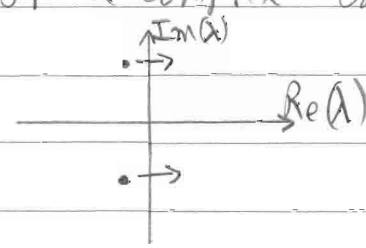
## * bifurcations in 2d: [chapter 8, strogatz]

+ As mentioned previously, the 1d bifurcations (saddle-node, transcritical, pitchfork) can occur in higher dimensional systems, along the center manifold. [p. 85-91 in first notebook].
+ There are also bifurcations that are inherently 2d:

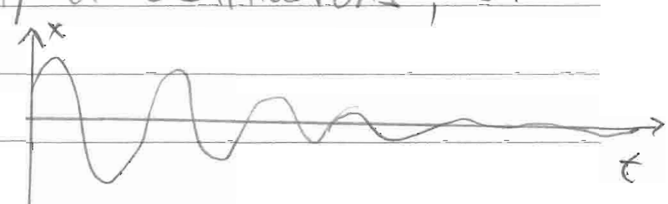
## * Hopf bifurcations:

+ bifurcations occur when $Re(\lambda)$ passes through zero. The 1d bifurcations mentioned above ocure when a real eigen value (in a possibly n-dim system) passes through zero.
+ consider now the case of a complex conjugate pair of eigen values:

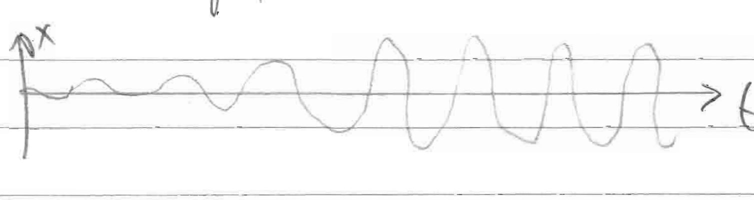



## * Super critical Hopf:

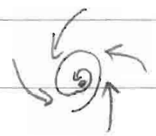
When $Re(\lambda) < 0$, we have damped osillations, or what we called a spiral:



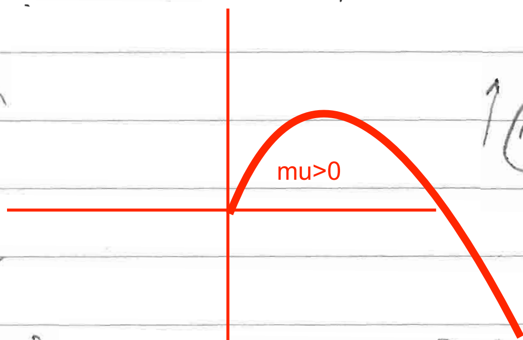

we also saw (weakly non linear van der pol), a case where an unstable spiral saturates at a limit cycle.

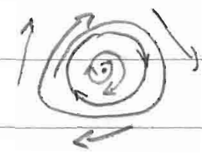


+ Hopf bifurcation in phase space:

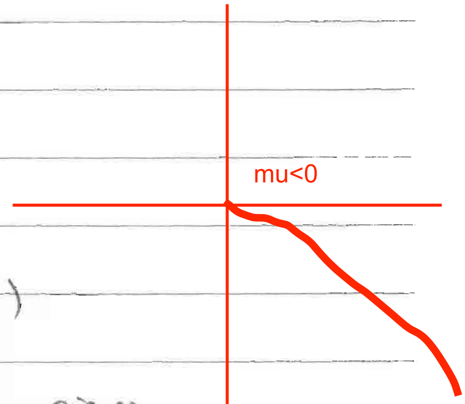


$\mu < \mu_c$



$\mu > \mu_c$




+ generic example:

$$\left.\begin{aligned} \dot{r} &= \mu r - r^3 \\ \dot{\theta} &= \omega + b r^2 \end{aligned}\right\} \quad (\#)$$

$\mu < 0 \Rightarrow \left\{\begin{matrix}\text{stable}\\ \text{origin}\end{matrix}\right\}$; $\mu > 0 \Rightarrow$ unstable origin.

$\omega$ = frequency at $0 < \mu \ll 1$

$b$ = nonlinear correction to frequency.

+ linearize in cartesian coordinates: $\dot{x} = f(x,y)$; $\dot{y} = g(x,y)$:

$$x = r\cos\theta \; ; y = r\sin\theta . \Rightarrow r^2 = x^2 + y^2$$

$$\Rightarrow \dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \; ; \; \dot{y} = \cdots$$

subst this in (#):

$$\dot{x} = (\mu - [x^2+y^2])x - (\omega + b[x^2+y^2])y$$

$$\Rightarrow \dot{x} = \mu x - \omega y + O(x^3, y^3)$$

$$\dot{y} = \omega x + \mu y + O(x^3, y^3)$$

$$\Rightarrow J = \begin{pmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{pmatrix} = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}$$

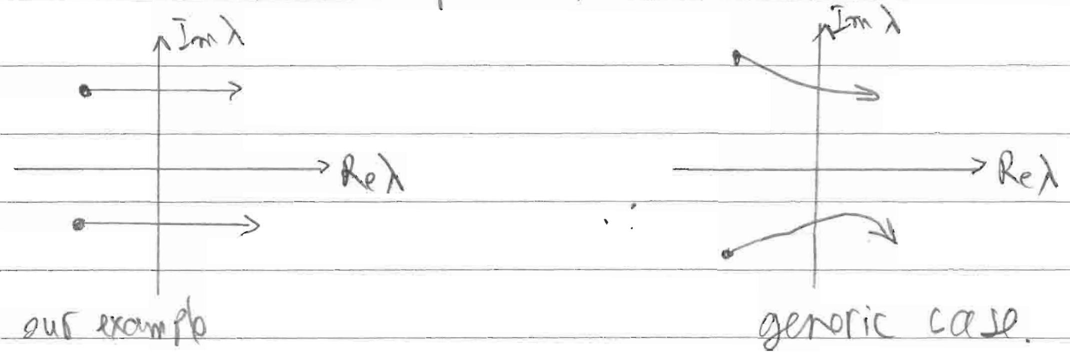
$$|J - \lambda I| = 0 \Rightarrow \underline{\lambda = \mu \pm i\omega}$$

+ conclusions:

1. size of limit cycle is prop to $(\mu-\mu_c)^{1/2}$ for $|\mu-\mu_c| \ll 1$.
2. frequency of limit cycle is $Im(\lambda)$. period is given by $T = 2\pi/Im(\lambda) + O(\mu-\mu_c)$.

+ Also:

1. As $\mu$ varies, $\lambda$ moves horizontally in the complex $\lambda$ plane in this example. In most cases it does not:



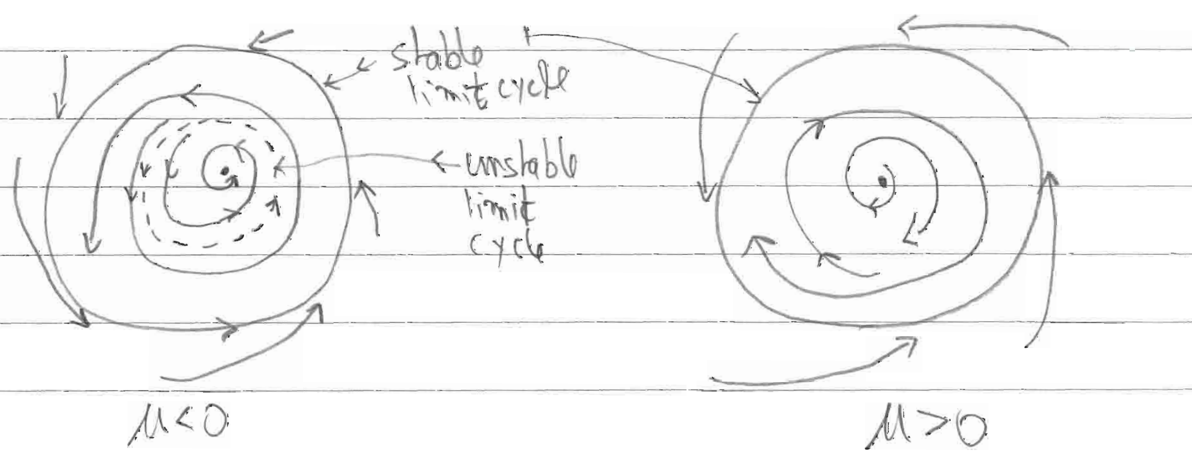

* subcritical Hopf bifurcation:

+ consider $\dot{r} = \mu r + r^3 - r^5$ ; $\dot{\theta} = \omega + b r^2$

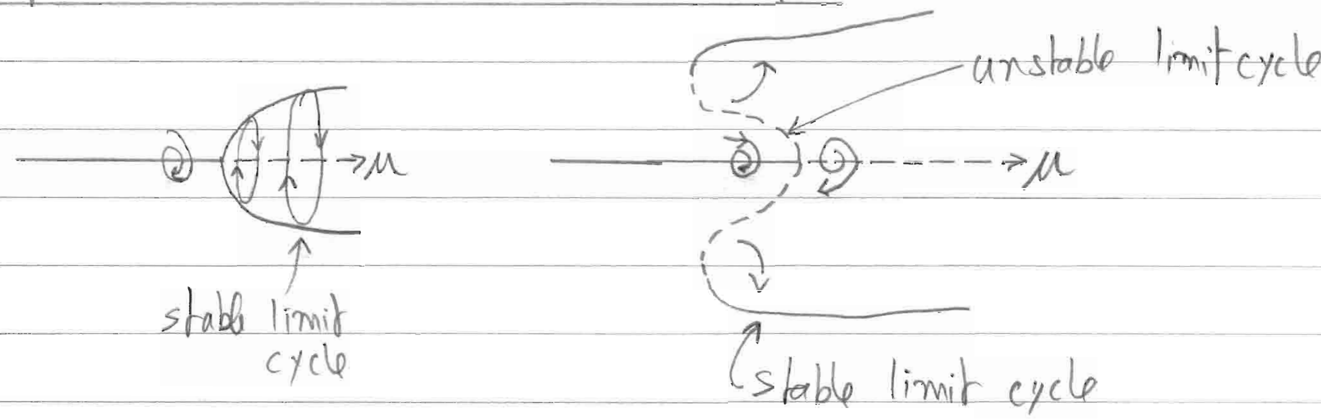
new..

$\Rightarrow$ the $r^3$ term is destabilizing at the origin.

+ for $\mu < 0$: origin is stable, $\mu > 0$: unstable



* super critical vs. subcritical Hopf:




+ Note that the $-r^5$ term in our model system for subcritical Hopf bifurcation is not typical of all such bifurcations. Beyond $\mu_c$, the system jumps to some distant attractor. may be a fixed pt, infinity, another limit cycle (as with $-r^5$), or chaos.

+ Note the hysteresis in the sub-critical case, as evident in above figure.

+ The super- & sub- versions may be differentiated by the appearance of a small, growing like $\sqrt{\mu}$ limit circle in the super-case, vs immediate appearance of finite-amplitude limit cycle in sub-case.