7 Non-normal dynamics and transient growth

We know now that if the real part of all eigenvalues of a set of linear constant coefficient ODEs is negative, the solution must be decaying to zero for long times. It turns out, though, that before decaying the solution in some cases sees a very dramatic amplification first. This has been used to explain the explosive development of some weather systems, and can be used to predict similar explosive growth in economic models, etc. To understand why and how this might happen, consider

$$\frac{d\mathbf{x}}{dt} = \mathsf{A}\mathbf{x}, \quad \mathbf{x}(t=0) = \mathbf{x}_0.$$

The solution is $\mathbf{x} = e^{\mathsf{A}t}\mathbf{x}_0 \equiv \mathsf{B}\mathbf{x}_0$. Given the eigenvectors/ values of the matrix,

$$\mathsf{A}\hat{\mathbf{e}}_i = \lambda_i \hat{\mathbf{e}}_i,$$

the solution may also be written as

$$\mathbf{x}(t) = \sum_i c_i \hat{\mathbf{e}}_i e^{\lambda_i t}.$$

where the coefficients c_i are chosen to satisfy the initial conditions such that

$$\mathbf{x}_0 = \sum_i c_i \hat{\mathbf{e}}_i.$$

The time dependent solution may also be written as,

$$\mathbf{x}(t) = e^{\mathsf{A}t}\mathbf{x}_0 \equiv \mathsf{B}\mathbf{x}_0.$$

If the eigenvalues all have negative real part, we expect the solution to decay to zero, $\mathbf{x}(t \to \infty) \to 0$. However, it turns out that when A is non normal, $\mathsf{A}\mathsf{A}^T \neq \mathsf{A}^T\mathsf{A}$ (that is, its eigenvectors are not orthogonal to each other) the solution may undergo an arbitrarily large amplification before decaying to zero.

Consider first a geometric view of the amplification using a 2×2 example where $\mathbf{x} = (x, y)$ shown in Fig. 4.

Next, derive the optimal i.c. that lead to a maximal $|\mathbf{x}|^2 = \mathbf{x}(\tau)^T\mathbf{x}(\tau)$ subject to the condition that the initial conditions are normalized $|\mathbf{x}_0|^2 = \mathbf{x}_0^T\mathbf{x}_0 = 1$. Use Lagrange multipliers to maximize

$$\begin{aligned} J(\mathbf{x}_0) &= \mathbf{x}(\tau)^T\mathbf{x}(\tau) + \lambda(1 - \mathbf{x}_0^T\mathbf{x}_0) \\ &= (\mathsf{B}\mathbf{x}_0)^T(\mathsf{B}\mathbf{x}_0) + \lambda(1 - \mathbf{x}_0^T\mathbf{x}_0) \\ &= \mathbf{x}_0^T(\mathsf{B}^T\mathsf{B})\mathbf{x}_0 + \lambda(1 - \mathbf{x}_0^T\mathbf{x}_0). \end{aligned}$$

Denote the components of the initial conditions as $x_{0,i}$, we require that at the maximum point $\partial J(\mathbf{x}_0)/\partial x_{0,i} = 0$. This, after some algebra, leads to

$$(\mathsf{B}^T\mathsf{B})\mathbf{x}_0 = \lambda\mathbf{x}_0$$

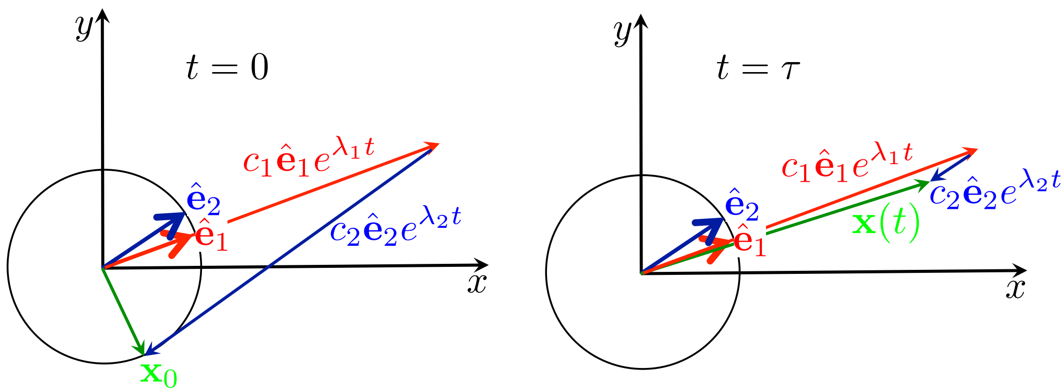

Figure 4: initial conditions and later development in a 2×2 non-normal transient growth case with $\lambda_2 \ll \lambda_1 < 0$.

so that the optimal initial conditions maximizing the state at a later time are the first eigenvector of $\mathsf{B}^T\mathsf{B}$ where B is the propagator $\mathsf{B} = \exp(\mathsf{A}\tau)$. We next show that the corresponding eigenvalue is the amplification factor from the initial conditions to the amplified state, by writing the amplification as follows,

$$\begin{aligned}\frac{|\mathbf{x}(\tau)|^2}{|\mathbf{x}_0|^2} = \frac{\mathbf{x}(\tau)^T\mathbf{x}(\tau)}{\mathbf{x}_0^T\mathbf{x}_0} = \frac{(\mathsf{B}\mathbf{x}_0)^T(\mathsf{B}\mathbf{x}_0)}{\mathbf{x}_0^T\mathbf{x}_0} \\ = \frac{\mathbf{x}_0^T(\mathsf{B}^T\mathsf{B})\mathbf{x}_0}{\mathbf{x}_0^T\mathbf{x}_0} = \frac{\mathbf{x}_0^T\lambda\mathbf{x}_0}{\mathbf{x}_0^T\mathbf{x}_0} \\ = \lambda.\end{aligned}$$

A numerical example based on

$$\mathsf{A} = \begin{bmatrix} -9.7945 & 60.2566 \\ -0.7395 & 4.2945 \end{bmatrix}$$

which has eigenvalues -5 and -0.5 is shown in Fig. 5, see Matlab/python codes non-normal_transient_amplification.m/py for matrix details.

For many more details, in particular for the implications on atmospheric dynamics and the explosive development of weather systems by this transient growth mechanism, please see Farrell and Ioannou (1996).

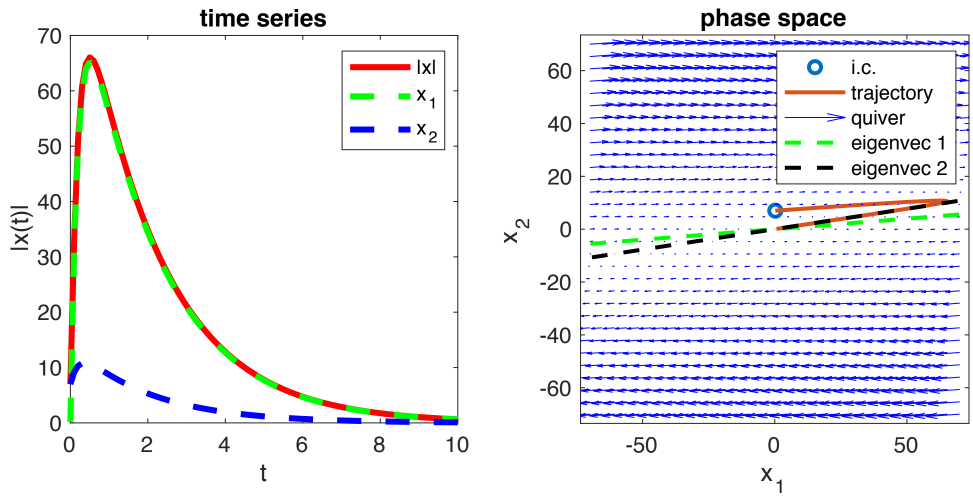

Figure 5: A numerical example of transient growth.