Summary of Climate Change Fingerprinting Methods

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1. Classical Climate Change Fingerprinting

Goal: Detect and attribute observed climate changes (e.g., surface temperature trends) to specific external forcings such as greenhouse gases, aerosols, and solar variability.

Linear Model

$$\mathbf{y} = \sum_{i=1}^{n} \beta_i \mathbf{x}_i + \boldsymbol{\varepsilon}$$

- $\mathbf{y}$: observed climate change vector
- $\mathbf{x}_i$: model-simulated fingerprint for forcing i
- β_i: scaling factor for fingerprint i
- $\boldsymbol{\varepsilon}$: internal climate variability, $\mathcal{N}(0, \boldsymbol{\Sigma})$

Whitening the Noise

To ensure valid regression, the internal variability is whitened:

$$\mathbf{y}' = \boldsymbol{\Sigma}^{-1/2}\mathbf{y}, \quad \mathbf{x}_i' = \boldsymbol{\Sigma}^{-1/2}\mathbf{x}_i$$

This transforms the problem into:

$$\mathbf{y}' = \sum_i \beta_i \mathbf{x}_i' + \boldsymbol{\varepsilon}', \quad \boldsymbol{\varepsilon}' \sim \mathcal{N}(0, \mathbf{I})$$

Estimation

Using generalized least squares (GLS):

$$\boldsymbol{\beta} = \left(\mathbf{X}'^T \mathbf{X}'\right)^{-1} \mathbf{X}'^T \mathbf{y}', \quad \mathbf{X}' = [\mathbf{x}_1', \ldots, \mathbf{x}_n']$$

Hypothesis testing:

- $\beta_i > 0$: fingerprint is detected
- $\beta_i \approx 1$: model and observation are consistent

2. Covariance Fingerprinting

Covariance fingerprinting focuses on changes in the **covariance structure** of climate variables, rather than their mean.

Procedure

1. Estimate the covariance matrix from observations: $\mathbf{S}_{\text{obs}}$
2. Estimate the covariance matrix from model simulations: $\mathbf{S}_{\text{model}}$
3. Test whether $\mathbf{S}_{\text{obs}} \approx \mathbf{S}_{\text{model}}$ using statistical tests (e.g., Frobenius norm, likelihood ratio)

Frobenius Norm Test

A common approach is to compare the sample covariances using the squared Frobenius norm:

$$D = \|\mathbf{S}_{\text{obs}} - \mathbf{S}_{\text{model}}\|_F^2 = \sum_{i,j}(S_{\text{obs},ij} - S_{\text{model},ij})^2$$

This test statistic quantifies the total squared difference between the observed and model covariance matrices. To assess significance:

- Generate a null distribution of D from internal variability (e.g., control runs)
- Compare the observed D to this distribution
- A large D indicates that the observed variability pattern is inconsistent with the model under a given forcing scenario

Application

- Used when mean changes are weak or unclear
- Captures changes in variability patterns due to external forcings

3. Covariance Matrix Estimation

Accurate estimation of the internal variability covariance $\mathbf{\Sigma}$ is crucial.

3.1 Sample Covariance Matrix

$$\mathbf{S} = \frac{1}{n-1}\sum_{i=1}^{n}(\mathbf{z}_i - \bar{\mathbf{z}})(\mathbf{z}_i - \bar{\mathbf{z}})^T$$

Limitations:

- Unstable or singular when number of variables p is large relative to sample size n

3.2 Shrinkage Estimator

Form a convex combination:

$$\widehat{\mathbf{\Sigma}} = \lambda\mathbf{T} + (1-\lambda)\mathbf{S}, \quad \lambda \in [0,1]$$

- $\mathbf{T}$: a stable target matrix (e.g., scaled identity, diagonal of $\mathbf{S}$)
- λ: shrinkage intensity

Benefits:

- Ensures positive definiteness and invertibility
- Reduces estimation error in high dimensions

3.3 Ledoit-Wolf Optimal Shrinkage

The optimal λ^* minimizes expected squared error:

$$\lambda^* = \frac{\mathbb{E}\left[\|\mathbf{S} - \mathbf{\Sigma}\|_F^2\right]}{\mathbb{E}\left[\|\mathbf{T} - \mathbf{\Sigma}\|_F^2 + \|\mathbf{S} - \mathbf{T}\|_F^2\right]}$$

Estimated using plug-in values directly from data.

3.4 Other Estimators

- **Tapering / Banding**: Forces covariances to decay with distance
- **Factor Models**: Assumes a low-rank structure plus noise
- **Sparse Precision Estimators (e.g., graphical lasso)**: Estimate sparse inverse covariance (conditional independence)

Conclusion

Climate fingerprinting is a robust statistical approach to attribute climate changes to specific forcings. It relies on:

- Linear regression of model-simulated fingerprints onto observations
- Whitening using estimated internal variability covariance
- Advanced techniques like covariance fingerprinting
- Regularized covariance estimation, especially shrinkage estimators, to ensure stability in high-dimensional settings