Solar Altitude and Azimuth Angle Calculation

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1 Solar altitude angle calculation

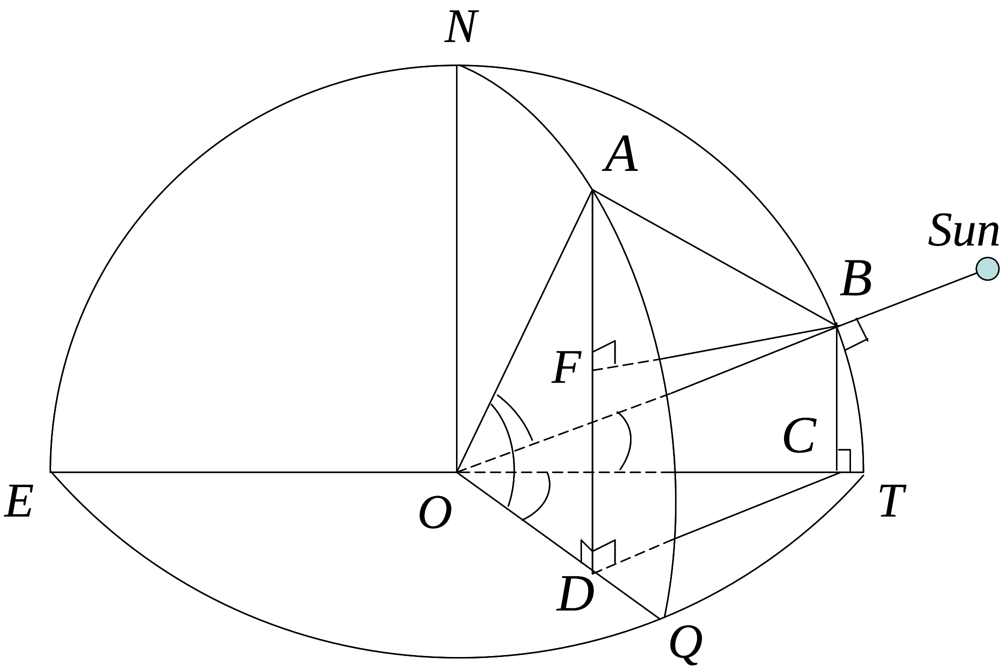

Figure 1: Schematic of the Earth North Hemisphere showing spatial geometric relationships of interested locations: Solar altitude

Refer to figure 1, where, O is the center of the Earth; N is the north pole of the Earth; arch NAQ and arch NBT are two meridians of the Earth; arch EQT is the equator; A is the location at the Earth surface where the solar altitude angle is to be calculated; B is the location at the Earth surface from which the Sun looks like right at the zenith of the sky; C is on line OT with $BC \perp OT$; D is on line OQ with $AD \perp OQ$; F is on line AD with $BF \perp AD$. It is not difficult to tell that, $\measuredangle BOC$ is the solar decline angle, $\measuredangle AOD$ is the latitude of point A, $\measuredangle COD$ is the hour angle before noon of point Q, and $\measuredangle AOB$ is the solar zenith angle of point A.

We know that the solar altitude angle is the complementary angle of the solar zenith angle. So the effort is to calculate the zenith angle $\measuredangle AOB$ at point A. This angle can be calculated by using the Law of Cosines:

$$c^2 = a^2 + b^2 - 2ab\cos(\measuredangle C) \tag{1}$$

where, a, b, and c are side lengths of the triangle; and $\measuredangle C$ denotes the angle by side a and b.

The Law of Cosines allows to calculate any of the three angle of a triangle if the lengths of the triangle's three sides are all known. For the triangle $\triangle AOB$ in figure (1), by setting the radius of the sphere to be unit, we have the length of 1 for both sides AO and BO. The length of side AB is to be calculated. We can tell the trapezoid $ABCD$ consists of a rectangle $BCDE$ and right triangle ABF. Now if we know the lengths of the right angle sides AF and BF of $\triangle ABF$, the length of the hypotenuse AB will be known by the Pythagorean theorem. Now we use the name of a side/line denotes its length as well, then it can be seen the following relations hold,

$$\begin{aligned}
BF^2 &= CD^2 = OC^2 + OD^2 - 2 \times OC \times OD \times \cos(\measuredangle COD) \\
&= [BO \times \cos(\measuredangle BOC)]^2 + [AO \times \cos(\measuredangle AOD)]^2 \\
&\quad - 2 \times [BO \times \cos(\measuredangle BOC)] \times [AO \times \cos(\measuredangle AOD)] \times \cos(\measuredangle COD) \\
&= [1 \times \cos(\measuredangle BOC)]^2 + [1 \times \cos(\measuredangle AOD)]^2 \\
&\quad - 2 \times [1 \times \cos(\measuredangle BOC)] \times [1 \times \cos(\measuredangle AOD)] \times \cos(\measuredangle COD) \\
&= [\cos(\measuredangle BOC)]^2 + [\cos(\measuredangle AOD)]^2 \\
&\quad - 2 \times \cos(\measuredangle BOC) \times \cos(\measuredangle AOD) \times \cos(\measuredangle COD) \qquad (2a) \\
AF^2 &= [AD - DF]^2 = [AD - BC]^2 \\
&= [AO \times \sin(\measuredangle AOD) - BO \times \sin(\measuredangle BOC)]^2 \\
&= [1 \times \sin(\measuredangle AOD) - 1 \times \sin(\measuredangle BOC)]^2 \\
&= [\sin(\measuredangle AOD) - \sin(\measuredangle BOC)]^2 \qquad (2b)
\end{aligned}$$

To simplify the notations, define $\alpha = \measuredangle BOC$, $\beta = \measuredangle AOD$, $\Omega = \measuredangle COD$, and $\psi = \measuredangle AOB$. With the new symbols, equations (2b)and (2b) are represented as,

$$BF^2 = \cos^2\alpha + \cos^2\beta - 2\cos\alpha\cos\beta\cos\Omega \qquad (3a)$$

$$AF^2 = [\sin\beta - \sin\alpha]^2 = \sin^2\beta + \sin^2\alpha - 2\sin\beta\sin\alpha \qquad (3b)$$

From equation (3a) and (3b), by Pythagorean theorem, the length of AB can be calculated as below,

$$\begin{aligned}
AB^2 &= BF^2 + AF^2 \\
&= [\cos^2\alpha + \cos^2\beta - 2\cos\alpha\cos\beta\cos\Omega] + [\sin^2\beta + \sin^2\alpha - 2\sin\beta\sin\alpha] \\
&= \sin^2\alpha + \cos^2\alpha + \sin^2\beta + \cos^2\beta - 2\cos\alpha\cos\beta\cos\Omega - 2\sin\alpha\sin\beta \\
&= 1 + 1 - 2\cos\alpha\cos\beta\cos\Omega - 2\sin\alpha\sin\beta \\
&= 2 - 2\cos\alpha\cos\beta\cos\Omega - 2\sin\alpha\sin\beta \qquad (4)
\end{aligned}$$

Now for triangle $\triangle AOB$, using Law of Cosine, yield,

$$\begin{aligned}
\cos\psi &= \frac{AO^2 + BO^2 - AB^2}{2 \times AO \times BO} = \frac{1^2 + 1^2 - AB^2}{2 \times 1 \times 1} = \frac{1}{2}[2 - AB^2] \\
&= \frac{1}{2}[2 - (2 - 2\cos\alpha\cos\beta\cos\Omega - 2\sin\alpha\sin\beta)] \\
&= \frac{1}{2}[2\cos\alpha\cos\beta\cos\Omega + 2\sin\alpha\sin\beta] \\
&= \sin\alpha\sin\beta + \cos\alpha\cos\beta\cos\Omega \qquad (5)
\end{aligned}$$

The hour angle Ω of point Q in the above equation decreases with the rotation of the Earth during the day time. Since Ω is a function of the planet angular velocity ω of the Earth and local time t of point Q, in order to conveniently use ωt to represent the hour angle, the following formula is considered,

$$\cos\Omega = \cos(\pi - \measuredangle QOE) = -\cos(\measuredangle QOE) = -\cos\omega t \qquad (6)$$

Substitute equation (6) into (5), get,

$$\cos\psi = \sin\alpha\sin\beta - \cos\alpha\cos\beta\cos\omega t \tag{7}$$

The solar altitude angle,Ψ, is the complementary angle of zenith angle ψ, which gives,

$$\cos\psi = \cos(\frac{\pi}{2} - \Psi) = \sin\Psi \tag{8}$$

Substitute equation (8) into (7), we finally get the solar altitude angle formula for point A as below,

$$\sin\Psi = \sin\alpha\sin\beta - \cos\alpha\cos\beta\cos\omega t \tag{9}$$

Where,

$\Psi =$	solar altitude angle	(rad)
$\alpha =$	solar declination angle	(rad)
$\beta =$	site latitude	(rad)
$\omega =$	earth angular velocity	$(\frac{\pi}{12}h^{-1})$
$t =$	local time $[0, 24)$	(h)

2 Solar azimuth angle calculation

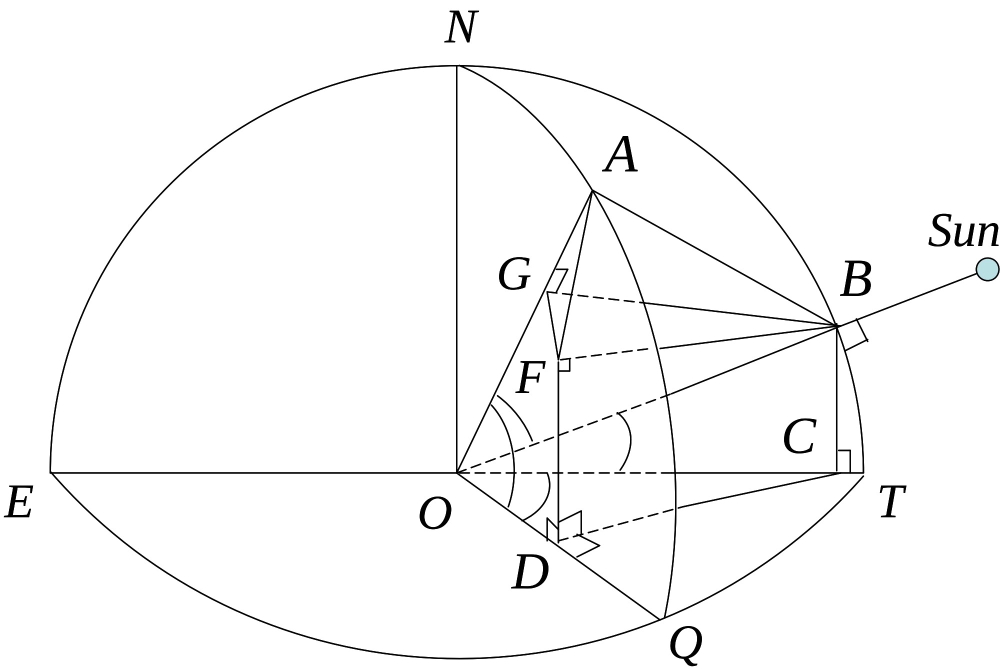

Figure 2: Schematic of the Earth North Hemisphere showing spatial geometric relationships of interested locations: Solar azimuth

Refer to figure 2, where, O is the center of the Earth; N is the north pole of the Earth; arch NAQ and arch NBT are two meridians of the Earth; arch EQT is the equator; A is the location at the Earth surface where the solar azimuth angle is to be calculated; B is the location at the Earth surface where the Sun looks like right at the zenith of the sky; F is the projection of point B on plane AOQ, e.g., $BF \perp AOQ$; C is on line OT with $BC \perp OT$; D is on line OQ with $FD \perp OQ$; by $BF \perp AOQ$, $BF \perp OQ$; by $FD \perp OQ$ and $BF \perp OQ$, $OQ \perp BCDF$; by $OQ \perp BCDF$, $CD \perp OQ$; G is on line AO with $BG \perp AO$; by $BF \perp AOQ$, $BF \perp FG$ and $BF \perp AO$; by $BG \perp AO$ and $BF \perp AO$, $AO \perp BFG$; by $AO \perp BFG$, $FG \perp AO$; by $BG \perp AO$ and $FG \perp AO$, the angle $\measuredangle BGF$ is the angle by planes AOQ and AOB, which is exactly the solar azimuth angle from the due south; meanwhile, it is easy to see that $\measuredangle BOC$ is the solar decline angle, $\measuredangle AOD$ is the latitude of point A, $\measuredangle COD$ is the hour angle before noon of point Q, and $\measuredangle AOB$ is the solar zenith angle of point A.

Our effort is to calculate $\measuredangle BGF$. By $BF \perp FG$, we know the $\triangle BFG$ is a right triangle. For the right triangle $\triangle BFG$, if any two of the three sides are known, then all the three internal angles will be known. It can be seen that, the hypotenuse side BG is the right angle side of another right triangle $\triangle BOG$, with $\measuredangle BOG$ being exactly the zenith angle of point A. Since the zenith angle becomes a known after using formula (9), then the length of BG is,

$$BG = BO \times \sin(\measuredangle BOG) = \sin\psi = \sin(\frac{\pi}{2} - \Psi) = \cos\Psi \tag{10}$$

It can also be seen that, the polygon $BFDT$ is a rectangle, which enables us to calculate right angle side BF as below,

$$BF = CD = CO \times \sin\Omega = BO \times \cos\alpha \times \sin\Omega = \cos\alpha\sin\Omega \tag{11}$$

From equation (10) and (11), the azimuth angle from due south at point A can be calculated as below,

$$\sin(\measuredangle BGF) = \frac{BF}{BG} = \frac{\cos\alpha \sin\Omega}{\cos\Psi} \tag{12}$$

Similarly, in order to conveniently use ωt to represent the hour angle, the following formula is considered,

$$\sin\Omega = \sin(\pi - \measuredangle QOE) = \sin(\measuredangle QOE) = \sin\omega t \tag{13}$$

Substitute equation (13) to (12), get,

$$\sin(\measuredangle BGF) = \frac{BF}{BG} = \frac{\cos\alpha \sin\omega t}{\cos\Psi} \tag{14}$$

In our case, we redefine the azimuth angle to be from due east instead of due south, and denote the angle with symbol Φ. It can be seen that Φ is the complementary angle of $\measuredangle BGF$, thus,

$$\sin(\measuredangle BGF) = \sin(\frac{\pi}{2} - \Phi) = \cos\Phi \tag{15}$$

Substitute equation (15) into (14), get the final azimuth equation,

$$\cos\Phi = \frac{\cos\alpha \sin\omega t}{\cos\Psi} \tag{16}$$

Where,

$\Phi =$	solar azimuth angle from due east	(rad)
$\alpha =$	solar declination angle	(rad)
$\omega =$	earth angular velocity	$(\frac{\pi}{12}h^{-1})$
$t =$	local time $[0, 24)$	(h)
$\Psi =$	solar altitude angle	(rad)