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Gross Thermodynamics of Heat Engines in Deep Interior of Earth

(mantle convection/geomagnetic dynamos/dissipation bounds)

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ABSTRACT **From the gross conservation laws of thermodynamics in a convecting material we derive a bound on the ratio of the rate of production of mechanical or magnetic energy to the rate of internal radioactive heating which drives the convection. Our bound for this "efficiency" depends on the temperatures in the material, and can exceed unity. Whether the bound can be attained by "efficiencies" in real fluids is not known, but a simple machine shows that "efficiencies" larger than unity are physically realizable. Our bound gives upper limits on the viscous dissipation in the earth's mantle and ohmic heating in the core, but these limits are too large to be physically interesting.**

Continental drift is probably a surface manifestation of convection in the earth's mantle, driven by radioactive heating. The motion of the fluid core which generates the geomagnetic field may also be convection driven by radioactive heating, although other causes of this motion have been suggested (1). At any rate, it is possible that the earth contains at least two heat engines, one in the mantle which produces kinetic energy dissipated by viscosity, and one in the core which produces magnetic energy dissipated by ohmic heating. These heat engines have been examined from a thermodynamic point of view (2) but not in sufficient detail to bring out a very interesting peculiarity which arises from the fact that they deposit their frictional heat losses in their hot reservoirs. In the present paper we will give the mathematic derivation and physical interpretation of the thermodynamic conservation laws for these two heat engines. Those laws give upper bounds for the ratio of total ohmic or viscous dissipation to total radioactive heating, but the bounds depend on temperature differences in the material, and may exceed unity. We exhibit a simple mechanical example to show that the bounds can be attained, giving heat engines with apparent "efficiencies" much larger than unity. Whether such "efficiencies" are possible in real fluid motions seems to be an open question.

This work was stimulated by a conversation with Robert Parker about some numerical calculations of Hewitt, McKenzie, and Weiss, reported by them at the Tenth Symposium on Mathematical Geophysics sponsored by the International Union of Geodesy and Geophysics at Cambridge, England in June 1974. Since carrying out this work, the author has learned of a manuscript by Hewitt, McKenzie, and Weiss (3) in which they independently, and before the present work, obtained the special case of inequality [**A9**] in which the material is a Newtonian viscous fluid in steady flow without internal radioactive heating and without a magnetic field.

MATHEMATIC DERIVATION

The relevant equations and inequalities can all be deduced by integrating over the volume of the core or the mantle the partial differential equations which describe local conservation of momentum, energy, and entropy. All the relevant local equations can be found, for example, in Malvern (4) and the derivations are now well known, so we will give only the results, with mention of the equations used to obtain them.

We denote all of the space by E. We consider a volume V (either the core or the mantle) with boundary ∂V and outward unit normal $\hat{\mathbf{n}}$. This volume contains a material with electrical conductivity σ, carrying an electric current density $\mathbf{J}$ which produces a magnetic field $\mathbf{B}$ and a Lorentz force $\mathbf{L} = \mathbf{J} \times \mathbf{B}$ on the material. The magnetic permittivity of free space and the material is μ. The material produces a gravitational potential ψ and the externally produced gravitational potential ψ_E is independent of time. There are no externally produced electromagnetic fields. The material is heated radioactively at a rate γ per unit volume, and has density ρ, internal energy U per unit mass, entropy S per unit mass, local absolute temperature θ, velocity $\mathbf{u}$, heat flux vector $\mathbf{H}$, and stress tensor $\mathbf{T}$. The stress tensor is the sum of a reversible part $\mathbf{T}^r$ and a dissipative part $\mathbf{T}^d$. In a Newtonian viscous fluid, $\mathbf{T}^r$ is $-p\mathbf{I}$ where $\mathbf{I}$ is the identity tensor and p is the thermostatic equilibrium pressure appropriate to the local values of ρ and U; and $\mathbf{T}^d$ is the viscous stress tensor.

If we neglect displacement current, then from Maxwell's equations for a moving conductor it is straight-forward, e.g., Bullard and Gellman (2), to show that

$$\frac{d}{dt}\int_E \frac{\mathbf{B}^2}{2\mu}\, dV = -\int_V \mathbf{u}\cdot\mathbf{L}\, dV - \int_V \mathbf{J}^2\sigma^{-1} dV. \qquad [1]$$

From Poisson's equation for ψ and from the local momentum equation for the material [Malvern (4)] some manipulation gives

$$\frac{d}{dt}\int_V \frac{\rho}{2}(\mathbf{u}^2 + \psi + 2\psi_E)\, dV = \int_{\partial V} \mathbf{u}\cdot\mathbf{T}\cdot\hat{\mathbf{n}}\, dA + \int_V \mathbf{u}\cdot\mathbf{L} dV - \int_V T_{ij}{}^r\partial_i u_j dV - \int_V T_{ij}{}^d\partial_i u_j dV \qquad [2]$$

where ∂_i is the derivative with respect to the i'th Cartesian coordinate, u_i and T_{ij} are the Cartesian coordinates of $\mathbf{u}$ and $\mathbf{T}$, and the repeated indices are to be summed from 1 to 3. By integrating the local internal energy equation [Malvern (4)] and combining the result with [1] and [2] we obtain

$$\frac{d}{dt}\left\{\int_E \frac{\mathbf{B}^2}{2\mu} dV + \int_V \rho\left(\frac{\mathbf{u}^2}{2} + \frac{\psi}{2} + \psi_E + U\right) dV\right\} + \int_{\partial V} \mathbf{H}\cdot\mathbf{n}\, dA = \int_{\partial V} \mathbf{u}\cdot\mathbf{T}\cdot\hat{\mathbf{n}}\, dA + \int_V \gamma\, dV. \qquad [3]$$

Finally, by integrating the local entropy equation [Malvern (4)] over V, we obtain

$$\frac{d}{dt}\int_V \rho S dV + \int_{\partial V} \theta^{-1}\mathbf{H}\cdot\hat{\mathbf{n}}\, dA = \int_V [\theta^{-1}(\gamma + \sigma^{-1}\mathbf{J}^2 + \mathbf{T}_{ij}{}^d\partial_i u_j) + \mathbf{H}\cdot\nabla\theta^{-1}]dV. \quad [4]$$

The physical interpretation of Eq [1] is that magnetic energy is created by the motion of the material against the Lorentz force, and is destroyed by ohmic heating. The physical interpretation of Eq. [2] is that mechanical (kinetic plus gravitational) energy is created by work done on the boundary of the material and by deformation of the material against the equilibrium stress $\mathbf{T}^r$, and is destroyed by work done against the Lorentz force and by the dissipative stress $\mathbf{T}^d$, e.g., viscosity. Eq. [3] indicates that total energy can be increased only by heat flowing into V across ∂V, by work done on ∂V, and by radioactive heating. Finally, Eq. [4] lists the possible causes of an entropy increase: flow across ∂V with current density $\theta^{-1}\mathbf{H}$, or injection locally at a rate $\theta^{-1}\gamma$ by radioactive heating, at a rate $\theta^{-1}\sigma^{-1}\mathbf{J}^2$ by ohmic heating, at a rate $\theta^{-1}T_{ij}{}^d\partial_i u_j$ by frictional (viscous) heating, and at a rate $\mathbf{H}\cdot\nabla\,\theta^{-1}$ by heat flow down a temperature gradient.

In the rest of the present discussion for simplicity we will neglect any mechanical work done on ∂V by asuming $\mathbf{u}\cdot\mathbf{T}\cdot\hat{\mathbf{n}} = 0$ there. This restriction makes our discussion inapplicable to Malkus's precessionally driven dynamo (1), but the interested worker can make the necessary extensions. We will also assume that all integrals in Eqs. [1] through [4] are bounded functions of time. We will denote the time average of a quantity f by Af. Let us denote the total average radioactive heating rate, $A\int_V \gamma dV$, by Q_R; the total average ohmic dissipation rate $A\int_V \sigma^{-1}\mathbf{J}^2\, dV$ by D_B; the total average frictional dissipation rate $A\int_V T_{ij}{}^d\,\partial_i u_j\, dV$ by D_f; the total average rate of production of magnetic energy, $-A\int_V \mathbf{u}\cdot\mathbf{L}\, dV$, by W_B; and the total average rate at which internal energy is converted to mechanical energy, $-A\int_V T_{ij}{}^r\,\partial_i u_j\, dV$, by W_M. Then taking the time averages of Eqs. [1] through [4] gives

$$W_B = D_B \quad [\mathbf{A1}]$$

$$W_m = W_B + D_f \quad [\mathbf{A2}]$$

$$A\int_{\partial V}\mathbf{H}\cdot\hat{\mathbf{n}}\, dA = Q_R \quad [\mathbf{A3}]$$

$$A\int_{\partial V}\theta^{-1}\mathbf{H}\cdot\hat{\mathbf{n}}\, dA = A\int_V [\theta^{-1}(\gamma + \sigma^{-1}\mathbf{J}^2 + T_{ij}{}^d\partial_i u_j) + \mathbf{H}\cdot\nabla\theta^{-1}]dV. \quad [\mathbf{A4}]$$

From its definition, $D_B \geq 0$, so $W_B \geq 0$. The strong form of the second law of thermodynamics [Malvern (4)] asserts that entropy cannot be destroyed even locally, i.e., that each of the source terms in [4] is separately nonnegative, so $T_{ij}{}^d\partial_i u_j \geq 0$ and $\mathbf{H}\cdot\nabla\theta^{-1} \geq 0$. From the first of these two inequalities, $D_f \geq 0$, so from [A2] we have $W_m \geq 0$. In a viscous fluid, $W_m = A\int_V p\,\nabla\cdot\mathbf{u}\, dV$, so if the motion does not die away (i.e. $W_m > 0$) then, on average, the pressure must be higher in expanding parcels of fluid than in contracting parcels. Eq. [A3] assures us that viscous and ohmic heat do not flow out the boundaries; only the radioactive heat does so.

None of [A1], [A2], or [A3] will yield a relation between W_m and either Q_R or $A\int_{\partial V}\mathbf{H}\cdot\hat{\mathbf{n}}dA$. If we want such a relation we must use the entropy Eq. [A4]. We suppose that ∂V consists of an outer part at temperature θ_o, across which heat leaves V at an average rate $Q_o \geq 0$, and an inner part at temperature θ_i, across which heat enters V at an average rate $Q_i \geq 0$. Then [A3] becomes

$$Q_o - Q_i = Q_R. \quad [\mathbf{A5}]$$

Let θ_M denote the largest value of θ in V. Then since each term in the integrand on the right in [A4] is positive, that equation implies

$$\theta_o^{-1}\, Q_o - \theta_i^{-1}\, Q_i \geq \theta_M^{-1}\,(Q_R + Q_m). \quad [\mathbf{A6}]$$

We can use [A5] to eliminate Q_R from [A6], obtaining

$$W_m \leq \left(\frac{\theta_M}{\theta_o} - 1\right)Q_o - \left(\frac{\theta_M}{\theta_i} - 1\right)Q_i. \quad [\mathbf{A7}]$$

Since $\theta_M \geq \theta_i$ and $Q_i \geq 0$, [A7] implies

$$W_m \leq \left(\frac{\theta_M}{\theta_o} - 1\right)Q_o. \quad [\mathbf{A8}]$$

We can also use [A5] to eliminate Q_o from [A6], obtaining

$$W_m \leq \left(\frac{\theta_M}{\theta_o} - 1\right)Q_R + \left(\frac{\theta_M}{\theta_o} - \frac{\theta_M}{\theta_i}\right)Q_i. \quad [\mathbf{A9}]$$

In case $\theta_M = \theta_i$ (which is usually assumed for the mantle but is not an obvious consequence of the convection equations) then [A9] simplifies to

$$W_m \leq \left(\frac{\theta_M}{\theta_o} - 1\right)(Q_R + Q_i). \quad [\mathbf{A10}]$$

COMPARISON WITH A CARNOT ENGINE

The foregoing inequalities can be derived from elementary thermodynamics if we accept [A1], [A2], and [A5] as intuitively or physically obvious. In that case, we have a heat engine which ejects heat at the average rate Q_o into a cold reservoir at temperature θ_o, and does useful work at average rate W_m. By the first law of thermodynamics, it must accept heat from its hot reservoir at the average rate $Q_h = W_m + Q_o$. From [A1], [A2], and [A5]

$$Q_h = Q_i + Q_R + D_f + D_B. \quad [\mathbf{A11}]$$

In other words, the heat source for the engine must include not only the heat Q_i entering its inner boundary and the radioactive heat Q_R but also the heat produced by mechanical and electrical dissipation. Failure to recognize that these last two sources of heat are available to the heat engine leads to a fallacious proof that

$$W_m \leq Q_R + Q_i. \quad [\mathbf{A12}]$$

Of the heat Q_h, Q_i is available at temperature θ_i, and $Q_R + W_m$ is available at various temperatures in V, none larger than θ_M. Thus, Q_R injects entropy into V at the average rate

$$S_h \geq \theta_i^{-1}\, Q_i + \theta_M^{-1}\,(Q_R + W_m).$$

Entropy is ejected from V with average rate $S_o = \theta_o^{-1}\, Q_o$. Since the engine cannot destroy entropy, $S_o \geq S_h$, which yields [A6].

APPLICATIONS

In mantle convection, [A8] gives an upper bound on the average viscous dissipation Q_f in terms of the average total surface heat flow Q_o. If the present measured surface heat flow is an approximation to Q_o, we have a bound on the viscous dissipation in the mantle in terms of observable quantities. As an illustration, let us suppose that the mantle is a Newtonian fluid with shear viscosity η (variable with position). Let us estimate the deformation rate $\partial_i u_j$ as u/L where L is the depth to which mantle convection extends and u is a typical continental drift velocity (to avoid inflating u by a non-dissipative rigid rotation, we could take u to be the maximum plate velocity in a reference frame chosen to minimize the maximum plate velocity). Then **[A8]** becomes

$$\langle\eta\rangle \leq \left(\frac{\theta_M}{\theta_o} - 1\right)\frac{ah}{2u^2}\left(\frac{\kappa}{1-\kappa+\kappa^2/3}\right); \qquad \textbf{[A13]}$$

here a is the radius of the earth, $\kappa = L/a$, h is the average surface heat flow per cm^3 per second, and $\langle\eta\rangle$ is the average mantle viscosity, weighted by $\partial_i u_j(\partial_i u_j + \partial_j u_i)$. With $h = 50$ ergs/cm^{2}sec, $u = 10$ cm/yr, **[A12]** is

$$\langle\eta\rangle \leq \left(\frac{\theta_M}{\theta_o} - 1\right)\left(\frac{\kappa}{1-\kappa+\kappa^2/2}\right) 1.4 \times 10^{23} \text{ poise.} \qquad \textbf{[A14]}$$

Available estimates of η lie between 10^{19} and 10^{22} poise [McConnell (5)], so **[A14]** is somewhat too large to be interesting, especially since θ_M/θ_o may be 10 or larger.

In the core, probably the dissipation is almost entirely ohmic [Gans (6)] so **[A8]** gives an upper bound on the average value of $\mathbf{J}^2/\sigma$. If we take $|\mathbf{J}| \simeq |\mathbf{B}|/\mu a$ where a is the radius of the core, if we suppose that $\theta_M/\theta_o \lessapprox 2$ for the core, and if we assume that Q_o is less than one-fifth of the geothermal heat flow at the surface of the earth, then **[A8]** gives $\mathbf{B}^2/\sigma < 50$, with $\mathbf{B}$ in gauss and σ in mhos/m. Jain *et al.* (7) and Johnston *et al.* (8) estimate $\sigma = 7 \times 10^5$ mhos/m. Then $|\mathbf{B}| < 6000$ G. If a more stringent limit on Q_o were available, possibly **[A8]** would give a physically interesting bound on $|\mathbf{B}|$.

For both the mantle and the core, **[A9]** gives a bound on the dissipation in terms of the heat available to drive the motion. The present writer knows of no estimates of Q_R in the core which are sufficiently convincing to provide physically interesting bound on $|\mathbf{B}|$. The estimates for viscous dissipation in the mantle obtained from **[A9]** differ only slightly from the estimates already given by **[A8]**. Thus **[A9]** at present is of only philosophical interest.

A MACHINE WHICH ACHIEVES THE BOUND

That we have proved **[A9]** and shown that one proof of **[A12]** is false does not make **[A12]** false, and whether **[A12]** can be violated in real fluids seems to be an open question at present. We can, however, construct a very simple machine, based on Bullard's (9) homopolar disc dynamo, which achieves the bound **[A9]**. Thus, we show that inequality **[A9]** cannot be improved without invoking more details of fluid motion than merely the gross conservation laws of thermodynamics; and we also show that **[A12]** can be violated in a physically realizable system, if not in a convecting fluid. Our machine is a gross oversimplification of a convective dynamo, but its simplicity may make it as useful an elementary tool in thermodynamic dynamo theory as Bullard's dynamo has been in kinematic dynamo theory.

The machine is shown in Fig. 1. The original disc dynamo (9) consists of two rigid pieces of nonmagnetic metal: one is the disc with its attached axle, and the other is the ring of electrical conductor which encircles the axle and makes brushing electrical contact with the axle and the edge of the disc. We have modified the dynamo by breaking the ring at G.

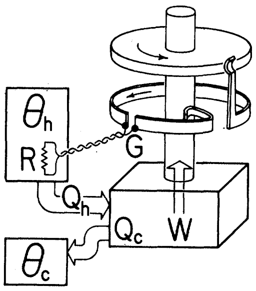

FIG. 1. Disc dynamo driven by a Carnot engine operating out of a hot reservoir which includes both radioactive heat and the ohmic heat from the dynamo current.

Without this break, when the disc rotates in the direction of its arrow, if there is any electrical current flowing in the ring, this current will generate a magnetic field which passes through the disc, so that the disc acts as a dynamo and drives the current around the ring. In our modified dynamo the current in the ring at G must pass out the wire and through resistor R before it can return and continue around the ring. The wires to R are wound together so as to produce no stray magnetic fields. Thus, Bullard's (9) analysis of the dynamo remains valid. Bullard postulated that an unspecified source of energy was available to turn the shaft either at a constant rate or with a constant torque. We will suppose that the source of energy is a Carnot engine operating between a hot reservoir at temperature θ_h and a cold reservoir at temperature θ_c. The hot reservoir will be supposed to contain radioactive heat sources producing heat at a rate Q_R. In addition, the resistor R will be placed in the hot reservoir. We will assume that the metals of which the dynamo disc and ring are made are nearly perfect electrical conductors. Then all of the resistance in the dynamo circuit is contained in the resistor R. If I is the current in the dynamo circuit, then RI^2 is the total ohmic heating rate in the dynamo circuit; hence, in the steady state, RI^2 is also W, the rate at which the Carnot engine must supply mechanical power to the shaft of the disc dynamo (and the rate at which that power is converted to magnetic energy).

The Carnot engine can extract heat from the hot reservoir at the rate $Q_h = Q_R + W$, and the rate at which the engine deposits heat in the cold reservoir is $Q_c = Q_h - W = Q_R$. In a Carnot engine, $Q_h/\theta_h = Q_c/\theta_c$, so $W = Q_R(\theta_h/\theta_c - 1)$. In short, the machine shown in Fig. 1 achieves the bound **[A9]**; and it violates **[A12]** if $\theta_h > 2\theta_c$. In any real convective motion, more hypotheses than simply the first and second laws of thermodynamics are available, so very likely **[A9]** can be improved. Real convective motions which achieve equality in **[A9]** will be rare if they exist at all; but at any rate they are not excluded by the gross laws of thermodynamics.

We can modify the machine of Fig. 1 to obtain a simple mechanical model for convection with viscous dissipation and no magnetic field. We retain the hot and cold reservoirs and the Carnot engine, and replace the disc dynamo by a frictionless rotating shaft which drives a frictionless gear train to a dash-pot or other frictional device entirely contained in the hot reservoir. Then we have a purely thermal and mechanical device, driven by radioactive heating, which achieves equality in **[A9]**.

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