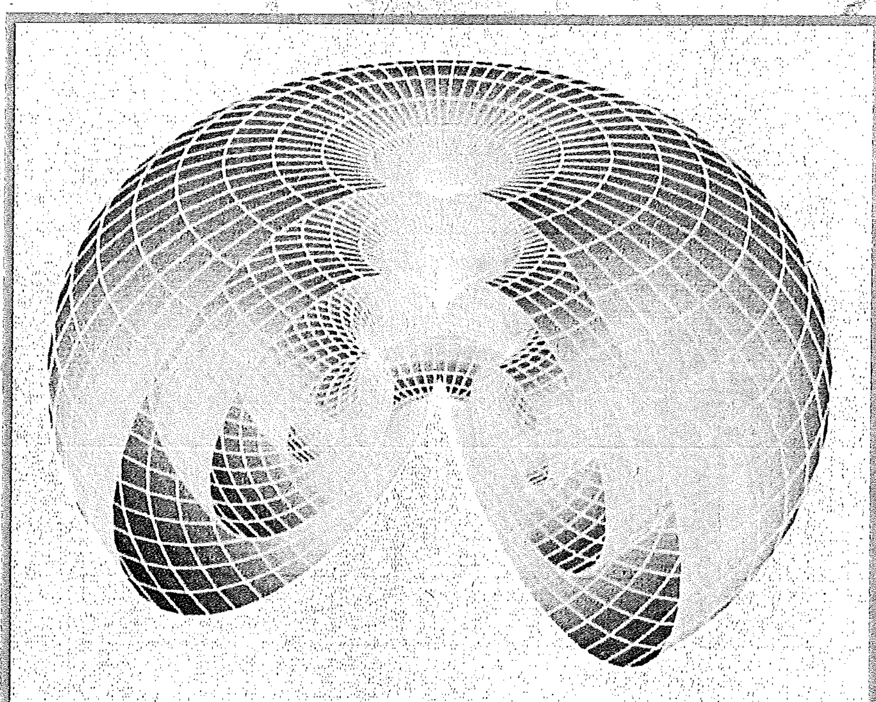
STUDIES IN NONLINEARITY

NONLINEAR DYNAMICS AND CHAOS



With Applications to Physics, Biology, Chemistry, and Engineering

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(1) $f(x)$ and $g(x)$ are continuously differentiable for all x;
(2) $g(-x) = -g(x)$ for all x (i.e., $g(x)$ is an *odd* function);
(3) $g(x) > 0$ for $x > 0$;
(4) $f(-x) = f(x)$ for all x (i.e., $f(x)$ is an *even* function);
(5) The odd function $F(x) = \int_0^x f(u)\,du$ has exactly one positive zero at $x = a$, is negative for $0 < x < a$, is positive and nondecreasing for $x > a$, and $F(x) \to \infty$ as $x \to \infty$.

Then the system (2) has a unique, stable limit cycle surrounding the origin in the phase plane.

This result should seem plausible. The assumptions on $g(x)$ mean that the restoring force acts like an ordinary spring, and tends to reduce any displacement, whereas the assumptions on $f(x)$ imply that the damping is negative at small $|x|$ and positive at large $|x|$. Since small oscillations are pumped up and large oscillations are damped down, it is not surprising that the system tends to settle into a self-sustained oscillation of some intermediate amplitude.

EXAMPLE 7.4.1:

Show that the van der Pol equation has a unique, stable limit cycle.

Solution: The van der Pol equation $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$ has $f(x) = \mu(x^2 - 1)$ and $g(x) = x$, so conditions (1)–(4) of Liénard's theorem are clearly satisfied. To check condition (5), notice that

$$F(x) = \mu\left(\tfrac{1}{3}x^3 - x\right) = \tfrac{1}{3}\mu x(x^2 - 3).$$

Hence condition (5) is satisfied for $a = \sqrt{3}$. Thus the van der Pol equation has a unique, stable limit cycle. ■

There are several other classical results about the existence of periodic solutions for Liénard's equation and its relatives. See Stoker (1950), Minorsky (1962), Andronov et al. (1973), and Jordan and Smith (1987).

7.5 Relaxation Oscillations

It's time to change gears. So far in this chapter, we have focused on a qualitative question: Given a particular two-dimensional system, does it have any periodic solutions? Now we ask a quantitative question: Given that a closed orbit exists, what can we say about its shape and period? In general, such problems can't be solved exactly, but we can still obtain useful approximations if some parameter is large or small.

We begin by considering the van der Pol equation

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$$

for $\mu \gg 1$. In this *strongly nonlinear* limit, we'll see that the limit cycle consists of an extremely slow buildup followed by a sudden discharge, followed by another slow buildup, and so on. Oscillations of this type are often called ***relaxation oscillations***, because the "stress" accumulated during the slow buildup is "relaxed" during the sudden discharge. Relaxation oscillations occur in many other scientific contexts, from the stick-slip oscillations of a bowed violin string to the periodic firing of nerve cells driven by a constant current (Edelstein–Keshet 1988, Murray 1989, Rinzel and Ermentrout 1989).

EXAMPLE 7.5.1:

Give a phase plane analysis of the van der Pol equation for $\mu \gg 1$.

Solution: It proves convenient to introduce different phase plane variables from the usual "$\dot{x} = y$, $\dot{y} = \ldots$". To motivate the new variables, notice that

$$\ddot{x} + \mu\dot{x}(x^2 - 1) = \frac{d}{dt}\left(\dot{x} + \mu\left[\tfrac{1}{3}x^3 - x\right]\right).$$

So if we let

$$F(x) = \tfrac{1}{3}x^3 - x, \qquad w = \dot{x} + \mu F(x), \tag{1}$$

the van der Pol equation implies that

$$\dot{w} = \ddot{x} + \mu\dot{x}(x^2 - 1) = -x. \tag{2}$$

Hence the van der Pol equation is equivalent to (1), (2), which may be rewritten as

$$\begin{aligned} \dot{x} &= w - \mu F(x) \\ \dot{w} &= -x. \end{aligned} \tag{3}$$

One further change of variables is helpful. If we let

$$y = \frac{w}{\mu}$$

then (3) becomes

$$\begin{aligned} \dot{x} &= \mu\left[y - F(x)\right] \\ \dot{y} &= -\tfrac{1}{\mu}x. \end{aligned} \tag{4}$$

Now consider a typical trajectory in the (x, y) phase plane. The nullclines are the key to understanding the motion. We claim that all trajectories behave like that shown in Figure 7.5.1; starting from any point except the origin, the trajectory zaps horizontally onto the ***cubic nullcline*** $y = F(x)$. Then it crawls down the nullcline until it comes to the knee (point B in Figure 7.5.1), after which it zaps over to the other branch of the cubic at C. This is followed by another crawl along the cubic

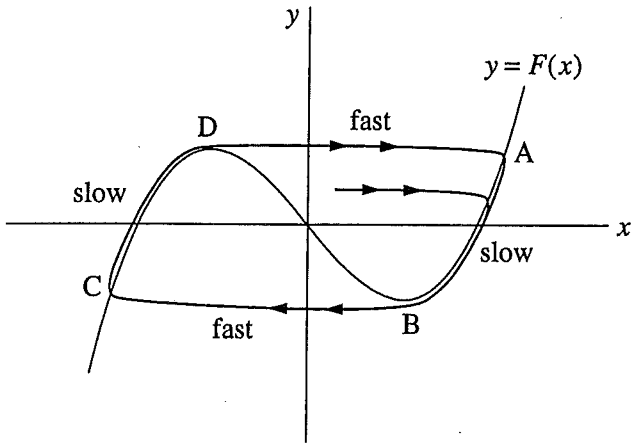

Figure 7.5.1

until the trajectory reaches the next jumping-off point at D, and the motion continues periodically after that.

To justify this picture, suppose that the initial condition is not too close to the cubic nullcline, i.e., suppose $y - F(x) \sim O(1)$. Then (4) implies $|\dot{x}| \sim O(\mu) >> 1$ whereas $|\dot{y}| \sim O(\mu^{-1}) << 1$; hence the velocity is enormous in the horizontal direction and tiny in the vertical direction, so trajectories move practically horizontally. If the initial condition is *above* the nullcline, then $y - F(x) > 0$ and therefore $\dot{x} > 0$; thus the trajectory moves sideways *toward* the nullcline. However, once the trajectory gets so close that $y - F(x) \sim O(\mu^{-2})$, then $\dot{x}$ and $\dot{y}$ become comparable, both being $O(\mu^{-1})$. What happens then? The trajectory crosses the nullcline vertically, as shown in Figure 7.5.1, and then moves slowly along the backside of the branch, with a velocity of size $O(\mu^{-1})$, until it reaches the knee and can jump sideways again. ∎

This analysis shows that the limit cycle has two ***widely separated time scales***: the crawls require $\Delta t \sim O(\mu)$ and the jumps require $\Delta t \sim O(\mu^{-1})$. Both time scales are apparent in the waveform of $x(t)$ shown in Figure 7.5.2, obtained by numerical integration of the van der Pol equation for $\mu = 10$ and initial condition $(x_0, y_0) = (2, 0)$.

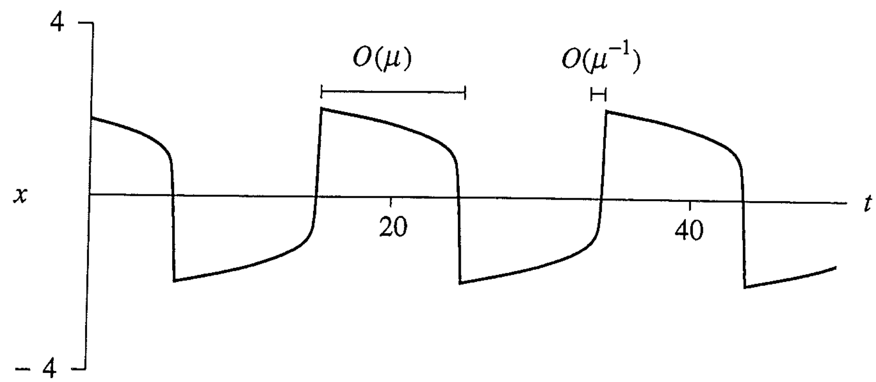

Figure 7.5.2

EXAMPLE 7.5.2:

Estimate the period of the limit cycle for the van der Pol equation for $\mu >> 1$.

Solution: The period T is essentially the time required to travel along the two ***slow branches***, since the time spent in the jumps is negligible for large μ. By symmetry, the time spent on each branch is the same. Hence $T \approx 2\int_{t_A}^{t_B} dt$. To derive an expression for *dt*, note that on the slow branches, $y \approx F(x)$ and thus

$$\frac{dy}{dt} \approx F'(x)\frac{dx}{dt} = (x^2-1)\frac{dx}{dt}.$$

But since $dy/dt = -x/\mu$ from (4), we find $dx/dt = -x/\mu(x^2-1)$. Therefore

$$dt \approx -\frac{\mu(x^2-1)}{x}\,dx \qquad (5)$$

on a slow branch. As you can check (Exercise 7.5.1), the positive branch begins at $x_A = 2$ and ends at $x_B = 1$. Hence

$$T \approx 2\int_2^1 \frac{-\mu}{x}(x^2-1)\,dx = 2\mu\left[\frac{x^2}{2} - \ln x\right]_1^2 = \mu\left[3 - 2\ln 2\right], \qquad (6)$$

which is $O(\mu)$ as expected. ■

The formula (6) can be refined. With much more work, one can show that $T \approx \mu[3-2\ln 2] + 2\alpha\mu^{-1/3} + \ldots$, where $\alpha \approx 2.338$ is the smallest root of $\mathrm{Ai}(-\alpha) = 0$. Here $\mathrm{Ai}(x)$ is a special function called the Airy function. This correction term comes from an estimate of the time required to turn the corner between the jumps and the crawls. See Grimshaw (1990, pp. 161–163) for a readable derivation of this wonderful formula, discovered by Mary Cartwright (1952). See also Stoker (1950) for more about relaxation oscillations.

One last remark: We have seen that a relaxation oscillation has two time scales that operate *sequentially*—a slow buildup is followed by a fast discharge. In the next section we will encounter problems where two time scales operate *concurrently*, and that makes the problems a bit more subtle.

7.6 Weakly Nonlinear Oscillators

This section deals with equations of the form

$$\ddot{x} + x + \varepsilon h(x, \dot{x}) = 0 \qquad (1)$$

where $0 \le \varepsilon << 1$ and $h(x, \dot{x})$ is an arbitrary smooth function. Such equations represent small perturbations of the linear oscillator $\ddot{x} + x = 0$ and are therefore called ***weakly nonlinear oscillators***. Two fundamental examples are the van der Pol equation

$$\ddot{x} + x + \varepsilon(x^2 - 1)\dot{x} = 0, \qquad (2)$$

(now in the limit of small nonlinearity), and the ***Duffing equation***

$$\ddot{x} + x + \varepsilon x^3 = 0. \qquad (3)$$

To illustrate the kinds of phenomena that can arise, Figure 7.6.1 shows a computer-generated solution of the van der Pol equation in the $(x, \dot{x})$ phase plane, for $\varepsilon = 0.1$ and an initial condition close to the origin. The trajectory is a slowly winding spiral; it takes many cycles for the amplitude to grow substantially. Eventually

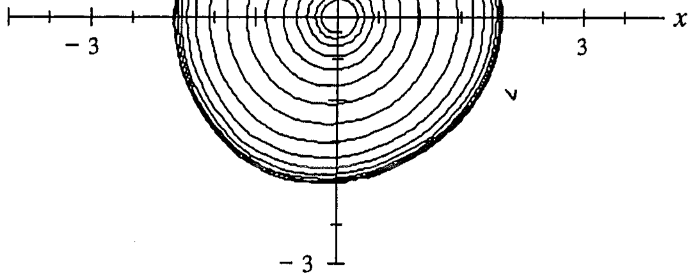

Figure 7.6.1