situations, and to suggest diagnostics that might be used to analyze both observations and numerical solutions of the fully nonlinear problem. We will almost exclusively concern ourselves with a *zonal* mean, for this is the simplest and often most useful case because of the presence of periodic boundary conditions. (With care some of our results can be extended to the case of a temporal mean.) We will also be mainly concerned with quasi-geostrophic dynamics on a β-plane.

7.1 QUASI-GEOSTROPHIC PRELIMINARIES

To fix our dynamical system and our notation, we write down the quasi-geostrophic potential vorticity equation

$$\frac{\partial q}{\partial t} + J(\psi, q) = D \tag{7.1}$$

where D represents any nonconservative terms and the potential vorticity in a Boussinesq system is

$$q = \beta y + \zeta + \frac{\partial}{\partial z}\left(\frac{f_0}{N^2} b\right), \tag{7.2}$$

where ζ is the relative vorticity and b is the buoyancy perturbation from the background state characterized by N^2, where $N^2 = \mathrm{d}\widetilde{b}/\mathrm{d}z$ where $\widetilde{b}$ is a reference profile. [In an ideal gas we have very similar equations, but with $q = \beta y + \zeta + (f_0/\rho_s)\partial/\partial z\, \left(\rho_s b/N^2\right)$.] We will refer to lines of constant b as isentropes (and also sometimes loosely refer to b as the temperature). In terms of streamfunction, the variables are

$$\zeta = \nabla^2\psi, \qquad b = f_0\frac{\partial \psi}{\partial z}, \qquad q = \beta y + \left[\nabla^2 + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial}{\partial z}\right)\right]\psi. \tag{7.3}$$

where $\nabla^2 \equiv (\partial_x^2 + \partial_y^2)$. The potential vorticity equation holds in the fluid interior; the boundary conditions on (7.3) are provided by the thermodynamic equation

$$\frac{\partial b}{\partial t} + J(\psi, b) + wN^2 = J, \tag{7.4}$$

where J represents heating terms. The vertical velocity at the boundary, w, is zero in the absence of topography and Ekman friction, and if J is also zero the boundary condition is just:

$$\frac{\partial b}{\partial t} + J(\psi, b) = 0. \tag{7.5}$$

Equations (7.1) and (7.5) are the evolution equations for the sytem and if both D and J are zero they conserve both the total energy and the total enstrophy:

$$\begin{aligned} \frac{\mathrm{d}\widehat{E}}{\mathrm{d}t} &= 0, \qquad \widehat{E} = \int_V (\nabla\psi)^2 + \frac{f_0^2}{N^2}\left(\frac{\partial\psi}{\partial z}\right)^2 \mathrm{d}V, \\ \frac{\mathrm{d}\widehat{Z}}{\mathrm{d}t} &= 0, \qquad \widehat{Z} = \int_V q^2\,\mathrm{d}V. \end{aligned} \tag{7.6}$$

where V is a volume bounded by surfaces at which the normal velocity is zero, or that has periodic boundary conditions. The enstrophy is also conserved layerwise — that is, the horizontal integral of q^2 is conserved at every level.

In the Eady problem there is no interior gradient of basic-state potential vorticity and all the terms in (7.10) are zero, but the perturbation grows at the boundary. If the waves are steady and adiabatic then, analogously to (7.11),

$$\overline{v'b'} = 0. \tag{7.14}$$

The boundary conditions and fluxes may be absorbed into the interior definition of potential vorticity and its fluxes by way of Bretherton's boundary layer construction, described in chapter 5. This can provide notational and conceptual advantages over dealing with boundary fluxes explicitly, but if an actual calculation is to be performed there is often little to be gained, for the boundary terms have to be dealt with one way or the other.

7.2 THE ELIASSEN-PALM FLUX

In terms of the flux of vorticty and buoyancy the eddy flux of potential vorticity is

$$v'q' = v'\zeta' + f_0 v' \frac{\partial}{\partial z}\left(\frac{b'}{N^2}\right) \tag{7.15}$$

The second term on the right-hand side can be written

$$\begin{aligned} f_0 v' \frac{\partial}{\partial z}\left(\frac{b'}{N^2}\right) &= f_0 \frac{\partial}{\partial z}\left(\frac{v'b'}{N^2}\right) - f_0 \frac{\partial v'}{\partial z}\frac{b'}{N^2} \\ &= f_0 \frac{\partial}{\partial z}\left(\frac{v'b'}{N^2}\right) - f_0 \frac{\partial}{\partial x}\left(\frac{\partial \psi'}{\partial z}\right)\frac{b'}{N^2} \\ &= f_0 \frac{\partial}{\partial z}\left(\frac{v'b'}{N^2}\right) - \frac{f_0^2}{N^2}\frac{\partial}{\partial x}\left(\frac{1}{2}\frac{\partial \psi'}{\partial z}\right)^2 \end{aligned} \tag{7.16}$$

using $b' = f_0 \partial\psi'/\partial z$.

Similarly, the flux of relative vorticity can be written

$$v'\zeta' = -\frac{\partial}{\partial y}u'v' + \frac{1}{2}\frac{\partial}{\partial x}(v'^2 - u'^2) \tag{7.17}$$

Using (7.16) and (7.17), (7.15) becomes

$$\boxed{v'q' = -\frac{\partial}{\partial y}(u'v') + \frac{\partial}{\partial z}\left(\frac{f_0}{N^2}v'b'\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}(v'^2 - u'^2) - \frac{b'^2}{N^2}\right)} \tag{7.18}$$

Thus the potential vorticity flux, in the quasi-geostrophic approximation, can be written as the divergence of a vector: $v'q' = \nabla\cdot\boldsymbol{\mathcal{E}}$ where

$$\boldsymbol{\mathcal{E}} \equiv \left(\frac{1}{2}(v'^2 - u'^2) - \frac{b'^2}{N^2}\right)\mathbf{i} - (u'v')\,\mathbf{j} + \left(\frac{f_0}{N^2}v'b'\right)\mathbf{k}. \tag{7.19}$$

A particularly useful form of this arises after zonally averaging, after which (7.18) becomes

$$\boxed{\overline{v'q'} = -\frac{\partial}{\partial y}\overline{u'v'} + \frac{\partial}{\partial z}\left(\frac{f_0}{N^2}\overline{v'b'}\right)}. \tag{7.20}$$

The vector defined by

$$\boldsymbol{\mathcal{F}} \equiv -\overline{u'v'}\,\mathbf{j} + \frac{f_0}{N^2}\overline{v'b'}\,\mathbf{k} \tag{7.21}$$

is called the *Eliassen-Palm flux*,[1] and its divergence, given by (7.20), gives the polewards flux of potential vorticity:

$$\overline{v'q'} = \nabla_x \cdot \boldsymbol{\mathcal{F}}, \tag{7.22}$$

where $\nabla_x \cdot \equiv (\partial/\partial y, \partial/\partial z)$ is the divergence in the meridional plane. Unless the meaning is unclear, the subscript x on the meridional divergence will be dropped.

For reference, in spherical coordinates and for an ideal gas the EP flux is (see also the appendix to chapter 12 on page 559):

$$\boldsymbol{\mathcal{F}} = -\cos\vartheta\,\overline{u'v'}\,\mathbf{j} + \cos\vartheta\,\frac{f_0}{\partial_z\overline{\theta}}\overline{v'\theta'}\,\mathbf{k}, \tag{7.23}$$

and multiplying by ρ_R and taking the divergence gives

$$\nabla\cdot\rho_R\boldsymbol{\mathcal{F}} = \frac{-\rho_R}{a\cos\vartheta}\frac{\partial}{\partial\vartheta}(\overline{u'v'}\cos^2\vartheta) + \frac{\partial}{\partial z}\left(\frac{\rho_R f_0}{\partial_z\overline{\theta}}\overline{v'\theta'\cos\vartheta}\right) = \rho_R\cos\vartheta\,\overline{v'q'}. \tag{7.24}$$

where ρ_R is a reference profile of density,

7.2.1 The Eliassen-Palm relation

On dividing by $\partial\overline{q}/\partial y$ and using (7.22), the enstrophy equation (7.10) becomes

$$\boxed{\frac{\partial\mathcal{A}}{\partial t} + \nabla\cdot\boldsymbol{\mathcal{F}} = \mathcal{D}}, \tag{7.25a}$$

where

$$\mathcal{A} = \frac{\overline{q'^2}}{2\partial\overline{q}/\partial y}, \qquad \mathcal{D} = \frac{\overline{D'q'}}{\partial\overline{q}/\partial y} \tag{7.25b}$$

Eq. (7.25a) is known as the *Eliassen-Palm relation,* and it is a conservation law for the the *wave activity density* $\mathcal{A}$, for if we integrate this expression over a meridional area A bounded by walls where the eddy activity vanishes, and if $\mathcal{D} = 0$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_A \mathcal{A}\,\mathrm{d}A = 0. \tag{7.26}$$

In general, a wave activity is a quantity that is quadratic in the amplitude of the perturbation and that is conserved in the absence of forcing and dissipation. More specifically, The second and third terms of this are the wave activities of each mode, and these are constants (to see this, consider the case when the disturbance is just a single mode). Now, because $\mathrm{d}P/\mathrm{d}t = 0$ the first term must vanish if $c_n \neq c_m$, implying the modes are orthogonal and in particular

$$\mathrm{Re} \int \frac{1}{\overline{q}_y} q_n q_m^* \,\mathrm{d}y = 0, \tag{7.49}$$

for $n \neq m$. Readers who would prefer a more direct derivation of the orthogonality condition directly from the eigenvalue equation (7.41) should see problem 7.6. Orthogonality is a useful result, for it means that the wave activity is a proper measure of the amplitude of a given mode unlike, for example, energy. The conservation of wave activity will lead to a more general derivation of the necessary conditions for stability, in section 7.6.

7.3 THE TRANSFORMED EULERIAN MEAN

The so-called *transformed Eulerian mean,* or TEM, provides a useful framework for discussing eddy effects under a wide range of conditions.[4] It is useful because, as we shall see, it is equivalent to a very natural form of averaging the equations that serves to eliminate eddy fluxes in the thermodynamic equation and collect them together, in a simple form, in the momentum equation, highlighting the role of potential vorticity fluxes. It also provides a natural separation between diabatic and adiabatic effects or between advective and diffusive fluxes and, in the case in which the flow is adiabatic, a nice simplification of the equations. In later chapters we will use the TEM to better understand the mid-latitude troposphere and the dynamics of the Antarctic Circumpolar current, and as a framework for the parameterization of eddy fluxes. Of course, there being no free lunch, the TEM brings with it its own difficulties, and in particular the implementation of boundary conditions can cause difficulties, especially in the actual numerical integration of the TEM equations.

7.3.1 Quasi-geostrophic form

Recall the conventional zonally averaged Eulerian mean equations for the zonally averaged zonal velocity and the buoyancy on the beta-plane:

$$\frac{\partial \overline{u}}{\partial t} - (f + \overline{\zeta})\overline{v} = -\frac{\partial}{\partial y}\overline{u'v'} - \frac{\partial}{\partial z}\overline{u'w'} + \overline{F} \tag{7.50a}$$

$$\frac{\partial \overline{b}}{\partial t} + w\frac{\partial \overline{b}}{\partial z} = -\frac{\partial}{\partial y}\overline{v'b'} - \frac{\partial}{\partial z}\overline{w'b'} + \overline{J} \tag{7.50b}$$

where $\overline{F}$ and $\overline{J}$ represent frictional and heating terms. Note that the only contribution to $\overline{v}$ is from the ageostrophic meridional velocity. Using quasi-geostrophic scaling we neglect the vertical eddy flux divergences and all ageostrophic velocities except when multiplied by f_0 or N^2. The equations then become

$$\frac{\partial \overline{u}}{\partial t} = f_0 \overline{v} - \frac{\partial}{\partial y}\overline{u'v'} + \overline{F}, \tag{7.51a}$$

$$\frac{\partial \overline{b}}{\partial t} = -N^2 w - \frac{\partial}{\partial y}\overline{v'b'} + \overline{J}, \tag{7.51b}$$

where $\overline{b}$ is in thermal wind balance with $\overline{u}$, $f_0 \partial\overline{u}/\partial z = -\partial\overline{b}/\partial y$ (in the Boussinesq approximation). One less-than-ideal aspect of these equations is that in the extratropics the dominant balance is usually between the first two terms on the right-hand sides of each equation, even in time-dependent cases. Thus, the Coriolis force closely balances the divergence of the eddy momentum fluxes, and the advection of the mean stratification ($N^2 w$, or 'adiabatic cooling') often balances the convergence of eddy heat flux, with heating being a small residual. This may lead to an underestimation of the importance of diabatic heating, for this is ultimately responsible for the mean meridional circulation. Thus, in the thermodynamic equation we might seek to combine the terms $N^2 w$ and the eddy flux into a single total (or 'residual') heat transport term that in a steady state is balanced by the diabatic term $\overline{J}$. The TEM provides this reformulation, and in doing so the eddy terms in the momentum equation also take a different form.

To begin, note that because $\overline{v}$ and $\overline{w}$ are related by mass conservation, we can define a mean meridional streamfunction ψ_m such that

$$(\overline{v}, \overline{w}) = \left(-\frac{\partial \psi_m}{\partial z}, \frac{\partial \psi_m}{\partial y}\right). \tag{7.52}$$

Then, if we define a 'residual' streamfunction by

$$\psi^* \equiv \psi_m + \frac{1}{N^2}\overline{v'b'} \tag{7.53}$$

the components of the *residual mean meridional circulation* are given by

$$(\overline{v}^*, \overline{w}^*) = \left(-\frac{\partial \psi^*}{\partial z}, \frac{\partial \psi^*}{\partial y}\right), \tag{7.54}$$

and

$$\overline{v}^* = \overline{v} - \frac{\partial}{\partial z}\left(\frac{1}{N^2}\overline{v'b'}\right), \qquad \overline{w}^* = \overline{w} + \frac{\partial}{\partial y}\left(\frac{1}{N^2}\overline{v'b'}\right). \tag{7.55}$$

Note that by construction, the residual overturning circulation satisfies

$$\frac{\partial \overline{v}^*}{\partial y} + \frac{\partial \overline{w}^*}{\partial z} = 0. \tag{7.56}$$

Substituting (7.55) into (7.51a) and (7.51b) the zonal momentum and buoyancy equations then take the simple forms

$$\boxed{\begin{aligned} \frac{\partial \overline{u}}{\partial t} &= f_0 \overline{v}^* + \overline{v'q'} + \overline{F} \\ \frac{\partial \overline{b}}{\partial t} &= -N^2 \overline{w}^* + \overline{J} \end{aligned}}, \tag{7.57}$$

Fig. **??**. We can see wavetrains emanating from both the Rockies and the Himalayas, but distinct polewards and equatorwards wavetrains are hard to discern.

13.3 * BAROCLINIC ROSSBY WAVES AND THEIR VERTICAL PROPAGATION

13.3.1 Forced and stationary waves in the atmosphere

Now consider the vertical propagation of Rossby waves in a stratified atmosphere. We will continue to use the stratified quasi-geostrophic equations, but we now allow the model to be compressible and semi-infinite, extending from $z = 0$ to ∞. The potential vorticity equation again describes motion in the fluid interior, with a surface boundary condition of vertical velocity being determined by the thermodynamic equation, and the upper boundary condition being determined by a radiation condition. Guided by the barotropic problem, we will allow for the possibility of Ekman friction and topography at the surface, but otherwise the flow is presumed inviscid and adiabatic. We will proceed using standard height coordinates.[11] The potential vorticity equation is

$$\frac{\partial q}{\partial t} + J(\psi, q) = 0, \qquad q = \nabla^2\psi + f + \frac{f_0^2}{\rho_R}\frac{\partial}{\partial z}\left(\frac{\rho_R}{N^2}\frac{\partial \psi}{\partial z}\right), \tag{13.45}$$

where we take $\rho_R = \rho_0 e^{-z/H}$ where H is a specified density scale height, typically $RT(0)/g$. We linearize this equation about a zonal wind that depends only on z; that is, we let

$$\psi = -\overline{u}(z)y + \psi', \tag{13.46}$$

and obtain

$$\frac{\partial q'}{\partial t} + \overline{u}\frac{\partial q'}{\partial x} + v'\frac{\partial \overline{q}}{\partial y} = 0, \qquad \frac{\partial \overline{q}}{\partial y} = \beta - \frac{f_0^2}{\rho_R}\frac{\partial}{\partial z}\left(\frac{\rho_R}{N^2}\frac{\partial \overline{u}}{\partial z}\right). \tag{13.47}$$

or equivalently, in terms of streamfunction,

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)\left[\nabla^2\psi' + \frac{f_0^2}{\rho_R}\frac{\partial}{\partial z}\left(\frac{\rho_R}{N^2}\frac{\partial \psi'}{\partial z}\right)\right] + \frac{\partial \psi'}{\partial x}\left[\beta - \frac{f_0^2}{\rho_R}\frac{\partial}{\partial z}\left(\frac{\rho_R}{N^2}\frac{\partial \overline{u}}{\partial z}\right)\right] = 0. \tag{13.48}$$

The lower boundary is obtained using the thermodynamic equation,

$$\frac{\partial \psi}{\partial t} + J\left(\psi, \frac{\partial \psi}{\partial z}\right) + \frac{N^2}{f_0}w = 0, \tag{13.49}$$

along with an equation for the vertical velocity, w, at the lower boundary. This is

$$w = \boldsymbol{u}\cdot\nabla h_b + r\zeta \tag{13.50}$$

where two terms represent the kinematic contribution to vertical velocity due to flow over topography and the contribution from Ekman pumping, with r a constant, and the effects are taken to be additive. Linearizing the thermodynamic equation about the zonal flow and using (13.50) gives

$$\frac{\partial}{\partial t}\left(\frac{\partial \psi'}{\partial z}\right) + \overline{u}\frac{\partial}{\partial x}\frac{\partial \psi'}{\partial z} - v'\frac{\partial \overline{u}}{\partial z} = -\frac{N^2}{f_0}\left(\overline{u}\frac{\partial h_b}{\partial x} + r\nabla^2\psi'\right), \qquad \text{at } z = 0. \quad (13.51)$$

Solution

We look for solutions of (13.47) and (13.51) in the form

$$\psi' = \text{Re}\,\widetilde{\psi}(z)\sin ly e^{ik(x-ct)}, \quad (13.52)$$

Solutions must then satisfy

$$\left[\frac{f_0^2}{\rho_R}\frac{\partial}{\partial z}\left(\frac{\rho_R}{N^2}\frac{\partial \widetilde{\psi}}{\partial z}\right)\right] = \widetilde{\psi}\left(K^2 - \frac{\partial \overline{q}/\partial y}{\overline{u}-c}\right) \quad (13.53)$$

in the interior, and the boundary condition

$$(\overline{u}-c)\frac{\partial \widetilde{\psi}}{\partial z} - \widetilde{\psi}\frac{\partial \overline{u}}{\partial z} + \frac{i\alpha N^2}{k f_0}\Psi = -\frac{N^2\overline{u}h_b}{f_0}, \qquad \text{at } z = 0, \quad (13.54)$$

as well as a radiation condition at plus infinity (and we must have that $\rho_0\Psi^2$ be finite). Let us simplify by considering the case of constant $\overline{u}$ and N^2 and setting $r = 0$. We then let

$$\Phi(z) = \widetilde{\psi}(z)\left(\frac{\rho_R}{\rho_R(0)}\right)^{1/2} = \widetilde{\psi}(z)e^{-z/2H} \quad (13.55)$$

and obtain the interior equation

$$\frac{d^2\Phi}{dz^2} + m^2\Phi = 0, \qquad \text{where} \quad m^2 = \frac{N^2}{f_0^2}\left(\frac{\beta}{\overline{u}-c} - K^2 - \gamma^2\right), \quad (13.56a,b)$$

and where $\gamma^2 = f_0^2/(4N^2H^2) = 1/(2L_d)^2$ where L_d is the deformation radius. The surface boundary condition is

$$(\overline{u}-c)\left(\frac{d\Phi}{dz} + \frac{\Phi}{2H}\right) = -\frac{N^2\overline{u}h_b}{f_0} \qquad \text{at } z = 0. \quad (13.57)$$

Oscillating Waves

From (13.56b) we obtain the dispersion relation for Rossby waves, namely

$$\omega = \overline{u}k - \frac{\beta k}{K^2 + \gamma^2 + m^2 f_0^2/N^2}. \quad (13.58)$$

The three components of the group velocity for these waves are:

$$c_g^x = \overline{u} - \frac{\beta[k^2 - (l^2 + m^2 f_0^2/N^2 + \gamma^2)]}{(K^2 + m^2 f_0^2/N^2 + \gamma^2)^2}, \quad (13.59a)$$

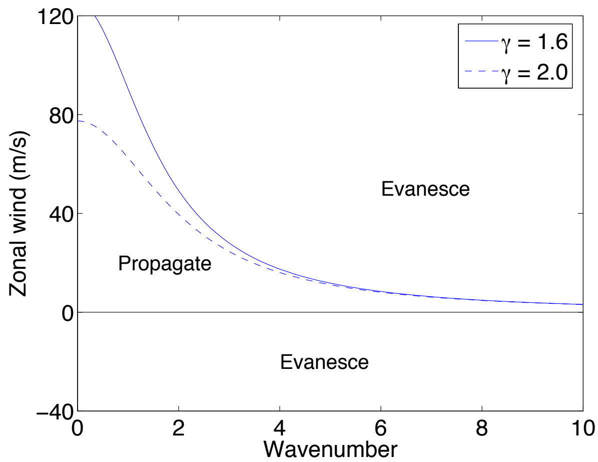

Figure 13.7 The boundary between propagating waves and evanescent waves as a function of zonal wind and wavenumber, using (13.61), for a couple of values of γ. With $N = 2 \times 10^{-2}\,\mathrm{s}^{-1}$, $\gamma = 1.6$ ($\gamma = 2$) corresponds to a scale height of 7.0 km (5.5 km) and a deformation radius NH/f of 1400 km (1100 km).

$$c_g^y = \frac{2\beta k l}{(K^2 + m^2 f_0^2/N^2 + \gamma^2)^2}, \tag{13.59b}$$

$$c_g^z = \frac{2\beta k m f_0^2/N^2}{(K^2 + m^2 f_0^2/N^2 + \gamma^2)^2}. \tag{13.59c}$$

The propagation in the horizontal is analogous to the propagation in a shallow water model, although note that higher baroclinic modes (bigger m) will have a more westward group velocity. The vertical group velocity is proportional to m, and therefore for waves that are excited at the surface we must choose m to be positive, for positive k.

Stationary waves

Stationary waves have $\omega = ck = 0$. In this case (13.56) has a solution $\Phi = \Phi_0 \exp(imz)$ provided m^2 is positive where

$$m = \pm \frac{N}{f_0}\left(\frac{\beta}{\overline{u}} - K^2 - \gamma^2\right)^{1/2}. \tag{13.60}$$

Furthermore, m itself must be positive. As we noted, for non-steady waves we must choose the sign of m to ensure that the group velocity, and hence the wave activity, is directed away from the energy source. This must still hold as $m \to 0$, and therefore the positive sign in (13.60) corresponds to the physically realizable solution.[12]

The condition $m^2 > 0$ holds if

$$\boxed{0 < \overline{u} < \frac{\beta}{K^2 + \gamma^2}}. \tag{13.61}$$

and this is illustrated in Fig. 13.7. Stationary, vertically oscillatory modes can exist only for zonal flows that are eastward and that are less than the critical velocity $U_c = \beta/(K^2 + \gamma^2)$. To interpret this condition, note that in a resting medium the Rossby wave frequency has a minimum value (and maximum absolute value) when $m = 0$ of

$$\omega = -\frac{\beta k}{K^2 + \gamma^2}. \tag{13.62}$$

Note too that in a frame moving with the speed $\overline{u}$ our Rossby waves (stationary in the Earth's frame) have the frequency $-\overline{u}k$, and this is the forcing frequency arising from the now-moving bottom topography. Thus, (13.61) is equivalent to saying that for oscillatory waves to exist *the forcing frequency must lie within the frequency range of vertically propagating Rossby waves.*

For westward flow, or for sufficiently strong eastward flow, the waves decay exponentially as $\Phi = \Phi_0 \exp(-\alpha z)$ where

$$\alpha = \frac{N}{f_0}\left(K^2 + \gamma^2 - \frac{\beta}{\overline{u}}\right)^{1/2}. \tag{13.63}$$

Note that the critical velocity U_c is a function of wavenumber, and that it increases with horizontal wavelength. Thus, for a given eastward flow long waves may penetrate vertically when short waves are trapped.[13] One important consequence of this is that the stratospheric motion is typically of longer wavelength than that of the troposphere, because waves tend to be excited first in the troposphere (by baroclinic instability and by flow over topography, among other things), but the shorter waves are trapped and only the longer ones reach the stratosphere. In the summer, the stratospheric winds are often westward and all waves are trapped in the troposphere; the eastward stratospheric winds that favour vertical penetration occur in the other thee seasons, although very strong eastward winds can suppress penetration in mid-winter.

Finally, the surface boundary condition, (13.57) gives

$$\Phi_0 = \frac{N^2 h_b / f_0}{(\alpha, im) - (2H)^{-1}} \tag{13.64}$$

where $(\alpha, -im)$ refers to the (trapped, oscillatory) case. Equation (13.64) indicates that resonance is possible when $\alpha = 1/(2H)$, and from (13.63) this occurs when $K^2 = \beta/\overline{u}$, that is when barotropic Rossby waves are stationary. This wave resonates because the wave is a solution of the unforced (and inviscid) equations and, because $\Psi = \Phi \exp[z/(2H)]$, it has uniform vertical structure. If $K > K_S$ then $\alpha > 1/(2H)$ and the forced wave (i.e., the amplitude of Ψ) decays with height with no phase variation. If $\alpha < 1/(2H)$ then Ψ increases with height, and this occurs when $(K_s^2 - \gamma^2)^{1/2} < K < K_S$. If $(K_s^2 - \gamma^2)^{1/2} > K$ then the amplitude of Ψ is again independent of height; their vertical structure is oscillatory, like $\exp(imz)$. The complete solutions are collected for convenience in the box on page 580.

13.3.2 Properties of the solution

The various dynamical fields associated with the solution can all be easily constructed from (T.1), and a few simple properties of the solution are worth noting explicitly. In some cases the explicit calculation is left as a problem to the reader — see problems 13.6 and 13.7.

Stationary, Topographically Forced Solutions

Collecting the results in section 13.3.1, the stationary solutions of (13.47) and (13.51) are:

$$\psi'(x, y, z) = \mathrm{Re}\, e^{imz} e^{z/2H} e^{ikx} \sin ly \frac{f_0 h_b \left[im - (2H)^{-1}\right]}{K_s^2 - K^2}, \qquad m^2 > 0 \qquad \text{(T.1a)}$$

$$\psi'(x, y, z) = \mathrm{Re}\, e^{[(2H)^{-1} - \alpha]z} e^{ikx} \sin ly \frac{N^2 h_b}{f_0 \left[\alpha - (2H)^{-1}\right]}, \qquad m^2 < 0 \qquad \text{(T.1b)}$$

where

$$m = +\frac{N}{f_0}\left(\frac{\beta}{\overline{u}} - K^2 - \gamma^2\right)^{1/2}, \qquad \text{(T.2)}$$

and

$$\alpha = +\frac{N}{f_0}\left(K^2 + \gamma^2 - \frac{\beta}{\overline{u}}\right)^{1/2}. \qquad \text{(T.3)}$$

and $\gamma = f_0/(2NH)$. If $m^2 > 0$ the solutions are propagating, or radiating, waves in the vertical. If $m^2 < 0$ the energy of the solution, $|\rho_R \psi'^2|$, is vertically evanescent. The condition $m^2 > 0$ is equivalent to

$$0 < \overline{u} < \frac{\beta}{K^2 + (f_0/2NH)^2}, \qquad \text{(T.4)}$$

so that vertical penetration is favoured when the winds are weakly eastward, and the range of $\overline{u}$ values that allows this is larger for longer waves.

Amplitudes and phases: The decaying solutions have no vertical phase variations (they are 'equivalent barotropic') and the streamfunction is exactly in phase or out of phase with the topography according as $K > K_s$ and $\alpha > (2H)^{-1}$, or $K < K_s$ and $\alpha < (2H)^{-1}$. In the latter case case the amplitude of the streamfunction actually increases with height, but the energy, proportional to $\rho_R|\psi'^2|$ falls. The oscillatory solutions have constant energy with height but a shifting phase. The phase of the streamfunction at the surface may be in or out of phase with the topography, depending on m, but the potential temperature, $\partial\psi/\partial z$ is always out of phase with the topography. That is, positive values of h_b are associated with cool fluid parcels.

Vertical energy propagation: As noted, the energy propagates upward for the oscillatory waves. This may be verified by calculating $\overline{p'w'}$ where p' is the pressure perturbation, proportional to ψ' and w' is the vertical velocity perturbation. To this end, linearize the thermodynamic equation (13.49) to give

$$\frac{\partial \psi'}{\partial t} + \overline{u}\frac{\partial}{\partial x}\frac{\partial \psi'}{\partial z} - \frac{\partial \overline{u}}{\partial z}\frac{\partial \psi'}{\partial x} + \frac{N^2}{f_0}w' = 0. \tag{13.65}$$

Then, multiplying by ψ' and integrating by parts gives a balance between the second and fourth terms,

$$N^2\overline{\psi' w'} = \overline{u}\,\overline{b' v'}, \tag{13.66}$$

where $b' = f_0 \partial\psi'/\partial z$ and $v' = \partial\psi'/\partial x$. Thus, the upwards transfer of energy is proportional to the poleward heat flux.

Meridional heat transport: The meridional heat transport associated with a wave is

$$\rho_R \overline{v' b'} = \rho_R f_0 \overline{\frac{\partial \psi'}{\partial x}\frac{\partial \psi'}{\partial z}} \tag{13.67}$$

For an oscillatory wave this can readily be shown to be positive. In particular, it is proportional to $km/(K_s^2 - K^2)$, and this is positive because $km > 0$ is the condition that energy is directed upwards, and $K_s^2 > K^2$ for oscillatory solutions. The meridional transport associated with a trapped solution is identically zero.

Form Drag: If the waves propagate energy upward, there must be a surface interaction to supply that energy. There is a force due to *form drag* associated with this interaction, given by

$$\text{Form drag} = \overline{p'\frac{\partial h_b}{\partial x}} \tag{13.68}$$

(see chapter 3). In the trapped case, the streamfunction is either exactly in or out of phase with the topography, so this interaction is zero. In the oscillatory case

$$\overline{\psi'\frac{\partial h_b}{\partial x}} = \frac{f_0 h_b^2 km}{4(K_s^2 - K^2)} \tag{13.69}$$

where the factor of 4 arises from the x and y averages of the squares of sines and cosines. The rate of doing work is $\overline{u}$ times this.

13.4 * EFFECTS OF THERMAL FORCING

How does thermal forcing influence the stationary eddies? To give an accurate answer for the real atmosphere is a little more difficult than for the orographic case where the forcing can be included reasonably accurately in a quasi-geostrophic model with a term $\overline{u} \cdot \nabla h_b$ at the lower boundary. Anomalous (i.e., variations from a zonal or temporal mean) thermodynamic forcing typically also arises initially at the lower boundary through, for example, variations in the surface temperature. However, such anomalies may be felt throughout the lower troposphere on a relatively short time-scale by way of such non-geostrophic phenomena as convection, so that the effective thermodynamic source that should be applied in a quasi-geostrophic calculation has a finite vertical extent. However, an accurate parameterization of this may depend on the structure of the atmospheric boundary layer and this cannot always be represented in a simple way.[14] Because of such uncertainties our treatment concentrates on the fundamental and qualitative aspects of thermal forcing.

The quasi-geostrophic potential vorticity equation, linearized around a uniform zonal flow, is [c.f., (13.48)]

$$\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)\left[\nabla^2\psi'+\frac{f_0^2}{\rho_R}\frac{\partial}{\partial z}\left(\frac{\rho_R}{N^2}\frac{\partial\psi'}{\partial z}\right)\right] + \frac{\partial\psi'}{\partial x}\left[\beta-\frac{f_0^2}{\rho_R}\frac{\partial}{\partial z}\left(\frac{\rho_R}{N^2}\frac{\partial\overline{u}}{\partial z}\right)\right]=\frac{f_0}{N^2}\frac{\partial Q}{\partial z}\equiv T \tag{13.70}$$

where Q is the source term in the (linear) thermodynamic equation,

$$\frac{\partial}{\partial t}\left(\frac{\partial\psi'}{\partial z}\right)+\overline{u}\frac{\partial}{\partial x}\frac{\partial\psi'}{\partial z}-v'\frac{\partial\overline{u}}{\partial z}+\frac{N^2}{f_0}w'=\frac{Q}{f_0} \tag{13.71}$$

A particular solution to (13.70) may be constructed if $\overline{u}$ and N^2 are constant, and if Q has a simple vertical structure. If we again write $\psi'=\mathrm{Re}\,\widetilde{\psi}(z)\sin ly\exp(ikx)$ and let $\Phi(z)=\widetilde{\psi}(z)\exp(-z/2H)$ we obtain

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}z^2}+m^2\Phi=\frac{T}{ik\overline{u}}e^{-z/2H},\qquad \text{where}\quad m^2=\frac{N^2}{f_0^2}\left(\frac{\beta}{\overline{u}}-K^2-\gamma^2\right). \tag{13.72}$$

If we let $T=T_0\exp(-z/H_Q)$, so that the heating decays exponentially away from the earth's surface, then the particular solution to the stationary problem is found to be

$$\widetilde{\psi}=\mathrm{Re}\,\frac{i\hat{T}e^{-z/H_Q}}{k\overline{u}\left[(N/f_0)^2(K_s-K^2)+H_Q^{-2}(1+H_Q/H)\right]} \tag{13.73}$$

where $\hat{T}$ is proportional to T. This solution does not satisfy the boundary condition at $z=0$, which in the absence of topography and friction is

$$\overline{u}\frac{\partial}{\partial x}\frac{\partial\psi'}{\partial z}-v'\frac{\partial\overline{u}}{\partial z}=\frac{Q(0)}{f_0}. \tag{13.74}$$

A homogeneous solution must therefore be added, and just as in the topographic case this leads to a vertically radiating or a surface trapped response, depending on the sign of m^2. One way to calculate the homogeneous solution is to first use the linearized thermodynamic equation (13.71), or the linearized vorticity equation (13.76), to calculate the vertical velocity at the surface implied by (13.73), $w_p(0)$ say. We then notice that the homogeneous solution is effectively forced by an equivalent topography given by $h_e=-w_p(0)/(ik\overline{u(0)})$, and so proceed as in the topographic case. The complete solution is rather hard to interpret, and is in any case available only in special cases, so it is useful to take a more qualitative approach.

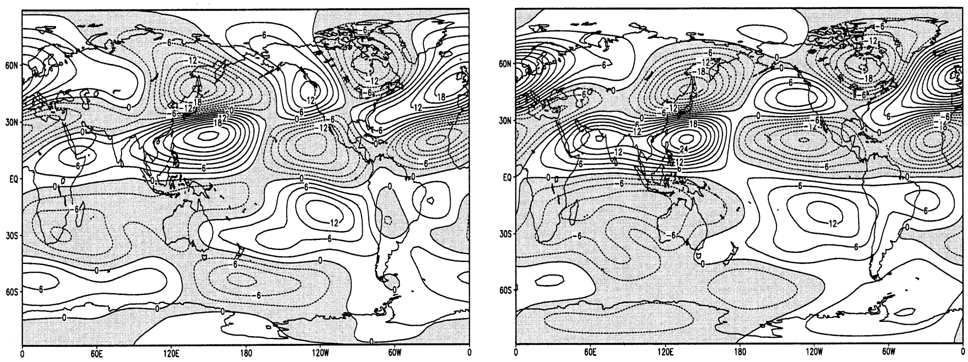

Fig. 13.10 Left: the observed stationary (i.e., time averaged) streamfunction at 300 mb (about 7 km altitude) in Northern Hemisphere winter. Right: the steady, linear response to forcing by orography, heat sources and transient eddy flux convergences, calculated using a linear model with the observed height-varying zonally averaged zonal wind. Note the generally good agreement, and also the much weaker zonal asymmetries in the southern hemisphere.[17]

13.5 STRATOSPHERIC DYNAMICS

In our final topic of this chapter we look, all too briefly, at the circulation in the stratosphere. (We draw on results from earlier starred sections but the less technically-inclined reader who may have skipped them can simply refer back as needed.) Its convenenient to divide this circulation into two components: (i) the meridional overturning circulation; (ii) the quasi-horizontal circulation. There is also a region of the lower stratosphere that interacts directly with the troposphere and where fluid properties are exchanged; however, the dynamics of this region are complex and we shall not explore them here.

13.5.1 A descriptive overview

The radiative forcing of the stratosphere is effectively illustrated in Fig. 13.11, and the observed zonally-averaged temperature and zonal-wind structure are plotted in Fig. 13.12. From these we note:

- ⋆ The stratosphere is very stably stratified, with a lapse rate corresponding to $N \approx 2 \times 10^{-2}$ s, about twice that of the troposphere on average. This is in part due to the absorption of solar radiation by ozone between 20 km and 50 km.

- ⋆ In the summer the solar absorption at high latitudes leads to a reversed temperature gradient (warmer pole than equator) and a negative shear. The temperature distribution is, in fact, not far from the radiative equilibrium distribution. Consistently, by thermal wind balance, over much of the summer stratosphere the mean winds are negative (westward).

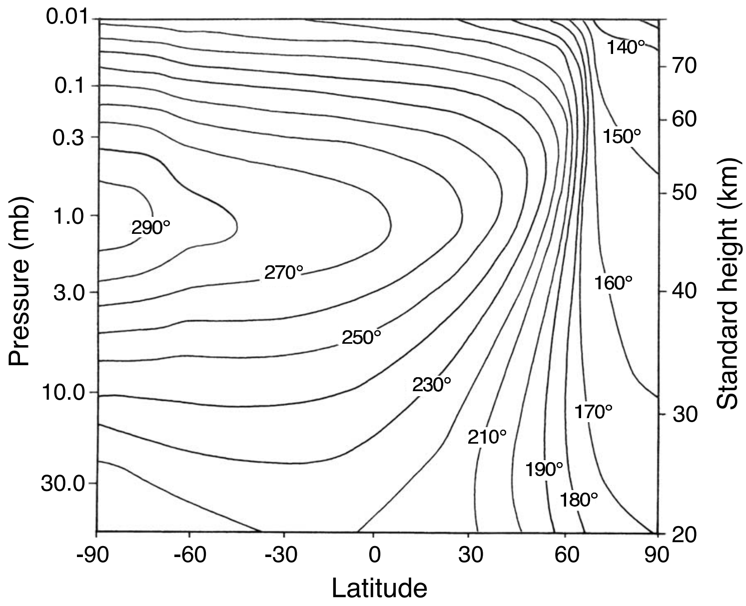

Fig. 13.11 The zonally-averaged radiative-equilibrium temperature in in January. The downwards solar radiation at the top of the atmosphere is given, and the upwards radiative flux into the stratosphere is based on observed properties, including temperature, of the troposphere.[18]

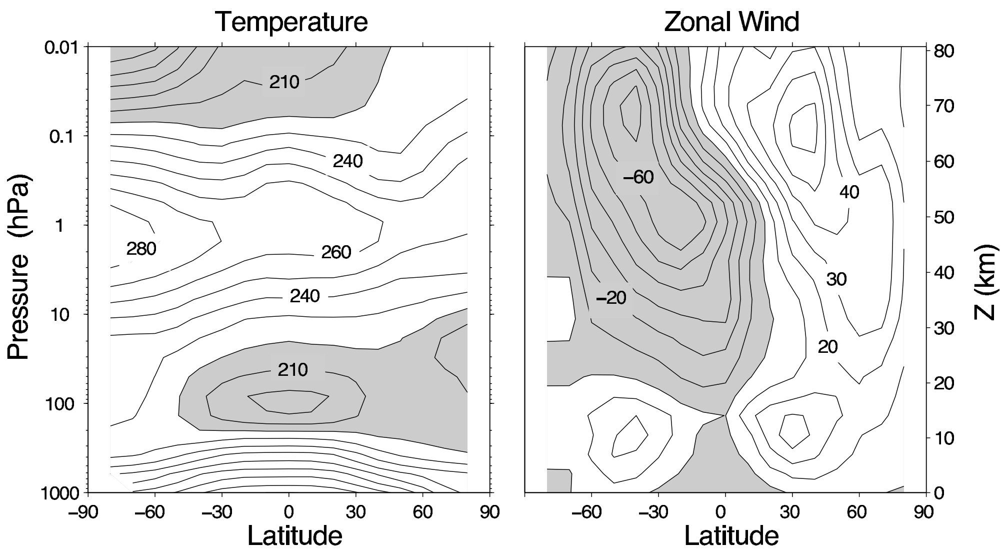

Fig. 13.12 The zonally averaged temperature and zonal wind in January. Temperature countour interval is 10 K, and values less than 220 K are shaded. Zonal wind contours are 10 m $s^{-1}$and negative (westward) values are shaded.[19]

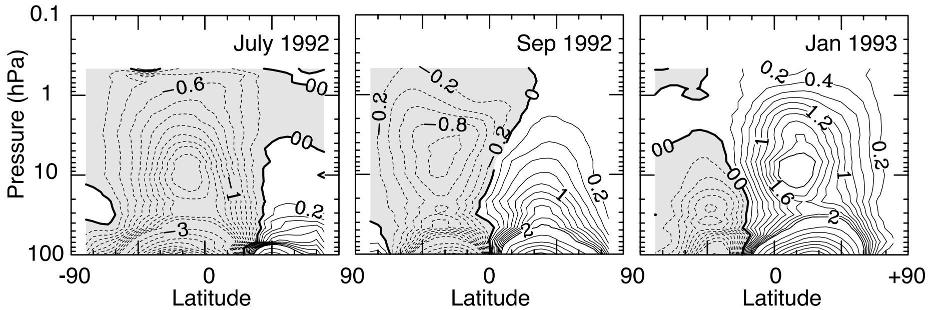

Fig. 13.13 The observed mass-weighted streamfunction in the stratosphere, in Sverdrups ($10^9\,\mathrm{kg\,s^{-1}}$). The circulation is clockwise where the contours are solid. Note the stronger circulation in the winter hemispheres, whereas the equinoctal circulation (September) is more interhemispherically symmetric.[21]

⋆ In winter high latitudes receive very little solar radiation and there is a stong meridional temperature gradient and consequently a strong vertical shear in the zonal wind. Nevertheless, this temperature gradient is significantly weaker than the radiative equilibrium temperature gradient, implying a poleward heat transfer by the fluid motions.

There must be, then, a circulation that keeps the stratosphere from radiative equilibrium, and one that is weakest in summer. In fact, a stratospheric meridional overturning circulation was inferred by A. Brewer and G. Dobson based on observations of water vapour and chemical transport, and is often called the *Brewer-Dobson circulation*.[20] It is depicted in Fig. 13.13; this shows the observed mass-weighted circulation, almost equivalent to the residual circulation, and so represents both the Eulerian mean and eddy-contributed components. It comprises a single, equator-to-pole cell in each hemisphere, stronger in the winter hemisphere where it goes high into the stratosphere. There is also a distinct lower branch to the circulation, present in all seasons although strongest in winter, that is confined to the lower stratosphere and is in some ways a vertical extension of (the residual circulation of) the tropospheric Ferrel Cell. Not all the upper circulation is ventilated by the troposphere — some of it recirculates within the stratosphere. This circulation and some of the associated dynamics is schematically illustrated in Fig. 13.14, and three three regions may usefully be delineated: (i) A tropical region; (ii) a mid-latitude region; (iii) the polar vortex. The tropical region is relatively quiescent, an area of generally upward motion where air is drawn up from the troposphere. In midlatitudes the residual flow is generally polewards before sinking at high latitudes. In winter the extreme cold leads to the formation of *polar vortex,* a strong cyclonic vortex that appears quite isolated from mid-latitudes although, especially in the Northern Hemisphere, it is not always centered over the pole.

13.5.2 Dynamics

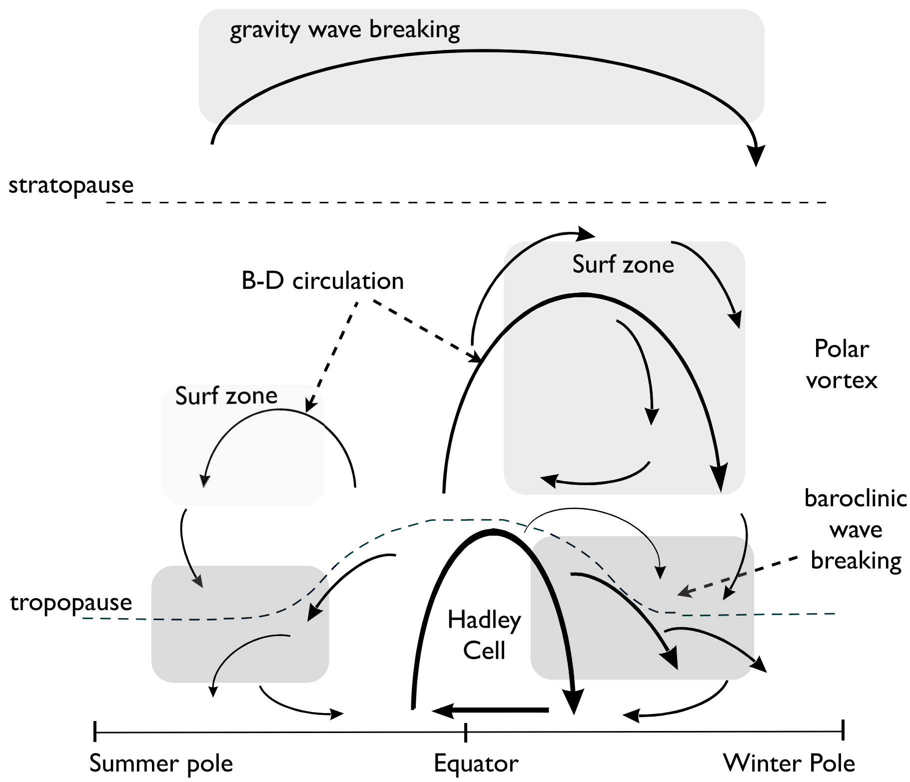

Fig. 13.14 A schema of the residual mean meridional circulation of the atmosphere. The solid arrows indicate the residual circulation (B-D for Brewer Dobson) and the shaded areas the main regions of wave breaking (i.e., enstrophy dissipation) associated with the circulation. In the surf zone the breaking is mainly that of planetary Rossby waves, and in the troposphere and lower stratosphere the breaking is that of baroclinic eddies. The surf zone and residual flow are much weaker in the summer hemisphere. Only in the Hadley Cell is the residual circulation comprised mainly of the Eulerian mean; elsewhere the eddy component dominates.[22]

Wave breaking and the residual circulation

Based on our discussion in the previous chapter of the wave-mean flow interaction and the (residual) Ferrel Cell, we might expect the Brewer-Dobson circulation to be a consequence of wave-breaking and enstrophy dissipation in the stratosphere. We ask:

(i) What is the source of such waves?
(ii) Does such wavebreaking give rise to a circulation of the right sense?
(iii) Why is the circulation weakest in summmer?

The equations of motion governing the mean fields are the zonally averaged momentum and thermodynamic equations, which in residual form may be written as

$$\frac{\partial \overline{u}}{\partial t} - f_0 \overline{v}^* = \nabla \cdot \boldsymbol{\mathcal{F}} + \overline{F}, \tag{13.84a}$$

$$\frac{\partial \overline{\theta}}{\partial t} + \frac{\partial \overline{\theta}}{\partial z} \overline{w}^* = \overline{J} \tag{13.84b}$$

where $\overline{F}$ represents frictional effects (for example, due to small scale turbulence) and $\overline{J}$ represents heating, and on the β-plane the residual velocities are related to the Eulerian means by

$$\overline{v}^* = \overline{v} - \frac{1}{\rho_R} \frac{\partial}{\partial z} \left(\rho_R \frac{\overline{v'\theta'}}{\partial_z \overline{\theta}} \right), \qquad \overline{w}^* = \overline{w} + \frac{\partial}{\partial y} \left(\frac{\overline{v'\theta'}}{\partial_z \overline{\theta}} \right). \tag{13.85}$$

The vector $\boldsymbol{\mathcal{F}}$ is the Eliassen-Palm flux, and this is related to the meridional flux of potential vorticity by $\nabla \cdot \boldsymbol{\mathcal{F}} = \overline{v'q'}$. The wave activity itself obeys the Eliassen-Palm relation

$$\frac{\partial \mathcal{A}}{\partial t} + \nabla \cdot \boldsymbol{\mathcal{F}} = \mathcal{D}, \tag{13.86}$$

where $\mathcal{A}$ is the wave activity, $\boldsymbol{\mathcal{F}}$ its flux and $\mathcal{D}$ its dissipation.

Now, in the stratosphere baroclinic instability is relatively weak, certainly compared to the troposphere (e.g., Fig. 6.21), and the main source of wave activity is upward propagation from the turbulent troposphere. From the autumn to the spring, the zonal wind in the stratosphere is generally receptive to planetary-scale Rossby waves propagating up from the troposphere (Fig. 13.7), although in at high latitudes in winter there may be a period when the eastward zonal winds are too strong for waves to propagate. If these waves break in the stratosphere then there will be an enstrophy flux to small scales and dissipation. In a quasi-statistically-steady state and with small frictional effects the dominant balance in the zonal momentum equation (13.84a) is

$$-f_0 \overline{v}^* \approx \overline{v'q'}, \tag{13.87}$$

where $\overline{v}^*$ is the residual velocity and the potential vorticity flux on the right-hand side is induced by the Rossby wave breaking. In dissipative regions the zonally-averaged potential vorticity flux will tend to be down its mean gradient and, if the potential vorticity gradient is polewards (largely because of the β-effect), the residual velocity will be positive if f_0 is positive. That is, the residual flow will be *polewards,* in both hemispheres, and the mechanism giving rise to this is called the 'Rossby wave pump.' Put another way, Rossby waves propagating up from the troposphere break and deposit westward momentum in the stratosphere, and this 'wave drag' is largely balanced by the Coriolis force on the residual meridional circulation.

This meridional circulation is weakest in summer mainly because linear Rossby waves cannot propagate upwards through the westward mean winds, as illustrated in Fig. 13.15. It is quite striking how the EP vectors avoid the region of westward winds in

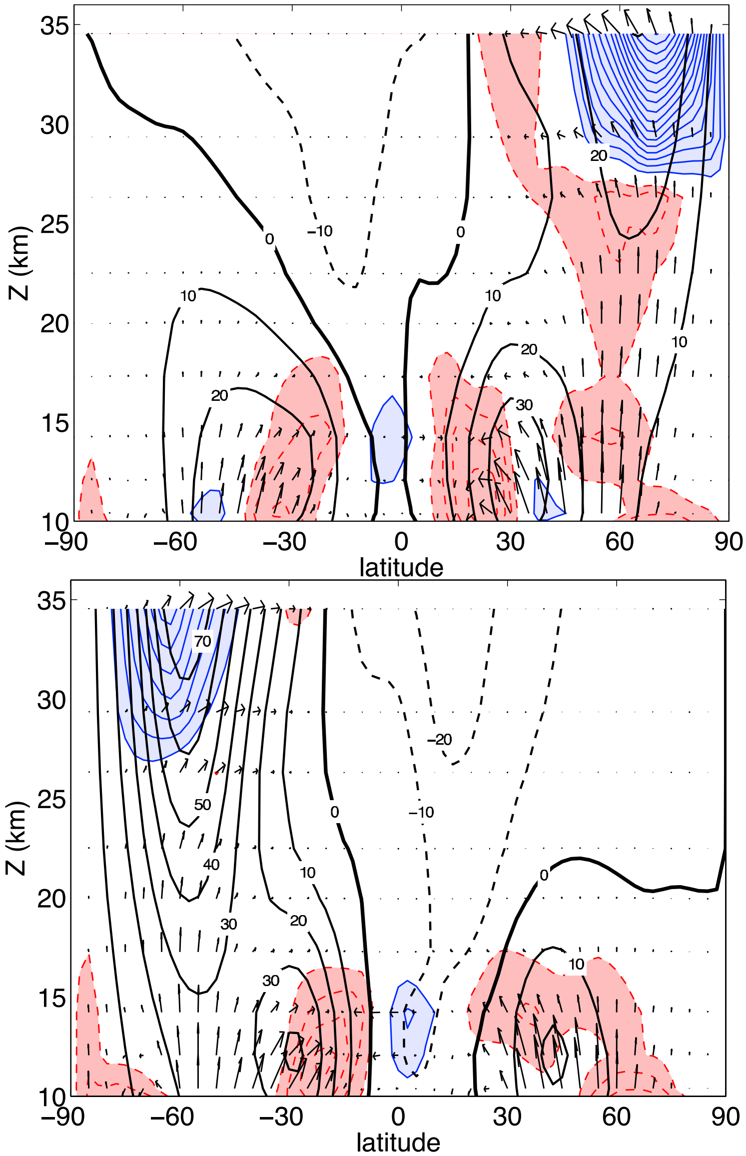

Fig. 13.15 The EP flux vectors (arrows), the EP flux divergence (shaded and light contours) and the zonally averaged zonal wind (heavy contours) for (a) Northern hemisphere winter; (b) Southern Hemisphere winter. Note the almost zero EP values in the summer hemispheres, and strong convergence at high latitudes in the winter hemispheres,leading to polewards residual flow and/or zonal flow acceleration. The EP divergence is shaded for values greater than $+1\,\mathrm{m\,s^{-1}}$/day, light solid contours) and for values less than $-1\,\mathrm{m\,s^{-1}}$/day (light dashed contours). The vertical coordinate is log pressure, extending between about 260 mb and 10 mb.

the summer hemisphere, even though the level of wave activity at low elevations is relatively similar in the summer and winter hemispheres (look between 10 km and 15 km in the figure). We can interpret this by noting that for nearly plane waves the EP flux obeys the group velocity property, meaning that $\boldsymbol{\mathcal{F}} = \boldsymbol{c}_g$; however, as discussed in section 13.3, if the mean winds are westward the waves evanesce instead of propagating, and thus almost the entire summer hemisphere is shielded from upwardly propagating waves, leaving it in a near-radiative equilibrium state. In the other seasons, the EP flux is able to propagate into the stratosphere and a circulation is generated. This acts to weaken the pole-equator temperature gradient, as we see by inspection of the thermodynamic equation: if the heating is represented by a simple relaxation to a radiative equilibrium state, θ_E, then in a steady state we have

$$N^2 \overline{w}^* = \frac{\theta_E - \theta}{\tau}. \tag{13.88}$$

Polewards flow in midlatitudes must be supplied by rising air at low latitudes, and sinking air at high. Thus, from autumn to spring, at low latitudes we have $\theta < \theta_E$ and at high latitudes $\theta > \theta_E$.

Although cause and effect can be very difficult to disentangle in fluid dynamical problems, and the ultimate cause of nearly all fluid motions in the atmosphere is the differential heating from the sun, it is important to realize that the meridional overturning in the stratosphere is not a direct response to differential heating. We see this simply by noting that the most intense heating is over the summer pole, yet here there is little or no ascent. Rather, the circulation is more directly a response to potential vorticity fluxes which in turn are determined by the upward propagation of Rossby waves from the troposphere and polewards gradient of potential vorticity in the stratosphere.

The polar vortex and the quasi-horizontal circulation

Let us now shift our perspective and consider the quasi-horizontal circulation in the stratosphere. Stratospheric dynamics are, in fact, rather more two-dimensional than those in the troposphere because the high stratification inhibits vertical motion, and the vortex stretching term in the quasi-geostrophic potential vorticity equation is relatively small. In any case, because diabatic effects occur on a somewhat longer timescale than advective processes, the flow may be characterized by the advection of potential vorticity on more slowly evolving isentropic surfaces. In midlatitudes this flow is forced by wave propagagation from below, and the upshot is that the midlatitude stratospheric circulation is a good example of geostrophic turbulence, as illustrated in Fig. 13.16 and Fig. 13.17. Both the potential vorticity and tracer are evocative the flows in chapters 8 and 9. We see Rossby waves breaking and vortices stretched into filaments and tendrils — in short, a region of an enstrophy cascade. We also perceive some idea of the spectral nonlocality of the enstrophy transfer — a single large vortex overturns and breaks and there is little sense of an orderly cascade of enstrophy to dissipative scales. For this reason, the mid-latitude region is known as the *surf zone.* It is precisely this wave breaking that gives rise to the enstrophy flux to small scales and it dissipation, and which in turn gives rise to the residual flow that is the Brewer-Dobson circulation.