4.4 Stochastically driven MOC variability

A review of classes of THC oscillations: small amplitude/ large amplitude; linear and stochastically forced/ nonlinear self-sustained; loop oscillations due to advection around the THC path, or periodic switches between convective and non convective states; relaxation oscillations; noise induced switches between steady state, stochastic resonance;

Details of noise-driven THC/MOC variability:

1. Linear Loop-oscillations due to advection around the circulation path:
   (a) Linearized stability analysis (write equations, explain linearization, writing of linearized equations in matrix form) and bifurcations (section 6.2.4 in Dijkstra (2000) or Figs 3, 4 from Tziperman et al. (1994)); Stability regimes in a 4-box model: stable, stable oscillatory, [Hopf bifurcation], unstable oscillatory, unstable; Note changes from 2-box Stommel model: oscillatory behavior and change to the point of instability on the bifurcation diagram; Stochastic forcing can excite this damped oscillatory variability.
   (b) The GCM study of Delworth et al. (1993) (DMS), Figs. 4, 5, 6, 8; this paper also demonstrates the link between the variability of meridional density gradients and of the THC; Note the proposed role of changes to the gyre circulation in this paper, mention related mechanisms based on ocean mid-latitude Rossby wave propagation;
   (c) A box model fit to the DMS GCM, showing that the horizontal gyre variability may not be critical and that the variability is due to the excitation of a damped oscillatory mode (Fig. 6, Griffies and Tziperman, 1995);
   (d) Useful and interesting analysis methods in DMS: composites (Figs. 6,7), and regression analysis between scalar indices (Figs. 8,9) and between scalar indices and fields (Figs. 10, 11, 12).
2. A complementary view of the above stochastic excitation of damped THC oscillatory mode: first, Hasselmann's model driven by white noise and leading to a red spectrum response

(to derive the spectrum of the response, Fourier transform the first equation, and multiply transformed equation and its complex conjugate).

$$\begin{aligned}\dot{x}+\gamma x &= \xi(t)\\ P(\omega)=|\hat{x}|^2 &= \xi_0^2/(\omega^2+\gamma^2).\end{aligned}$$

Compare this to a damped oscillatory mode excited by noise that results in a spectral peak, using the following derivation of the spectral response,

$$\begin{aligned}\ddot{x}+\gamma\dot{x}+\Omega^2 x &= \xi(t)\\ -\omega^2\hat{x}-i\gamma\omega\hat{x}+\Omega^2\hat{x} &= \xi_0\\ \hat{x} &= \xi_0/(\Omega^2-\omega^2-i\gamma\omega)=\xi_0(\Omega^2-\omega^2+i\gamma\omega)/((\Omega^2-\omega^2)^2+\gamma^2\omega^2)\\ \hat{x}^* &= \xi_0(\Omega^2-\omega^2-i\gamma\omega)/((\Omega^2-\omega^2)^2+\gamma^2\omega^2)\\ P(\omega)=|\hat{x}|^2=\hat{x}\hat{x}^* &= \frac{\xi_0^2}{(\Omega^2-\omega^2)^2+\gamma^2\omega^2}.\end{aligned}$$

3. Stochastic variability due to noise induced transitions between steady states. First, the GCM study showing jumping between three very different MOC equilibria under sufficiently strong stochastic forcing: Figs. 2, 4 from Weaver and Hughes (1994). Then Fig. 5 from Cessi (1994) showing a careful analysis of the same type of variability in the Stommel box model.

4. (Time permitting:) details of Cessi analysis: Stommel 2 box model from section 2 with model derivation and in particular getting to eqn 2.9 with temperature fixed and salinity difference satisfying an equation of a particle on a double potential surface; section 3 with deterministic perturbation;

5. Stochastic resonance: periodic FW forcing plus noise. Matlab code `Stommel_stochastic_resonance.m`, and jpeg figures with results: `Stochastic_Resonance_a,b,c.jpg`;

6. (Time permitting:) THC oscillations due to "Thermal" Rossby waves analyzed by Te Raa and Dijkstra (2002), using equations 8-10 and Figure 7 of Zanna et al. (2011).

7. Stochastic forcing, non normal THC dynamics, transient amplification; first the basic derivation: Stochastic optimals: the derivation from Tziperman and Ioannou (2002): consider a stochastically forced linear system:

$$\dot{\mathbf{P}}=A\mathbf{P}+\mathbf{f}(t)$$

solution is

$$\mathbf{P}(\tau)=e^{A\tau}\mathbf{P}(0)+\int_0^\tau ds\,e^{A(\tau-s)}\mathbf{f}(s)=B(\tau,0)\mathbf{P}(0)+\int_0^\tau ds\,B(\tau,s)\mathbf{f}(s)$$

where the first term is a response to the initial conditions which decays in time and can therefore ignored, and the second is the response to the stochastic forcing. Assuming for simplicity a zero-mean state variable $\mathbf{P}(t)$, the variance is given by the following expression (summation convention assumed),

$$
\begin{aligned}
var(\|\mathbf{P}\|) &= \langle P_i(\tau)P_i(\tau)\rangle \\
&= \left\langle \int_0^\tau ds \int_0^\tau dt B_{il}(\tau,s) f_l(s) B_{in}(\tau,t) f_n(t) \right\rangle \\
&= \int_0^\tau ds \int_0^\tau dt B_{il}(\tau,s) B_{in}(\tau,t) \langle f_l(s) f_n(t)\rangle
\end{aligned}
$$

Specifying the noise statistics as separable in space and time, letting the ith component of the noise be, say, $f_i \nu(t)$, and with $C_{ln} = f_l f_n$ being the noise spatial correlation matrix and $D(t-s) = \langle \nu(t)\nu(s)\rangle$ the temporal correlation function (delta function for white noise),

$$\langle f_l(s) f_n(t)\rangle = C_{ln} D(t-s)$$

we have

$$
\begin{aligned}
var(\|P\|) &= \int_0^\tau ds \int_0^\tau dt B_{il}(\tau,s) B_{in}(\tau,t) C_{ln} D(t-s) \\
&= Tr\left(\int_0^\tau ds \int_0^\tau dt [B^T(\tau,s) B(\tau,t) D(t-s)] C \right) \\
&\equiv Tr(ZC)
\end{aligned}
$$

This implies that the most efficient way to excite the variance is to make the noise spatial structure be the first eigenvector of Z. To show this, show that eigenvectors of Z maximize $J = Tr(CZ) = Z_{ij} C_{ji}$; given the above expression for $C_{ij} = f_i f_j$ and we need to maximize $Z_{ij} f_i f_j + \lambda(1 - f_k f_k)$; differentiating with respect to f_n we get that $\mathbf{f} = \{f_n\}$ is an eigenvector of the matrix Z. The vector that maximizes the variance is, as usual, the one corresponding to the largest eigenvalue of Z.

Then more specifically to THC/MOC: The 3-box model of Tziperman and Ioannou (2002), or the spatially resolved 2d model of Zanna and Tziperman (2005)). Figures for the amplification and mechanism, taken from a talk on this subject (file nonormal_THC.pdf).

8. (Time permitting:) Use of eigenvectors for finding the instability mechanism: linearize, solve eigenvalue problem, substitute spatial structure of eigenvalue into equations and see which equations provide the positive/ negative feedbacks; results for THC problem (e.g.,

Tziperman et al., 1994, section 3): destabilizing role of $v'\nabla\bar{S}$ and stabilizing role of $\bar{v}\nabla S'$; difference in stability mechanism in upper ocean ($v'\nabla\bar{S}$) vs that of the deep ocean (where $\nabla\bar{S} = 0$ and $\bar{v}\nabla S'$ is dominant); for temperature, also ($v'\nabla\bar{T}$ is more important, but from the eigenvectors one can see that v' is dominated by salinity effects; GCM verification and the distance of present-day THC from stability threshold: Figs 4,5,6 from Tziperman et al. (1994); Fig 3 from Toggweiler et al. (1996); Figs 1, 2, 3 from Tziperman (1997).