1 field it causes the lines of force to follow closely wound spirals going round the axis in opposite directions in the two hemispheres; the start of this process is shown in figure 21*a*. The essential feature of a dynamo of the kind considered here is to distort this T_2 field, which runs almost along circles of latitude into loops in meridian planes which reproduce the S_1 field. Consider the distortion of a line of force running from west to east in the northern hemisphere by an upward motion rising near a point P (figure 21*b*) to the south of the line of force and spreading out in all directions (this could be one of the two rising currents of the S_2^{2c} motion). The line of force will be bent upwards and to the north into an arch leaning out of the vertical plane towards the north. If this loop can be twisted towards the meridian plane it will provide what is required. This could be done by introducing a suitable motion such as T_3^{2c}, and it is possible that this would provide a simpler dynamo than the one we have considered. In the absence of such an additional motion, the twisting may be done by the combined effect of the T_1 and S_2^{2c} motions. Suppose the T_1 differential rotation carries the arch westwards relative to the rising current, the northward component will then tip its eastern end to the north. This process of making a bulge in a line of force and then twisting the bulge produces a loop of lines of force having a component in the meridian plane. It may be seen that an east to west field in the southern hemisphere is distorted into a loop whose projection on the meridian plane has the same sense as that in the northern hemisphere. Alternate rising and sinking currents spaced round the equator will all distort the field into loops whose meridian projections have the same sign. The mutual repulsion of the lines of force will push them through the surface of the core and build up the external field, or in equilibrium just prevent it from collapsing inwards. Close to the core the field will be quite complicated and will contain S_1, S_3, S_3^{2c}, S_3^{2s}, etc., components. At a distance the S_1 field will predominate. Dr E. Parker has independently developed the theory of dynamos from this point of view and will shortly publish a much more detailed discussion.

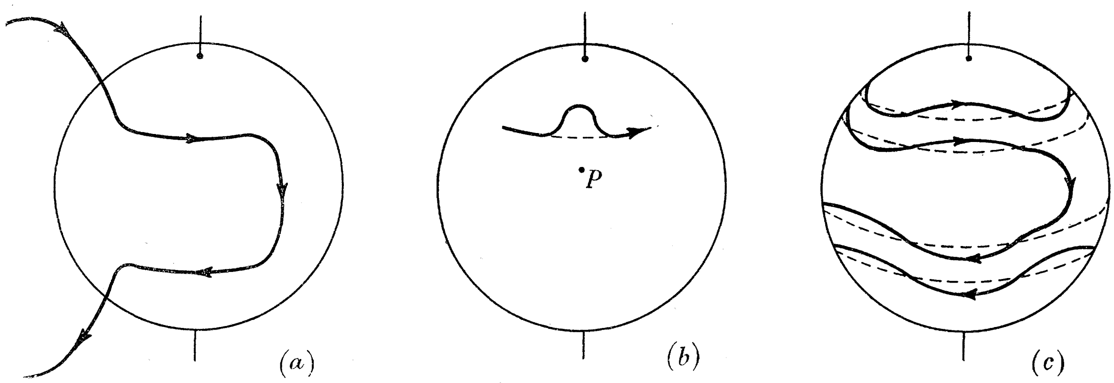


FIGURE 21. Lines of force.

Arguments of this kind give a physical insight into the process of regeneration and help to render the mathematical argument intelligible, but do not in themselves prove the possibility of the process.

The detailed calculation of the forms of the lines of force for the dynamo with T_1 and S_2^{2c} motions would be a considerable task and has not been undertaken. From the qualitative arguments given above it seems likely that they will form closely wound spirals round the axis as in figure 21 *c*, and that the turns of the spiral will themselves be twisted into very open spirals round a line of latitude, the pitch of this latter spiral being 90° of longitude; if an $S_\alpha^{\alpha c}$ motion with $\alpha > 2$ were substituted for S_2^{2c}, the field would be similar but the pitch of the spirals along a line of latitude would be reduced.

It appears that such systems of lines of force do not violate Cowling's theorem, though their topological properties are not fully understood and would repay further study.

8. CONVERGENCE OF THE SOLUTIONS

In § 7, solutions have been obtained to the equations of a dynamo with S_2^{2c} and T_1 motions subject to two simplifications. First, the equations solved are not the differential equations, but a finite-difference approximation to them in which the range is divided into ten parts, and, secondly, all harmonic components in the field beyond the fourth have been neglected. In this section we discuss the effect of these approximations. The treatment is necessarily somewhat involved, and the section can be ignored by readers not concerned with questions of convergence.

The main point of interest is whether the approximations can affect the existence of solutions. For a proof of existence, the following possibilities must be excluded:

(*a*) on proceeding to a finer division in r with a finite number of equations, no real V might be found or V might not tend to a limit,

(*b*) on proceeding to a finer division, the radial functions might not tend to a limit at some points in the range of r,

(*c*) on increasing the number of harmonics considered, no real V might be found or V might not tend to a limit,

(*d*) on increasing the number of harmonics, the series (17) for the field might not converge at all points.

The numerical evidence, given in § 7, strongly suggests the convergence of the V found from four equations as the fineness of division of the range in r is increased; the matter has not been investigated for the twelve equations, but the similarity of the radial functions to those from the four equations and their smoothness leaves little doubt that division of the range of r into ten parts is sufficient to give a fair approximation to V. This disposes of possibility (*a*).

If V tends to a finite limit as the fineness of division is increased, then the radial functions must do likewise, since every point in $0<r\leqslant 1$ is an ordinary point for the differential equations and the solutions must therefore be bounded and continuous in this interval. The origin is a singular point of the equations, but the boundary conditions require the radial functions to be zero there. Possibility (*b*) is therefore excluded.

The discussion of possibilities (*c*) and (*d*) is necessarily more complicated. The easiest procedure is probably to suppose that a solution has been obtained including all terms up to and including the T terms of degree $2p$, where p is an integer (the T terms are all of even degree and order), and the S terms of degree up to $(2p-1)$ (the S terms are all of odd degree and even order). The total number of terms, and therefore of differential equations, is then easily seen from figure 8 to be $2p(p+1)$. A new group of T terms of degree $(2p+2)$ and of S terms of degree $(2p+1)$ is now added and a new solution obtained. There are $4(p+1)$ of these additional terms and $4(p+1)$ new equations; the $4p$ equations with terms of degrees $2p$ and $(2p+1)$ on their left-hand sides will also be altered by the inclusion of extra terms in the $4(p+1)$ new radial functions, but the $2p(p-1)$ equations, whose left-hand sides contain terms of lower degree, will be unaltered. To exclude possibilities (*c*) and (*d*) it must be shown that if this procedure of adding blocks of terms is repeated indefinitely, V tends to a limit and (17) converges. The calculations of § 7 consisted of two steps in this process: first the four equations containing terms of degrees one and two were solved, and then the eight terms of degrees three and four were added to give twelve equations in all. It is of course impossible to deduce anything about convergence by numerical methods without going at least one step further and adding a further twelve equations of degrees five and six; this would give additional terms in the eight equations having terms of degree three and four on their left-hand sides, but would leave the four equations having terms of degree one and two on their left sides unchanged. The work involved in this is almost prohibitive, and an attempt has been made instead to estimate the orders of magnitude of the quantities involved. This leads to a formal proof that (*d*) is excluded if (*c*) is, that is, that (17) converges if V tends to a limit. It has not been possible to prove that V does tend to a limit, but a consideration of the orders of magnitude of the terms added at each stage suggests that it does.

An essential preliminary to the discussion is to discover how the coefficients in (24) depend on the degree of the radial functions involved. With only S_2^{2c} and T_1 motions, an equation such as (25) with terms of degree γ on the left-hand side can contain only terms of degrees γ, $\gamma\pm 1$ and $\gamma\pm 2$ on the right. The integrals K and L can be evaluated from Bird's general expressions, or by use of recurrence relations and known integrals of the P_n^m, and all the terms found that correspond to connexions shown in figure 8 and its extension to the right. The results are given in table 7. Each term on the right-hand side may be written in the form

$$\beta^2 V[\mathscr{A}\ddot{U}_\beta + \mathscr{B}\dot{U}_\beta + \mathscr{C}U_\beta], \tag{52}$$

where U_β stands for S_β or T_β. $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ are functions of β and r (the factor β^2 is included for subsequent convenience).

For the subsequent argument, it is convenient to use functions based on the partially normalized spherical harmonics introduced by Schmidt and not on the unnormalized functions used in the numerical work. If the partially normalized harmonic P_{nm} given by

$$P_{nm} = \{\delta(n-m)!/(n+m)!\}^{\frac{1}{2}} P_n^m,$$

where $\delta = 1$ if $m\neq 0$, $\delta = 2$ if $m = 0$, is used, every term in (24) and on the left of table 7 must be multiplied by

$$\left[\frac{\delta_\gamma(\beta-m_\beta)!\,(\gamma+m_\gamma)!}{\delta_\beta(\beta+m_\beta)!\,(\gamma-m_\gamma)!}\right]^{\frac{1}{2}}, \tag{53}$$

where δ_β and δ_γ are the δ's corresponding to the spherical harmonics concerned in the field components U_β and U_γ.

Only the orders of magnitude of the terms as functions of β are required. These are given on the right of table 7. They have been obtained by multiplying the expressions on the left of table 7 by (53), letting β become large and selecting the leading term. The orders of magnitude are different when the order of the harmonics is large and when it is small. The results given for large m are to be used when $\beta\to\infty$ in such a way that $(\beta-m)$ is bounded, and those for small m when it is $O(\beta)$; intermediate cases, such as $\beta-m = O(\beta^{\frac{1}{2}})$, have orders of magnitude lying between the two given. The behaviour of $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ as functions of r depends on Q_S and Q_T; for the forms of these functions used in § 7 they are always simple polynomials.

The results of table 7 will now be used to establish the convergence of (17) assuming a real, finite V to exist. Let U_γ represent either S_γ or T_γ. The boundary conditions require all U_γ to have at least one maximum. Near a maximum $\ddot{U}_\gamma$ is of opposite sign to U_γ (or is zero). Thus at a maximum of U_γ the absolute value of the left-hand side of (22) or (23) is greater than $\gamma^2\,|\,U_\gamma(\text{max.})\,|$. Hence from (52), putting $\beta^2/\gamma^2 = 1$, which is near enough since $|\,\gamma-\beta\,|\leqslant 2$,

$$|\,U_\gamma(\text{max.})\,| < V\sum_\beta |\mathscr{A}\ddot{U}_\beta + \mathscr{B}\dot{U}_\beta + \mathscr{C}U_\beta\,|. \tag{54}$$

An examination of table 7 shows that the orders of $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ as functions of β are never greater than $\beta^{-\frac{5}{2}}$, $\beta^{-\frac{3}{2}}$ and $\beta^{-\frac{1}{2}}$. Thus

$$U_\gamma(\text{max.}) = O\sum_\beta (\beta^{-\frac{5}{2}}\ddot{U}_\beta + \beta^{-\frac{3}{2}}\dot{U}_\beta + \beta^{-\frac{1}{2}}U_\beta).$$

If $\ddot{U}_\beta/U_\beta$ does not increase more rapidly than β^2 or $\dot{U}_\beta/U_\beta$ more rapidly than β, this implies

$$U_\gamma(\text{max.})/U'_\beta = O(\beta^{-\frac{1}{2}}),$$

where U'_β is the particular U_β that contributes the largest term to the right-hand side of (54). Thus $U_\gamma(\text{max.})$ cannot exceed $C_0\beta^{-\frac{1}{2}}$ times the largest of the U_β occurring on the right of (54), where C_0 is a constant independent of β and γ. This applies to all the S's and T's, the maximum of each of them is at most $C_0\beta^{-\frac{1}{2}}$ times the greatest of those of degree β, $\beta\pm1$, or $\beta\pm2$. Thus for large β the sequence of S_β and T_β will decrease more rapidly than any geometrical progression.

The assumptions about $\ddot{U}_\beta/U_\beta$ and $\dot{U}_\beta/U_\beta$ are not severe, but we have not succeeded in determining what their true orders of magnitude as functions of β are.

From (17) and (13) the r component of the field is

$$H_r = \sum_\beta \frac{\beta(\beta+1)}{r^2} S_\beta P_{\beta m} \begin{matrix}\cos\\ \sin\end{matrix} m\phi,$$

where the S_β are those corresponding to the partly normalized spherical harmonics $P_{\beta m}$. Let the terms be divided as before into groups containing $1, 3, 5, \ldots$ terms of degrees $1, 3, 5, \ldots$. An upper limit to the maximum value of a term in a group may be obtained by taking the maximum value of each factor. The maximum value of β in the pth group is $(2p-1)$, the maximum value of $P_{\beta m} \begin{matrix}\cos\\ \sin\end{matrix} m\phi$ for a partly normalized harmonic is 1 and the maximum value of a term is therefore $2p(2p-1)\,(S_\beta/r^2)_{\text{max.}}$. The number of S terms in the pth group is $(2p-1)$, and their sum is thus less than $8p^3(S_\beta/r^2)_{\text{max.}}$. The series is therefore less than $8\sum_p p^3(S_\beta/r^2)_{\text{max.}}$. For large p the ratio of successive terms of this series is equal to the ratio of the successive S_β/r^2, which has been shown to be $C_0\beta^{-\frac{1}{2}}$, and is therefore less than unity for sufficiently large β. The series for H_r therefore converges absolutely. The convergence of the series for the θ and ϕ components can be established in a similar way.

From (48) it is clear that some of the terms added when the third and fourth harmonics are included will be as large as those coupling the first and second harmonics. It is only when β gets above about four that the powers of β on the right of table 7 reduce them substantially. Thus it is not surprising that adding third and fourth harmonics produced large changes, but it is likely that higher terms will have only a small effect.

The foregoing argument proves that if there is a real characteristic value V, then the series for the field converges. All that is now needed is a proof that adding terms of degree above the fourth will not greatly disturb the V found in § 7 from the terms of degrees one to four. It is unlikely that a general theorem can be found, for if there are radial velocity functions Q_S and Q_T for which there is a solution and others for which there is not, there will be critical forms for which a solution just does or just does not exist, and the inclusion of small additional terms may then greatly affect the solution or destroy one that would otherwise exist.

Table 7. Coupling terms for T_1 and S_2^{2c} motions

terms in (24) with non-normalized harmonics		order of magnitude for large β with normalized harmonics: small m			large m		
		$\mathscr{A}$	$\mathscr{B}$	$\mathscr{C}$	$\mathscr{A}$	$\mathscr{B}$	$\mathscr{C}$
$(T_1 S_\beta^{mc} S_\beta^{ms}) = -(T_1 S_\beta^{ms} S_\beta^{mc})$	$= -mQ_T S_\beta^m$	—	—	β^{-2}	—	—	β^{-1}
$(T_1 T_\beta^{mc} T_\beta^{ms}) = -(T_1 T_\beta^{ms} T_\beta^{mc})$	$= -mQ_T T_\beta^m$	—	—	β^{-2}	—	—	β^{-1}
$(T_1 S_\beta^{mc} T_{\beta+1}^{mc}) = (T_1 S_\beta^{ms} T_{\beta+1}^{ms})$	$= -\dfrac{\beta(\beta-m+1)}{2\beta+1}(\dot{Q}_T - 2Q_T/r)\, S_\beta^m$	—	—	β^{-1}	—	—	$\beta^{-\frac{3}{2}}$
$(T_1 S_\beta^{mc} T_{\beta-1}^{mc}) = (T_1 S_\beta^{ms} T_{\beta-1}^{ms})$	$= \dfrac{(\beta+1)(\beta+m)}{2\beta+1}(\dot{Q}_T - 2Q_T/r)\, S_\beta^m$	—	—	β^{-1}	—	—	$\beta^{-\frac{3}{2}}$
$(S_2^{2c} S_\beta^{mc} S_\beta^{(m+2)c}) = (S_2^{2c} S_\beta^{ms} S_\beta^{(m+2)s})$	$= -\dfrac{9\delta}{\beta(\beta+1)(2\beta+3)(2\beta-1)}[2(\beta^2+\beta-3)\, Q_S \dot{S}_\beta^m - \beta(\beta+1)\, \dot{Q}_S S_\beta^m]$	—	β^{-2}	β^{-2}	—	β^{-3}	β^{-3}
$(S_2^{2c} S_\beta^{mc} S_\beta^{(m-2)c}) = (S_2^{2c} S_\beta^{ms} S_\beta^{(m-2)s})$	$= -\dfrac{9(\beta+m)(\beta+m-1)(\beta-m+2)(\beta-m+1)}{\beta(\beta+1)(2\beta+3)(2\beta-1)}[2(\beta^2+\beta-3)\, Q_S \dot{S}_\beta^m - \beta(\beta+1)\, \dot{Q}_S S_\beta^m]$	—	β^{-2}	β^{-2}	—	β^{-3}	β^{-3}
$(S_2^{2c} S_\beta^{mc} S_{\beta+2}^{(m+2)c}) = (S_2^{2c} S_\beta^{ms} S_{\beta+2}^{(m+2)s})$	$= \dfrac{3\beta\delta}{(\beta+2)(2\beta+1)(2\beta+3)}[3Q_S \dot{S}_\beta^m - (\beta+1)\, \dot{Q}_S S_\beta^m]$	—	β^{-2}	β^{-1}	—	β^{-2}	β^{-1}
$(S_2^{2c} S_\beta^{mc} S_{\beta+2}^{(m-2)c}) = (S_2^{2c} S_\beta^{ms} S_{\beta+2}^{(m-2)s})$	$= \dfrac{3\beta(\beta-m+4)(\beta-m+3)(\beta-m+2)(\beta-m+1)}{(\beta+2)(2\beta+1)(2\beta+3)}[3Q_S \dot{S}_\beta^m - (\beta+1)\, \dot{Q}_S S_\beta^m]$	—	β^{-2}	β^{-1}	—	β^{-4}	β^{-3}
$(S_2^{2c} S_\beta^{mc} S_{\beta-2}^{(m+2)c}) = (S_2^{2c} S_\beta^{ms} S_{\beta-2}^{(m+2)s})$	$= \dfrac{3(\beta+1)\,\delta}{(\beta-1)(2\beta+1)(2\beta-1)}[3Q_S \dot{S}_\beta^m + \beta \dot{Q}_S S_\beta^m]$	—	β^{-2}	β^{-1}	—	β^{-4}	β^{-3}
$(S_2^{2c} S_\beta^{mc} S_{\beta-2}^{(m-2)c}) = (S_2^{c} S_\beta^{ms} S_{\beta-2}^{(m-2)s})$	$= \dfrac{3(\beta+1)(\beta+m)(\beta+m-1)(\beta+m-2)(\beta+m-3)}{(\beta-1)(2\beta-1)(2\beta+1)}[3Q_S \dot{S}_\beta^m + \beta \dot{Q}_S S_\beta^m]$	—	β^{-2}	β^{-1}	—	β^{-2}	β^{-1}
$(S_2^{2c} T_\beta^{mc} S_{\beta+1}^{(m+2)s}) = -(S_2^{2c} T_\beta^{ms} S_{\beta+1}^{(m+2)c})$	$= \dfrac{18 Q_S \delta}{(2\beta+1)(\beta+1)(\beta+2)}\, T_\beta^m$	—	—	β^{-3}	—	—	$\beta^{-\frac{7}{2}}$
$(S_2^{2c} T_\beta^{mc} S_{\beta+1}^{(m-2)s}) = -(S_2^{2c} T_\beta^{ms} S_{\beta+1}^{(m-2)c})$	$= -\dfrac{18(\beta+m)(\beta+m-1)(\beta+m-2)(\beta-m+1)}{(\beta+1)(\beta+2)(2\beta+1)}\, Q_S T_\beta^m$	—	—	β^{-3}	—	—	$\beta^{-\frac{5}{2}}$
$(S_2^{2c} T_\beta^{mc} S_{\beta-1}^{(m+2)s}) = -(S_2^{2c} T_\beta^{ms} S_{\beta-1}^{(m+2)c})$	$= -\dfrac{18\delta}{\beta(\beta-1)(2\beta+1)}\, T_\beta^m$	—	—	β^{-3}	—	—	$\beta^{-\frac{9}{2}}$
$(S_2^{2c} T_\beta^{mc} S_{\beta-1}^{(m-2)s}) = -(S_2^{2c} T_\beta^{ms} S_{\beta-1}^{(m-2)c})$	$= \dfrac{18(\beta+m)(\beta+m-1)(\beta+m-2)(\beta-m+1)}{\beta(\beta-1)(2\beta+1)}\, Q_S T_\beta^m$	—	—	β^{-3}	—	—	$\beta^{-\frac{7}{2}}$

terms in (24) with non-normalized harmonics	order of magnitude for large β with normalized harmonics: small m $\mathscr{A}$	small m $\mathscr{B}$	small m $\mathscr{C}$	large m $\mathscr{A}$	large m $\mathscr{B}$	large m $\mathscr{C}$
$(S_2^{2c} S_\beta^{mc} T_{\beta+1}^{(m+2)s}) = -(S_2^{2c} S_\beta^{ms} T_{\beta+1}^{(m+2)c}) = -\dfrac{3\delta}{(\beta+1)(\beta+2)(2\beta+1)}[6Q_S \ddot{S}_\beta^m - 2\{(\beta-2)\dot{Q}_S + 6Q_S/r\}\dot{S}_\beta^m + \beta(\beta+1)(\ddot{Q}_S - 2\dot{Q}_S/r) S_\beta^m]$	β^{-3}	β^{-2}	β^{-1}	$\beta^{-\frac{7}{2}}$	$\beta^{-\frac{5}{2}}$	$\beta^{-\frac{3}{2}}$
$(S_2^{2c} S_\beta^{mc} T_{\beta+1}^{(m-2)s}) = -(S_2^{2c} S_\beta^{ms} T_{\beta+1}^{(m-2)c}) = \dfrac{3(\beta+m)(\beta+m-1)(\beta+m-2)(\beta-m+1)}{(\beta+1)(\beta+2)(2\beta+1)}[6Q_S \ddot{S}_\beta^m - 2\{(\beta-2) Q_S + 6Q_S/r\}\dot{S}_\beta^m + \beta(\beta+1)(\ddot{Q}_S - 2\dot{Q}_S/r) S_\beta^m]$	β^{-3}	β^{-2}	β^{-1}	$\beta^{-\frac{5}{2}}$	$\beta^{-\frac{3}{2}}$	$\beta^{-\frac{1}{2}}$
$(S_2^{2c} S_\beta^{mc} T_{\beta-1}^{(m+2)s}) = -(S_2^{2c} S_\beta^{ms} T_{\beta-1}^{(m+2)c}) = \dfrac{3\delta}{\beta(\beta-1)(2\beta+1)}[6Q_S \ddot{S}_\beta^m + 2\{(\beta+3)\dot{Q}_S - 6Q_S/r\}\dot{S}_\beta^m + \beta(\beta+1)(\ddot{Q}_S - 2Q_S/r) S_\beta^m]$	β^{-3}	β^{-2}	β^{-1}	$\beta^{-\frac{9}{2}}$	$\beta^{-\frac{7}{2}}$	$\beta^{-\frac{5}{2}}$
$(S_2^{2c} S_\beta^{mc} T_{\beta-1}^{(m-2)s}) = -(S_2^{2c} S_\beta^{ms} T_{\beta-1}^{(m-2)c}) = -\dfrac{3(\beta+m)(\beta+m-1)(\beta+m-2)(\beta-m+1)}{\beta(\beta-1)(2\beta+1)}[6Q_S \ddot{S}_\beta^m + 2\{(\beta+3)\dot{Q}_S - 6Q_S/r\}\dot{S}_\beta^m + \beta(\beta+1)(\ddot{Q}_S - 2\dot{Q}_S/r) S_\beta^m]$	β^{-3}	β^{-2}	β^{-1}	$\beta^{-\frac{7}{2}}$	$\beta^{-\frac{5}{2}}$	$\beta^{-\frac{3}{2}}$
$(S_2^{2c} T_\beta^{mc} T_\beta^{(m+2)c}) = (S_2^{2c} T_\beta^{ms} T_\beta^{(m+2)s}) = -\dfrac{9\delta}{\beta(\beta+1)(2\beta+3)(2\beta-1)}[2(\beta^2+\beta-3) Q_S \dot{T}_\beta^m + \{3(\beta+2)(\beta-1)\dot{Q}_S - 4(\beta^2+\beta-3) Q_S/r\} T_\beta^m]$	—	β^{-2}	β^{-2}	—	β^{-3}	β^{-3}
$(S_2^{2c} T_\beta^{mc} T_\beta^{(m-2)c}) = (S_2^{2c} T_\beta^{ms} T_\beta^{(m-2)s}) = -\dfrac{9(\beta+m)(\beta+m-1)(\beta-m+2)(\beta-m+1)}{\beta(\beta+1)(2\beta+3)(2\beta-1)}[2(\beta^2+\beta-3)\dot{T}_\beta^m + \{3(\beta+2)(\beta-1)\dot{Q}_S - 4(\beta^2+\beta-3) Q_S/r\} T_\beta^m]$	—	β^{-2}	β^{-2}	—	β^{-3}	β^{-3}
$(S_2^{2c} T_\beta^{mc} T_{\beta+2}^{(m+2)c}) = (S_2^{2c} T_\beta^{ms} T_{\beta+2}^{(m+2)s}) = \dfrac{3\delta}{(\beta+2)(\beta+3)(2\beta+1)(2\beta+3)}[3\beta(\beta+3) Q_S \dot{T}_\beta^m - \{\beta(\beta+3)(\beta-1)\dot{Q}_S + 6\beta(\beta+3) Q_S/r\} T_\beta^m]$	—	β^{-2}	β^{-1}	—	β^{-2}	β^{-1}
$(S_2^{2c} T_\beta^{mc} T_{\beta+2}^{(m-2)c}) = (S_2^{2c} T_\beta^{ms} T_{\beta+2}^{(m-2)s}) = \dfrac{3(\beta-m+4)(\beta-m+3)(\beta-m+2)(\beta-m+1)}{(\beta+2)(\beta+3)(2\beta+1)(2\beta+3)}[3\beta(\beta+3) Q_S \dot{T}_\beta^m - \{\beta(\beta+3)(\beta-1)\dot{Q}_S + 6\beta(\beta+3) Q_S/r\} T_\beta^m]$	—	β^{-2}	β^{-1}	—	β^{-4}	β^{-3}
$(S_2^{2c} T_\beta^{mc} T_{\beta-2}^{(m+2)c}) = (S_2^{2c} T_\beta^{ms} T_{\beta-2}^{(m+2)s}) = \dfrac{3(\beta+1)\delta}{(\beta-1)(2\beta+1)(2\beta-1)}[3Q_S \dot{T}_\beta^m + \{(\beta+2)\dot{Q}_S - 6Q_S/r\} T_\beta^m]$	—	β^{-2}	β^{-1}	—	β^{-4}	β^{-3}
$(S_2^{2c} T_\beta^{mc} T_{\beta-2}^{(m-2)c}) = (S_2^{2c} T_\beta^{ms} T_{\beta-2}^{(m-2)s}) = \dfrac{3(\beta+m)(\beta+m-1)(\beta+m-2)(\beta+m-3)}{(\beta-1)(2\beta+1)(2\beta-1)}[3Q_S \dot{T}_\beta^m + \{(\beta+2)\dot{Q}_S - 6Q_S/r\} T_\beta^m]$	—	β^{-2}	β^{-1}	—	β^{-2}	β^{-1}

$\delta = 1$ if $m \neq 0$, $\quad \delta = 2$ if $m = 0$.

Although no general theorem is to be expected, the matter is clarified by an examination of the size of the terms involved; suppose, as before, that a solution has been found including all the $2\gamma(\gamma+1)$ terms up to $T^m_{2\gamma}$ and $S^m_{2\gamma-1}$. If the matrix method of § 7 is used, V is given as the latent root of a matrix and satisfies an equation that may be reduced to the form

$$\begin{bmatrix} \lambda \mathsf{I} & \mathsf{A} \\ \mathsf{B} & \lambda \mathsf{I} \end{bmatrix} \begin{bmatrix} \mathsf{X}_1 \\ \mathsf{X}_2 \end{bmatrix} = 0, \tag{55}$$

where $\lambda = 1/V$, and A and B are matrices whose elements represent the interactions of one-half of the $2\gamma(\gamma+1)$ harmonics with the other half (cf. (33) and (34)). If the range $0 \leqslant r \leqslant 1$ is divided into M parts, the matrices A and B will each consist of $\gamma(\gamma+1)$ submatrices each with M or $M-1$ rows and columns. As before, let the harmonics be considered in groups such that the pth group contains all the T^m_{2p} and S^m_{2p-1}. Then the harmonics of the pth group are coupled only to each other and to those of groups $p \pm 1$. The matrices A and B may then be written

$$\mathsf{A} = \begin{bmatrix} \mathsf{A}_{11} & \mathsf{A}_{12} & 0 & 0 & 0 & \dots & \dots \\ \mathsf{A}_{21} & \mathsf{A}_{22} & \mathsf{A}_{23} & 0 & 0 & \dots & \dots \\ 0 & \mathsf{A}_{32} & \mathsf{A}_{33} & \mathsf{A}_{34} & 0 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & 0 & \mathsf{A}_{\gamma,\gamma-1} & \mathsf{A}_{\gamma\gamma} \end{bmatrix}, \quad \mathsf{B} = \begin{bmatrix} \mathsf{B}_{11} & \mathsf{B}_{12} & 0 & 0 & 0 & \dots & \dots \\ \mathsf{B}_{21} & \mathsf{B}_{22} & \mathsf{B}_{23} & 0 & 0 & \dots & \dots \\ 0 & \mathsf{B}_{32} & \mathsf{B}_{33} & \mathsf{B}_{34} & 0 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & 0 & \mathsf{B}_{\gamma,\gamma-1} & \mathsf{B}_{\gamma\gamma} \end{bmatrix},$$

where each of the A_{ij} and B_{ij} are submatrices which include the interactions of half the terms in the ith group with half those in the jth group. The introduction of the group of harmonics $T^m_{2\gamma+2}$ and $S^m_{2\gamma+1}$ provides A and B each with an extra row and an extra column, so that their bottom right-hand corners become

$$\begin{bmatrix} \dots & \dots & \dots & \dots & \dots \\ \dots & 0 & \mathsf{A}_{\gamma,\gamma-1} & \mathsf{A}_{\gamma\gamma} & \mathsf{A}_{\gamma,\gamma+1} \\ \dots & 0 & 0 & \mathsf{A}_{\gamma+1,\gamma} & \mathsf{A}_{\gamma+1,\gamma+1} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \dots & \dots & \dots & \dots & \dots \\ \dots & 0 & \mathsf{B}_{\gamma,\gamma-1} & \mathsf{B}_{\gamma\gamma} & \mathsf{B}_{\gamma,\gamma+1} \\ \dots & 0 & 0 & \mathsf{B}_{\gamma+1,\gamma} & \mathsf{B}_{\gamma+1,\gamma+1} \end{bmatrix}.$$

If normalized harmonics are used, the arguments employed above show that the new terms $\mathsf{A}_{\gamma,\gamma+1}$, $\mathsf{A}_{\gamma+1,\gamma}$ and $\mathsf{A}_{\gamma+1,\gamma+1}$ have elements that are at most of order $\gamma^{-\frac{1}{2}}$.

What is needed is an estimate of the effect of the added terms at the lower right-hand corners of A and B. These terms are small, and if the matrix AB were symmetrical their effect could be estimated by perturbation theory and shown to be small. As AB is not symmetrical, there is no reliable way of estimating their effect except by actually computing the latent roots of matrices including more and more terms. This is impracticable, but there is some numerical evidence that the latent roots are not unduly sensitive to small changes in the elements. Owing to an error, a large part of table 6*b* was computed with two neighbouring columns of (49) interchanged; none of the V's was affected by as much as $\frac{1}{2}$ % by this.

The problem can be looked at from a more physical point of view. Suppose the field given by a solution including harmonics up to the pth to be substituted in the time-dependent equations (22) and (23). The harmonics up to the pth will be instantaneously steady, that is, all the $\partial S/\partial t$ and $\partial T/\partial t$ up to those of degree p will vanish, and the higher harmonics will be zero. The higher harmonics will start to grow and will influence the ones of lower degree. If this latter influence is small, a small adjustment of V and of the radial functions will give a steady state. If V does not tend to a limit as the number of terms included increases, the reaction of the terms of higher degree on the lower ones can never be small, and terms of indefinitely high degree are important. The motion may be said to tear any large-scale field into indefinitely small shreds instead of supporting it. It seems unlikely that a motion of finite velocity will do this, but it has been demonstrated by Takeuchi and Bullard that it actually happens when $\epsilon \to \infty$ and that the eigenvalue V is then $O(n^{\frac{3}{2}})$, where n is the degree of the highest harmonic included in the calculation. The proof of this is a little involved and does not contribute to the main argument; it will therefore be given elsewhere.

The expressions given in table 7 can also be used to justify the assumption that the dynamo action resides in the terms of low degree. The diagram of figure 8 when extended to the right contains a series of quadrilaterals with corners $S_{2\gamma-1}$, $T_{2\gamma}$, $T_{2\gamma}^{2c}$, $T_{2\gamma}^{2s}$. The effect of each of these taken by itself may be investigated. If the four equations considered in § 7, for which $\gamma = 1$, represent the most important quadrilateral, it would be expected that the critical V's for those with higher γ would be greater than for $\gamma = 1$. A V that will just cause the $\gamma = 1$ quadrilateral to regenerate will then give only weak interactions in the higher ones, which will contribute little to the regeneration.

Each quadrilateral taken by itself gives four equations similar to (38), but with different coefficients. If $\epsilon = \infty$ these reduce to a form similar to (45) and become

$$\begin{aligned} r^2\ddot{S}_{2\gamma-1} - (2\gamma-1)\,S_{2\gamma-1} &= -\frac{324(\gamma+1)\,V}{\gamma(4\gamma+3)\,(4\gamma-1)\,(4\gamma+1)}\,\frac{Q_S}{Q_T}\,[(4\gamma^2+2\gamma-3)\,Q_S\dot{T}_{2\gamma} \\ &\qquad + \{3(\gamma+1)\,(2\gamma-1)\,\dot{Q}_S - 2(4\gamma^2+2\gamma-3)\,Q_S/r\}\,T_{2\gamma}], \\ r^2\ddot{T}_{2\gamma} - 2\gamma(2\gamma+1)\,T_{2\gamma} &= -\frac{2\gamma(2\gamma-1)\,V}{(4\gamma-1)}\,(\dot{Q}_T - 2Q_T/r)\,S_{2\gamma-1}. \end{aligned}$$

Approximate values of V have been found by dividing the range into three parts and using the matrix method. The results are

γ	1	2	3	4	large
V	18·8	70·4	161	290	$19{\cdot}1\gamma^2$

The result for large γ has been obtained by neglecting all but the leading terms in γ; it agrees quite well with the exact latent root even for γ as small as one, but this is something of a coincidence. The V's given by the higher quadrilaterals are much greater than that from the first, and the higher ones can contribute little to the dynamo action.

There are many other quadrilaterals in figure 8, and it is impracticable to examine all of them. The set $S_{2\gamma-1}^{(2\gamma-2)c}$, $T_{2\gamma}^{(2\gamma-2)c}$, $T_{2\gamma}^{2\gamma c}$, $T_{2\gamma}^{2\gamma s}$ has been worked out for large γ. Using this result for $\gamma = 2$, 3 and 4, and the previous result for $\gamma = 1$, gives

γ	1	2	3	4	γ large
V	18·8	153	632	1731	$13{\cdot}5\gamma^{\frac{7}{2}}$

The quadrilateral S_3, T_2, T_2^{2c}, T_2^{2s} has also been considered. It gives $V = 84$.

The solution of these quadrilaterals may be regarded as a sample investigation of closed loops among the higher degree terms. It suggests that for terms of degree greater than four the V's found in § 7 are 'small', that the interactions between them will therefore be weak and that the convergence of V and of the series (17) will be rapid when proceeding to terms of degree greater than four.

The physical reason for the unimportance of the higher terms is that the fields $\boldsymbol{S}_\gamma^m$ and $\boldsymbol{T}_\gamma^m$ reverse their direction in an angular distance of about π/m in longitude and $\pi/(\gamma-m)$ in latitude. For large γ, the field has therefore a lesser distance to diffuse, and if unsupported decays more rapidly than for small γ. Equations (22) and (23) represent a balance between decay of field by diffusion and its production by electromagnetic induction. If the velocity is too low for the higher harmonics to maintain themselves by their mutual interactions alone, they must be parasitic on the terms of lower degree, and the more rapid their free decay the smaller will be the field in the high harmonics associated with a given dipole field.

The foregoing argument refers to the dynamo with S_2^{2c} and T_1 motions, but it seems likely that it could be generalized by using Bird's (1949) expressions for K and L in place of those of table 7. We have refrained from attempting this in order not to complicate further an already somewhat intricate discussion.

9. Dynamical considerations

Apart from any doubts that may remain about convergence, the preceding work establishes the existence of steady, non-zero solutions of Maxwell's equations for a conducting liquid moving in a specified way. It is not intended in this paper to discuss the dynamics of such motions in detail; a brief review of the subject as it affects the core of the earth and terrestrial magnetism will, however, be given.

The fluid core is the only part of the earth that can reasonably be supposed to be the seat of the motions required to produce a dynamo. Possible causes of such motions have been discussed in a previous paper (Bullard 1949*a*), where it is concluded that thermal convection is the only likely cause. The conditions necessary for radioactive heating in the core to produce convection have been considered and lead to some difficulties which appear not to be insuperable (Bullard 1950); these difficulties have recently been reduced by a more careful consideration of the quantities involved (Jacobs 1953). Elsasser (1950) has suggested that the heat causing convection might be produced in the inner core and not in the larger outer part. Benfield has suggested that some or all of the heat might be provided by compression and not by radioactivity. Urey (1952) has suggested that motion might be produced by gravitational settling of iron from the mantle into the core or of nickel or iron sulphide from the outer core to the inner. Such questions are difficult to settle with certainty, but the assumption of thermal convection caused by heating of the fluid core by radioactivity distributed uniformly through it appears to be the most likely, to involve the fewest arbitrary assumptions and to be the view that is most easily stated precisely. We tentatively adopt it in this paper, but do not regard the alternatives as impossible.

A complete treatment would involve combining the equations of heat conduction and hydrodynamics with Maxwell's equations and finding a steady solution. It would be allowable to linearize the hydrodynamic equations expressed in co-ordinates rotating with the earth and thus remove the terms involving squares and products of the velocities, but it would be necessary to retain the quadratic terms in the field that give the electromagnetic forces. The products of velocities and fields in Maxwell's equations are also, of course, essential. The resulting non-linear characteristic value problem is of considerable complexity and has not so far been attempted. However, even without a solution of this, it is possible to make some progress in understanding the motion.

Chandrasekhar (1952) has shown that in a non-rotating sphere in the absence of a magnetic field, the form of convection current that is most easily excited is S_1 (or S_1^1). This gives a motion along an axis and a return flow in the opposite direction near the outside of the sphere. The motion is symmetrical about an axis and thus violates Cowling's theorem and cannot act as a dynamo. There is no published investigation of the form taken by convection currents in a rotating sphere, though we understand that papers on the subject by Chandrasekhar and by Takeuchi will appear shortly. The S_1 motion involves large circuits in meridian planes, and it seems unlikely that such motions will be easily excited in a rotating body. Gyroscopic forces will tend to turn them into a plane parallel to the equator in the same way that the gyro-compass is forced to rotate with its axis parallel to the earth's axis. Chandrasekhar's (1953) investigation of convection in a rotating disk gives an example of this behaviour. It seems likely that the most easily excited motion will be one in which the movement is predominantly in planes at right angles to the axis. Such a motion is most easily attained by spacing the axes of the convection cells around the equator to give an S_α^α motion of which the S_2^{2c} motion considered in § 7 is an example.

It is well known (Proudman 1916; Taylor 1921, 1923) that slow motions in a rotating fluid are in some circumstances confined to planes at right angles to the axis of rotation, and it has been suggested that this will be true of convective motion in the earth's core. The motion in meridian planes plays an essential part in the $T_1 S_2^{2c}$ dynamo discussed above; it is this component that produces the T_2^{2c} field from the T_2 field. We now consider the circumstances in which three-dimensional motions can be produced in a rotating sphere of fluid.

Consider the equations of motion of a rotating, incompressible fluid body in which the motions are slow relative to axes rotating with the body with angular velocity Ω. The quadratic terms in the velocity components can be neglected and the terms in Ω^2 combined with the pressure. The main effect of gravity can also be combined with the hydrostatic pressure to leave only the forces driving the motion and the corresponding departure from a symmetrical pressure distribution. The resulting equations in right-handed Cartesian co-ordinates $(x_1 x_2 x_3)$ with rotation about the x_3 axis are

$$\left.\begin{aligned} \rho\,\partial v_1/\partial t - \rho\Omega v_2 &= X_1 - \partial P/\partial x_1, \\ \rho\,\partial v_2/\partial t + \rho\Omega v_1 &= X_2 - \partial P/\partial x_2, \\ \rho\,\partial v_3/\partial t \qquad\quad &= X_3 - \partial P/\partial x_3, \end{aligned}\right\} \tag{56}$$

where

$$P = p - \tfrac{1}{2}\rho\Omega^2 r^2 + \int_r^a g\rho\,\mathrm{d}r;$$

v_1, v_2 and v_3 are the components of velocity, ρ is the density of the fluid, which may, to a sufficient approximation, be taken as that at the mean temperature, p is the pressure, and g is the acceleration due to gravity. $X_1 X_2 X_3$ are the components of the driving force, $\boldsymbol{X}$, per unit volume. In a motion due to thermal convection $\boldsymbol{X}$ will be the force of gravity acting on the thermally induced density differences. If $\boldsymbol{X}$ is zero and the motion is steady

$$\begin{aligned} \rho\Omega v_2 &= \partial P/\partial x_1, \\ \rho\Omega v_1 &= -\partial P/\partial x_2, \\ 0 &= \partial P/\partial x_3. \end{aligned}$$

Taking the curl of this gives $$\frac{\partial v_1}{\partial x_3} = \frac{\partial v_2}{\partial x_3} = \frac{\partial v_3}{\partial x_3} = 0. \tag{57}$$

If the fluid is contained in a rigid spherical container, the normal component v_A of the velocity must vanish at the point A in figure 22, and thus from (57) the velocity component in the direction of v_A must vanish at every point on a line AB parallel to x_3. Similarly, the component v_B parallel to the normal at B must vanish along AB, and thus everywhere along AB the velocity must be constant and at right angles to the meridian plane. The equation of continuity then requires the velocity to be the same at all points of a cylindrical shell with axis O and containing AB. The only slow free motions possible in a body of rotating fluid contained in a rigid spherical envelope are therefore motions in which cylindrical shells

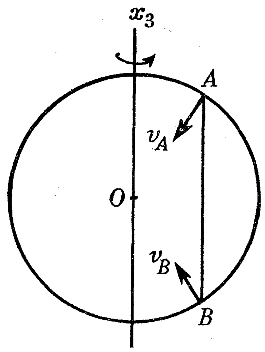


FIGURE 22

rotate like rigid bodies about the axis of the main rotation; the motion is even more restricted than that considered by Proudman and Taylor. The restriction on the motion is due to the impossibility of balancing the Coriolis forces $-\rho\Omega v_2$ and $\rho\Omega v_1$ in (56) by a pressure gradient. If a fluid is compelled by impulsive forces to start on a motion that has radial and meridional components, it will escape from the difficulty by performing a rapidly changing and highly curved motion in which the Coriolis forces are balanced by the inertia forces. In such a motion the gradients of the velocities are comparable with Ω, and the neglected quadratic terms in (56) become comparable with the Coriolis forces.

Such a motion is of no use for the present purpose. To produce a dynamo with a large-scale motion, the Coriolis forces must be balanced by the buoyancy forces and the electromagnetic forces, both of which have a non-zero curl. If energy is to be conserved, the buoyancy and electromagnetic forces must themselves be of the same order of magnitude. In a previous paper (Bullard 1949*a*) it has been shown that the electromagnetic forces in the core are sufficient to stop a free motion in the core in a few years, and that if motion is to continue it must be continually driven by the action of gravity on thermally induced density differences or by some other force.

From these considerations it is possible to get a rough estimate of the quantities involved in the dynamo action and to test the reasonableness of the whole scheme.

The calculations of § 7 give the non-dimensional number V. The actual velocity is related to this by (9) and (13), and its radial component v_r is

$$v_r = \frac{3VQ_S P_2^2(\cos\theta)\cos 2\phi}{2\pi\kappa a r^2}.$$

If $Q_S = r^3(1-r)^2$ this has its maximum at $r = \frac{1}{3}$, $\theta = \frac{1}{2}\pi$, $\phi = 0$, where it is

$$v_r(\text{max.}) = 9VQ_S/2\pi\kappa a r^2 = 2V/3\pi\kappa a. \tag{58}$$

The solutions of the seven and twelve equations in § 7 all give values of V between 50 and 100; we take 70 as a typical value. The radius of the core, a, is $3{\cdot}47 \times 10^8$ cm, and κ will be taken as 3×10^{-6} e.m.u. as in former papers. With these values, (58) gives

$$v_r(\text{max.}) = 0{\cdot}014\,\text{cm/s}.$$

As the conductivity is uncertain by a factor of three, this can only be regarded as an order-of-magnitude estimate. It is possible that a better value for κ could be obtained by using the recent work of Powell (1953) and Bridgeman (1952), but it is thought better to postpone a change till the temperature coefficient of the resistance of molten iron has been redetermined. It is believed that the value adopted for κ is more likely to be below than above the true value.

At a speed of $0{\cdot}014$ cm/s, a particle would take 800 years to travel a distance equal to the radius of the core. Such a speed seems very reasonable in relation to the time scale of secular variation.

The S_2^{2c} motion in a rotating sphere involves a continuous flow of angular momentum from the outside to the inside of the core. This causes the inside to rotate faster than the outside and gives the T_1 motion of § 7. In a steady motion this transfer of angular momentum must be balanced by the electromagnetic couples which transfer it back again to the outer part of the core. These couples are principally due to the interaction of the S_1 field with the current producing the T_2 field; they are therefore proportional to the product of the S_1 and T_2 fields. The S_1 field is known from observation, and the angular momentum balance may therefore be expected to determine the T_2 field. The strength of the T_2 field will then determine the T_1 velocity (the S_2^{2c} velocity has already been fixed).

Since the radial velocity functions Q_S and Q_T have been arbitrarily chosen and not made consistent with the equations of motion, this calculation cannot be carried out in a completely satisfactory way; it should, however, be possible to obtain the correct order of magnitude. In a previous paper (Bullard 1949*a*, p. 443), the core was divided into two parts of equal volume and the electromagnetic couple between the two halves balanced against the convection of angular momentum. A very similar result may be obtained more easily by equating the electromagnetic and Coriolis forces in the ϕ direction. This gives

$$\rho\Omega v_r = \frac{1}{4\pi}(\boldsymbol{H}_0 \times \text{curl}\,\boldsymbol{H}_\phi)_\phi, \tag{59}$$

where $\boldsymbol{H}_0$ is the dipole field and $\boldsymbol{H}_\phi$ is the T_2 field. This relation will not hold exactly at every point, but will be roughly satisfied if the terms are treated as space averages. If $Q_S = r^3(1-r)^2$, the average value of v_r is $(9/20\pi)\, v_r(\text{max.})$. The r and θ components of $\boldsymbol{H}_0$ are proportional to $(2S_1/r^2)\cos\theta$ and $-(\mathrm{d}S_1/r\,\mathrm{d}r)\sin\theta$. From tables 3 and 6 *a*, *b* it seems likely that the averages of these will be greater than the surface value by perhaps a factor of two. Curl $\boldsymbol{H}_\phi$ will be taken as $3H_\phi(\text{max.})/a$. With these rough estimates (58) and (59) give

$$H_\phi(\text{max.}) = \frac{9a\rho\Omega v_r(\text{max.})}{20H_0} \doteqdot \frac{0{\cdot}1\,V\rho\Omega}{\kappa H_0}, \tag{60}$$

34-2

where H_0 is the value of $\boldsymbol{H}_0$ for $\theta = 0$ at the surface of the core. Taking $V = 70$, $\rho = 10{\cdot}7$ g/cm^3, $\Omega = 7{\cdot}3 \times 10^{-5}\,\mathrm{s}^{-1}$, $\kappa = 3 \times 10^{-6}$ e.m.u. and $H_0 = 3{\cdot}8$ G, this gives

$$H_\phi(\text{max.}) = 480\,\text{G},$$
$$H_\phi(\text{max.})/H_0 = 130.$$

The magnitude of the T_1 motion required to produce this T_2 field will now be found.

From (12) and (13) the ratio of the dipole and T_2 fields is given by

$$\frac{H_\phi(\text{max.})}{H_0} = \frac{3}{4}\frac{(T_2/r)_{\text{max.}}}{S_1(1)}.$$

From the results from seven equations given in table 6*a* for $\epsilon = 5$, $Q_T = \epsilon r^2(1-r)$, the right-hand side of this is found to be 12·8; if the calculations were repeated for other values of ϵ until a value was found for which the ratio was 130, this would be the value of ϵ necessary to give an H_ϕ agreeing with (60). In fact, accurate radial functions for a series of values of ϵ are not available; the arguments of § 7 give $T_2/S_1 = O(\epsilon)$ for large ϵ which would suggest that ϵ should be increased about ten times and is about 50. A similar argument from the solution of twelve equations for $\epsilon = 100$ gives $\epsilon = 18$. The true variation of T_2/S_1 with ϵ is not known, but the order of magnitude of the two results being the same suggests taking $\epsilon = 30$ as a rough estimate. The maximum T_1 velocity is

$$v_T(\text{max.}) = \frac{\epsilon V(Q_T/r)_{\text{max.}}}{4\pi\kappa a},$$

which gives

$$v_T(\text{max.}) = 0{\cdot}04\,\text{cm/s}.$$

These values can only be regarded as the roughest order of magnitude estimates. The conductivity could well be three times greater than the value used; the above values of ϵ, H_ϕ and v_r would then be divided by three and v_T by nine. In the present argument the velocity radial functions Q_S and Q_T have been arbitrarily assumed and are not solutions of the equations of motion; it is not even known whether the S_2^{2c} motion is the one most easily generated. In place of the full equations of motion, only the conservation of angular momentum has been satisfied, and this has been done only approximately.

Some further quantities can be roughly estimated on the same basis. The magnetic energy per unit volume is $\boldsymbol{H}^2/8\pi$. Most of this is supplied by the T_2 field. On integrating through the core, the magnetic energy per unit volume is found to be 2300 erg/cm^3. This energy must be replenished in a time about equal to the free decay period of the current system, which is $4\kappa a^2/\pi = 4{\cdot}6 \times 10^{11}\,\text{s} = 14\,000$ years. The rate of dissipation of energy is therefore 5×10^{-9} erg/cm^3 s. This is less than the heat conducted away by the adiabatic temperature gradient and less than that generated in meteorites by radioactivity. It therefore raises no difficulty beyond those previously discussed (Bullard 1950; Jacobs 1953).

The kinetic energy per unit volume relative to an observer rotating with the earth is approximately $\frac{1}{2}\rho v_T^2$; if $\rho = 10{\cdot}7$ g/cm^3, the mean value of this is 3×10^{-3} erg/cm^3. The magnetic energy thus greatly exceeds the kinetic. It has sometimes been supposed that the two should be equal. This follows from balancing the electromagnetic forces against the inertia forces; there seems no reason to do this when considering a large-scale motion, in fact on any theory the motions would have to be improbably fast to make the two energies equal. Even if there were no toroidal field the dipole field of 3·8 G would give 0·57 erg/cm^3, which would require velocities of 0·33 cm/s if the magnetic and kinetic energies were equal; a toroidal field of 50 G would need 4·3 cm/s.

The Coriolis forces and the electromagnetic forces in a horizontal plane have been made to balance each other by (59). Each is of order 1×10^{-5} dyn/cm^3. If the viscosity is η, the viscous forces are $\eta\nabla^2 v$, which is a few times $\eta v_T/a^2 = 10^{-18}\eta$ dyn/cm^3. This is much less than the electromagnetic forces for any reasonable viscosity.

In a place where the temperature is δT greater than the surroundings, the buoyancy forces are $\rho\alpha g\delta T$, where α is the coefficient of cubical expansion. Putting $\rho = 10{\cdot}7$ g/cm^3, $\alpha = 4{\cdot}5 \times 10^{-6}\,°\mathrm{C}^{-1}$, $g = 800$ cm/s^2, this gives $0{\cdot}04\delta T$. If energy is to be conserved the buoyancy forces must be comparable with the electromagnetic forces. If this is so, δT is 3×10^{-4} °C. An alternative estimate can be made by regarding the whole process as a heat engine. If the difference in temperature between rising and falling convective currents is δT, the rate of transfer of heat across a surface of radius r is $2\pi r^2 \rho v_r c\,\delta T$, where c is the specific heat. The ideal thermodynamic efficiency of the engine will be equal to the adiabatic difference between the top and bottom of the core divided by the temperature. This will be about $\frac{1}{10}$. If, as seem likely, the efficiency is of this order, a useful lower limit to the value of δT necessary to supply 5×10^{-9} erg/cm^3 s to the field can be calculated. It is $0{\cdot}4 \times 10^{-4}$ °C, which is consistent with the previous estimate.

Two checks are possible on the reasonableness of the estimates. The first concerns the westward drift of the non-dipole and secular variation fields. In a previous paper (Bullard *et al.* 1950) this has been explained as the consequence of a leakage into the mantle of the current producing the T_2 field. This current interacts with the dipole field to drive the mantle eastwards relative to the core. Any feature of the motion and field in the core therefore moves slowly westward relative to the mantle. The angular velocity of this westward motion is about half the maximum angular velocity in the core due to the T_1 motion. The maximum velocity was found above to be 0·04 cm/s. Half of this at the radius $0{\cdot}79a$, which divides the core into two parts of equal volume, gives a westward drift of 0·13°/year. The observed value is 0·18°/year from the non-dipole field, 0·14°/year from the second and third harmonics in the non-dipole field, and 0·32°/year from the secular variation. These agree with the calculated value within the uncertainty introduced by lack of knowledge of the conductivity. Halving κ would multiply the calculated drift by four. The agreement is gratifying, since the only magnetic quantity used in the calculation is the magnitude of the dipole field.

A further check on the reasonableness of the orders of magnitude concerned can be obtained from the secular variation. The natural explanation of this is to suppose that it is due to irregularities in the motion of the material in the core. Such irregularities will cause currents by electromagnetic induction which will produce transient magnetic fields at the surface. Several authors have investigated the possibility of representing the secular variation field by a limited number of dipoles near the surface of the core (McNish 1940; Bullard 1948; Lowes & Runcorn 1951). Such a representation is possible. The largest known change involves a dipole whose moment changes in a hundred years by an amount not exceeding 2×10^{24} G cm^3. If a change is to occur in a hundred years its cause must not lie too deep in the core or its effect will be screened from observation. The thickness of material needed to reduce an effect of period τ to $1/e$ of its unscreened amplitude is $(\tau/\kappa)^{\frac{1}{2}}/2\pi$. If the field increases for a hundred years and then decreases, τ may be taken roughly as 200 years; with $\kappa = 0{\cdot}3 \times 10^{-5}$ this gives a 'skin depth' of 73 km. Since all the T fields vanish at $r = a$, the field in the outermost 73 km of the core will not differ much from the dipole field, which has a maximum of 3·8 G. This is known (Bullard 1948) to be insufficient to produce the secular variation by induction. It seems likely that a much larger disturbance at the surface could be produced by a local rising movement which brings up with it part of the T_2 field from the interior of the core. No detailed investigation has been made, but a volume W inside which the field is H gauss in excess of the normal will have a dipole moment σWH, where σ is a numerical constant which is $3/8\pi$ for a sphere and unity for a long rod

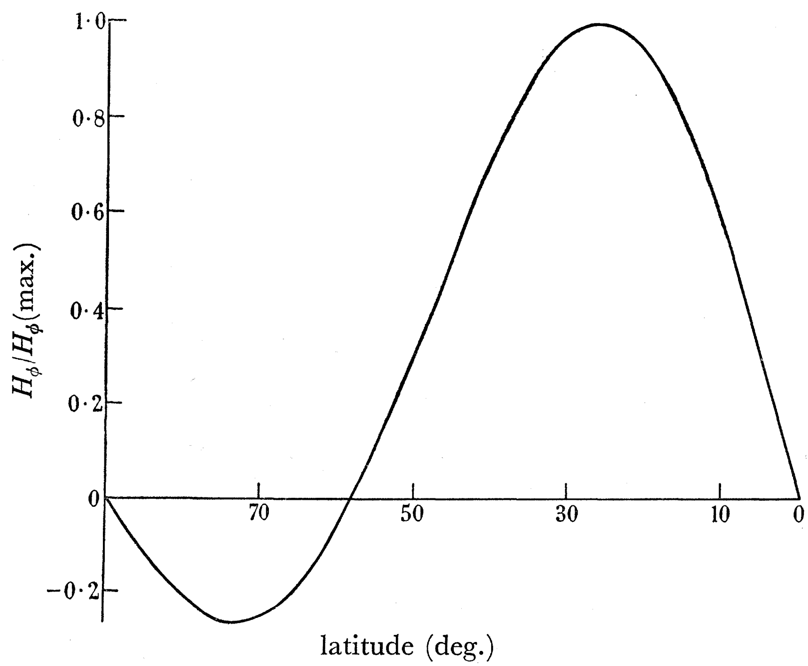


FIGURE 23. Variation of H_ϕ with latitude, including components T_2 and T_4 for $Q_T = \epsilon r^2(1-r)$, $\epsilon \to \infty$.

parallel to H. With $H = 240$ G (half the maximum found above) and $\sigma = 3/8\pi$, the volume needed for a moment of 2×10^{24} G cm³ is 7×10^7 km³. A disk 75 km thick and 1100 km in diameter would give this volume. The analysis of the secular variation by Lowes & Runcorn (1951) and by Vestine, Laporte, Lange & Scott (1947, figure 28) suggests diameters of 2000 to 4000 km for the current circuits giving the secular variation. Our result is thus of the right order of magnitude and leaves some margin to allow for the effect of screening by the conductivity of the mantle and for the effect of currents induced in the core when the source is at the outside. These points will not be considered in detail here; they have been treated by Rikitake (1952 and earlier papers). He concludes that any conductivity less than 10^{-5} e.m.u. is acceptable. It is desirable that the disturbance produced by a few types of motion should be investigated, since it is not clear how big the effects of screening will be. Even in a perfect conductor, a motion that extends to the surface will produce a distortion in a field whose lines of force cross the surface. It therefore seems likely that the screening will be less than considerations of 'skin depth' would suggest.

Any theory that will explain the secular variation will also explain the non-dipole field, since the latter is of the order of the former integrated over a period of a hundred years.

It might have been hoped that not merely the order of magnitude but also something of the general form of the secular variation and non-dipole fields could be explained, as has been possible for the sunspot field (Bullard 1954). In particular, Lowes & Runcorn (1951) have pointed out that the centres of secular variation and of the non-dipole field should not occur near the equator or the poles, as the T_2 toroidal field vanishes there. The positions of the disturbances in the core corresponding to the foci of the non-dipole and secular variation field can be found with the least possibility of personal bias from Vestine *et al.* (1947)

TABLE 8. ESTIMATE OF QUANTITIES CONNECTED WITH THE CORE OF THE EARTH

quantity	symbol	value	dependence on κ
radius	a	3470 km	—
density	ρ	10·7 g/cm^3	—
temperature	T	5000° C	—
specific heat	c_v and c_p	0·16 cal/g °C	—
coefficient of expansion	α	$4{\cdot}5\times10^{-6}$ °C^{-1}	—
compressibility	χ	$1{\cdot}2\times10^{-13}$ dyne/cm^2	—
Grüneisen's constant	$\alpha/\chi\rho c_v$	0·8	—
adiabatic gradient at the surface of the core	$g\alpha T/c_p$	0·26° C/km	—
mean adiabatic gradient	—	0·13° C/km	—
temperature difference between top and bottom of core	—	450° C	—
electrical conductivity	κ	3×10^{-6} e.m.u.	κ
thermal conductivity	—	0·10 cal/cm° C s	κ
heat flow at surface of core due to adiabatic gradient	—	$0{\cdot}26\times10^{-6}$ cal/cm^2 s	κ
heat conducted per unit volume	—	9×10^{-8} erg/cm^3 s	κ
radial velocity function for S_2^{2c} motion (arbitrarily assumed)	Q_S	$r^3(1-r)^2$	—
radial velocity function for T_1 motion (arbitrarily assumed)	Q_T	$\epsilon r^2(1-r)$	—
maximum radial velocity	v_r (max.)	0·014 cm/s	κ^{-1}
time for 1000 km radially	—	230 yr	κ
maximum ϕ velocity	v_T (max.)	0·04 cm/s	κ^{-2}
westward drift (calc.)	—	0·13°/yr	κ^{-2}
—	V	70	—
—	ϵ	30	κ^{-1}
dipole field at the surface of the core	H_0	3·8 G	—
maximum of T_2 field	H_ϕ (max.)	480 G	κ^{-1}
ratio toroidal to dipole field	$H_\phi(\text{max.})/H_0$	130	κ^{-1}
free decay period	$4\kappa a^2/\pi$	14000 yr	κ
mean magnetic energy density	$\bar{H}_\phi^2/8\pi$	2300 erg/cm^3	κ^{-2}
mean kinetic energy density	$\frac{1}{2}\rho v_T^2$	3×10^{-3} erg/cm^3	κ^{-2}
rate of dissipation	$\bar{H}_\phi^2/32\kappa a^2$	5×10^{-9} erg/cm^3 s	κ^{-3}
Coriolis force	$\rho\Omega v_r$	1×10^{-5} dyne/cm^3	κ^{-1}
electromagnetic force	$(H_0\times\text{curl}\,H_\phi)/4\pi$		
temperature difference between rising and falling currents	—	3×10^{-4}° C	κ^{-1}

The data not discussed in this paper are from Bullard (1950), revised as suggested by Jacobs (1953).

figures 8 and 15. These figures give the lines of current flow near the surface of the core that will produce the observed fields. The latitude of the centre of each closed set of lines of flow in these figures has been found. The numbers in 20° ranges of latitude are:

latitude	10° S–10° N	10°–30°	30°–50°	50°–70°	70°–90°
number	4	7	5	6	1

The centres show no marked tendency to avoid the equator. This difficulty is somewhat reduced by the inclusion of the T_4 field. When this is combined with the T_2 field, as in figure 23, the maxima of the east-west field for $Q_T = r^2(1-r)$, $\epsilon \to \infty$, is in latitude 26° instead of latitude 45°, and the toroidal field is less than half the maximum only in latitudes less than 9°. The $S_2^{2c}\, T_1$ dynamo has been worked out in detail primarily because it is arithmetically the simplest. There is no reason to suppose that it is very closely related to the actual motions in the core. These may well be of greater complexity, and the plane of zero east-west field may depart considerably from the equator. There is also the possibility of the field from within the core being dragged over the equator as it is brought to the surface by a current rising from the interior.

The dipole obtained in § 7 is parallel to the earth's axis. A dipole at a small angle to the axis can be obtained by using a less symmetrical motion, for example, by adding a small S_2^c component to the motion (see figure 7).

The numerical values obtained in this section are collected in table 8. They are intended only as order of magnitude estimates, but considerable difficulties would be encountered if they were changed by more than a factor of ten. The small temperature differences and radial velocities are necessary if the transport of heat is to be kept within the range of what can easily be supplied. On the other hand, a minimum velocity is set by the general time scale of terrestrial magnetism. It is remarkable that the velocities calculated from the dynamo theory fit so well into these rather definite limits.

10. Further developments

An attempt has been made in this paper to determine whether homogeneous dynamos are in principle possible, and to provide a numerical example in which the velocities and fields can be estimated. When the results are applied to the core of the earth, the orders of magnitude appear reasonable and there seems no reason why the earth's magnetic field should not be explained by the fluid core acting as a dynamo. It is, of course, impossible to prove directly that the field is produced in this way; the apparent reasonableness of the dynamo might be a coincidence and some other mechanism might, in fact, exist. The situation is somewhat like that of the theory of the production of energy in stars by the carbon-nitrogen cycle. It is only the study of the process and its alternatives over a long period that brings either contradictions or a reasonable probability of correctness.

The present paper leaves many things to be done. It would be easy but tedious to work out dynamos with other types of motion, but it is doubtful if anything of importance would be found. For the sake of completeness, the system of figure 9 should be solved to determine whether its critical velocity is really greater than that of figure 8, as has been tacitly assumed in § 7. An example of a dynamo with a pure S motion or a proof of its impossibility would also be interesting. Methods similar to those of this paper might be applied in Cartesian or cylindrical co-ordinates. The results would not have a direct relevance to terrestrial magnetism, but might be of use elsewhere.

It is probably too much to expect a formal proof that the V derived from (25) remains bounded as the number of equations is increased, but it may well be possible to strengthen the results of § 8.

A much more important and difficult problem is the simultaneous solution of the equations of electromagnetism, hydrodynamics and heat conduction. If this is to be brought within the scope of existing computing machinery, some drastic simplification will be necessary. The easiest approach may be to consider not a stationary state, but the stability of a small disturbance starting from a state with no field. The time would then appear in the equations, but it would not be necessary to find the critical value of the velocity.

It is possible that the magneto-hydrodynamics of the secular variation would prove more tractable, since the initial field could be assumed and the changes regarded as small. The secular variation and the non-dipole field appear to provide the only direct evidence on the pattern of motions in the core. The stability of the system is of particular interest, since it is possible that undamped oscillations may occur; the disk dynamo is known to be capable of undamped oscillations, but nothing is known about oscillations in systems of the kind considered here.

The magnetic fields accompanying sunspots may prove simpler to interpret than the terrestrial secular variation, since the surface of the conducting fluid of the sun is visible and that of the earth is concealed by the mantle. Also, the time scale of the visible changes on the sun seems to be measured in tens rather than hundreds of years and is thus better adapted to the rapid accumulation of data.

Experimental and theoretical evidence on the constitution, temperature and properties of the interior of the earth is also of importance, since dynamo theories of terrestrial magnetism are only possible for a certain range of values of electrical and thermal conductivity, viscosity and heat production.

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