around this reference state and become

$$\frac{\partial u_*}{\partial t_*} - \beta_0 y_* v_* = -g' \frac{\partial h_*}{\partial x_*} \quad (7.7a)$$

$$\frac{\partial v_*}{\partial t_*} + \beta_0 y_* u_* = -g' \frac{\partial h_*}{\partial y_*} \quad (7.7b)$$

$$\frac{\partial h_*}{\partial t_*} + H(\frac{\partial u_*}{\partial x_*} + \frac{\partial v_*}{\partial y_*}) = 0 \quad (7.7c)$$

It is convenient to introduce nondimensional quantities by

$$t_* = \frac{L}{c_o} t \; ; \; x_* = Lx \; ; \; y_* = \lambda_o y \quad (7.8a)$$

$$h_* = Hh \; ; \; u_* = c_o u \; ; \; v_* = \frac{\lambda_o}{L} c_o v \quad (7.8b)$$

Here, L is the zonal basin length, c_o is a shallow water gravity wave speed and λ_o is a characteristic meridional length scale, the equatorial Rossby radius of deformation, given by

$$c_o = \sqrt{g'H} \; ; \; \lambda_o = \sqrt{\frac{c_o}{\beta_0}} \quad (7.9)$$

Using these scales, the dimensionless equations become

$$\frac{\partial u}{\partial t} - yv + \frac{\partial h}{\partial x} = 0 \quad (7.10a)$$

$$\zeta_o^2 \frac{\partial v}{\partial t} + yu + \frac{\partial h}{\partial y} = 0 \quad (7.10b)$$

$$\frac{\partial h}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (7.10c)$$

with $\zeta_o = \lambda_0 / L$.

Travelling wave solutions are sought of the form

$$u(x, y, t) = \hat{u}(y) e^{i(kx - \sigma t)} \quad (7.11a)$$

$$v(x, y, t) = \hat{v}(y) e^{i(kx - \sigma t)} \quad (7.11b)$$

$$h(x, y, t) = \hat{h}(y) e^{i(kx - \sigma t)} \quad (7.11c)$$

with k being the nondimensional wavenumber and σ the angular frequency. The boundary conditions are

$$y \to \pm\infty : \hat{u}, \hat{v}, \hat{h} \to 0 \quad (7.12)$$

The solutions with $\hat{v} \equiv 0$ have a dispersion relation

$$\sigma^2 = k^2 \quad (7.13)$$

and the meridional structure of the wave is

$$\hat{u}(y) = \hat{u}(0) e^{\frac{-ky^2}{2\sigma}} \quad (7.14a)$$

$$\hat{h}(y) = \frac{\sigma}{k} \hat{u}(y) \quad (7.14b)$$

with $\hat{u}(0)$ being an arbitrary amplitude. The solutions which are bounded for $y \to \pm\infty$ exist only when $\sigma = +k$. Hence, the phase velocity of these waves is positive and the waves only move eastward. These are the well-known Kelvin waves with a dimensional wavelength and phasespeed (σ/k) given by

$$\lambda_* = \frac{2\pi L}{k} \; ; \; c_* = c_o \tag{7.15}$$

Patterns of the the thermocline field h of a Kelvin wave are plotted in Fig. 7.10 for four stages during the propagation. The dimensionless wavenumber is chosen $k = \pi$, corresponding to a wavelength of exactly twice the basin $\lambda_* = 2L$. For the Kelvin wave, the dimensionless period $\mathcal{P}$ is $2\pi/\sigma = 2$ and the pictures in Fig. 7.10 are at times $t = 0, t = 1/8, t = 1/4, t = 3/8$, which covers a quarter of a period. The maximum amplitude of the thermocline field for the Kelvin wave is located just at the equator.

Also free wave solutions with $\hat{v} \neq 0$ exist. In (7.10), $\hat{u}$ and $\hat{h}$ can be eliminated (see e.g., Pedlosky (1987), section 8.3) to give a scalar equation for $\hat{v}$, i.e.

$$\hat{v}'' + \hat{v}\left[\zeta_0^{-2}(\sigma^2 - k^2) - \frac{k}{\sigma} - y^2\right] = 0 \tag{7.16}$$

where the $'$ indicates the differentiation to y. Equation (7.16) has only bounded solutions when

$$\zeta_0^{-2}(\sigma^2 - k^2) - \frac{k}{\sigma} = 2j + 1 \tag{7.17}$$

for integers $j = 0, 1, \cdots,$. These solutions are of the form

$$\hat{v}_j(\eta) = \psi_j(y) = \frac{e^{\frac{-y^2}{2}} H_j(y)}{(2^j j! \pi^{1/2})^{1/2}} \tag{7.18}$$

with H_j being the Hermite polynomials and the ψ_j are called the Hermite functions. The first couple of Hermite polynomials are

$$H_0(y) = 1 \quad ; \quad H_1(y) = 2y \tag{7.19a}$$
$$H_2(y) = 4y^2 - 2 \quad ; \quad H_3(y) = 8y^3 - 12y \tag{7.19b}$$

First, we consider the full spectrum by putting $\zeta_o = 1$, which is equivalent to use L also as a meridional length scale. The dispersion relation (7.17) can be written as

$$k = -\frac{1}{2\sigma} \pm \frac{1}{2}\left[(\frac{1}{\sigma} - 2\sigma)^2 - 8j\right]^{1/2} \tag{7.20}$$

For $j > 0$, two real roots exist provided $(1/\sigma - 2\sigma)^2 \geq 8j$ in which case σ satisfies

$$0 < \sigma < \frac{1}{\sqrt{2}}((j+1)^{1/2} - j^{1/2}) \tag{7.21a}$$

or

$$\sigma > \frac{1}{\sqrt{2}}(j^{1/2} + (j+1)^{1/2}) \tag{7.21b}$$

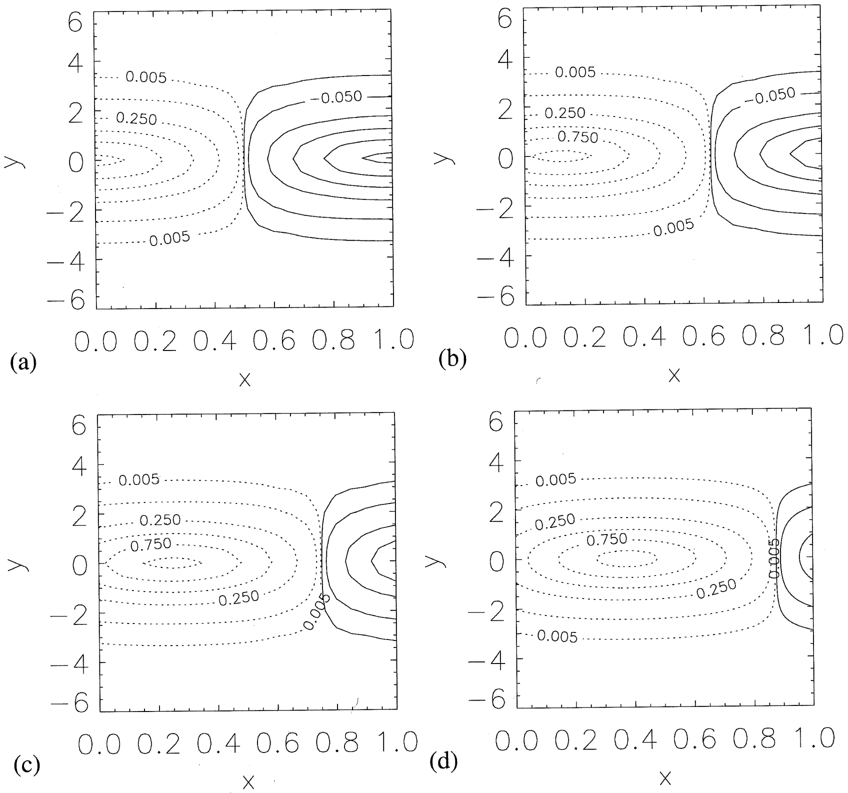

Figure 7.10. *Patterns of the dimensionless thermocline field for the Kelvin wave for four different times during one period $\mathcal{P} = 2$ of evolution (a) $t = 0$ (b) $t = \mathcal{P}/8$, (c) $t = \mathcal{P}/4$ and $t = 3\mathcal{P}/8$. The wavenumber $k = \pi$ and plotted is $\psi_0(y)cos(\pi(x-t))/\sqrt{2}$, where ψ_0 is the Hermite function in (7.18). Note that x and y are scaled according to (7.8).*

The first interval of σ is in the low frequency range and the waves are called equatorial Rossby waves. The second interval represents the high frequency so-called 'inertia-gravity' waves.

For the case $j = 0$, two roots are found from (7.20), the first one being $\sigma = -k$ which leads to a westward travelling Kelvin wave which becomes unbounded far from the equator. The second root is

$$k = -\frac{1}{\sigma} + \sigma \qquad (7.22)$$

which gives a bounded wave called the Yanai wave. For large σ, the character of the wave becomes Kelvin like, whereas for small σ it becomes Rossby like. A classical picture of the dispersion relation for the Kelvin wave, the Yanai wave and $j = 1$ Rossby and inertia-gravity waves is plotted in Fig. 7.11. The Yanai and

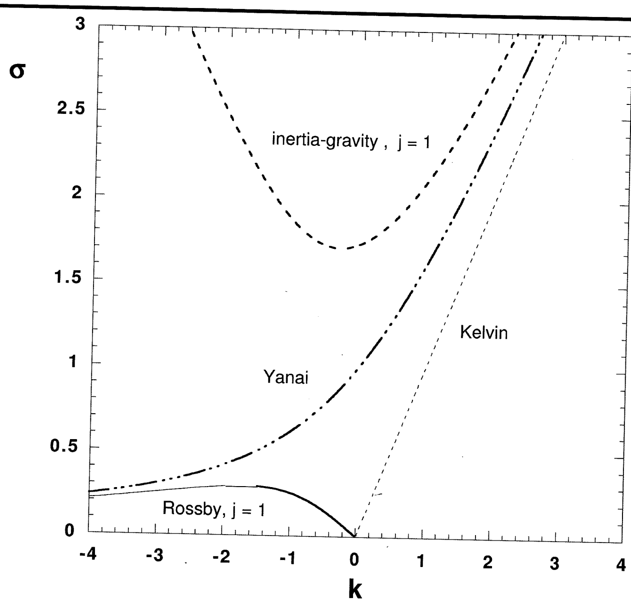

Figure 7.11. Dispersion relation of equatorial free waves. Shown are the Kelvin wave, the Yanai wave and the $j = 1$ Rossby wave.

Kelvin waves have a positive group velocity and for inertia-gravity and Rossby waves, the group velocity c_g becomes

$$c_g = \frac{\partial \sigma}{\partial k} = \frac{1 + 2\sigma k}{2\sigma^2 + \frac{k}{\sigma}} \tag{7.23}$$

For long, low frequency Rossby waves (Fig. 7.11), the group velocity is negative and the approximate dispersion relation is (note that both $k^2 << 1$ and $\sigma^2 << 1$)

$$\sigma = -\frac{k}{2j+1} \tag{7.24}$$

Their dimensional phase velocity is given by

$$c_* = -\frac{c_o}{2j+1}$$

and depends the meridional wavenumber j. These long waves only remain as non-dispersive waves (in addition to the Kelvin wave) in the limit $\zeta_o \to 0$, which can be immediately concluded from (7.17). This limit is therefore called the long wave limit. The first long Rossby wave (j = 1) travels westward with a phase velocity which is 1/3 of that of the Kelvin wave. From the expressions of the Hermite functions in (7.18), one can see that the amplitude is restricted to a relatively small meridional interval around the equator; these waves are therefore called "equatorially trapped".

Patterns of the thermocline field for the $j = 1$ Rossby wave, with again a dimensionless wavenumber $k = \pi$, are plotted Fig. 7.12 for four stages during the propagation. The dimensionless period of the $j = 1$ Rossby wave is $\mathcal{P} = 6$, and the pictures are shown at $t = 0, t = 3/8, t = 3/4, t = 9/8$, which again covers a quarter of the period. The maximum amplitude of the j = 1 Rossby wave is

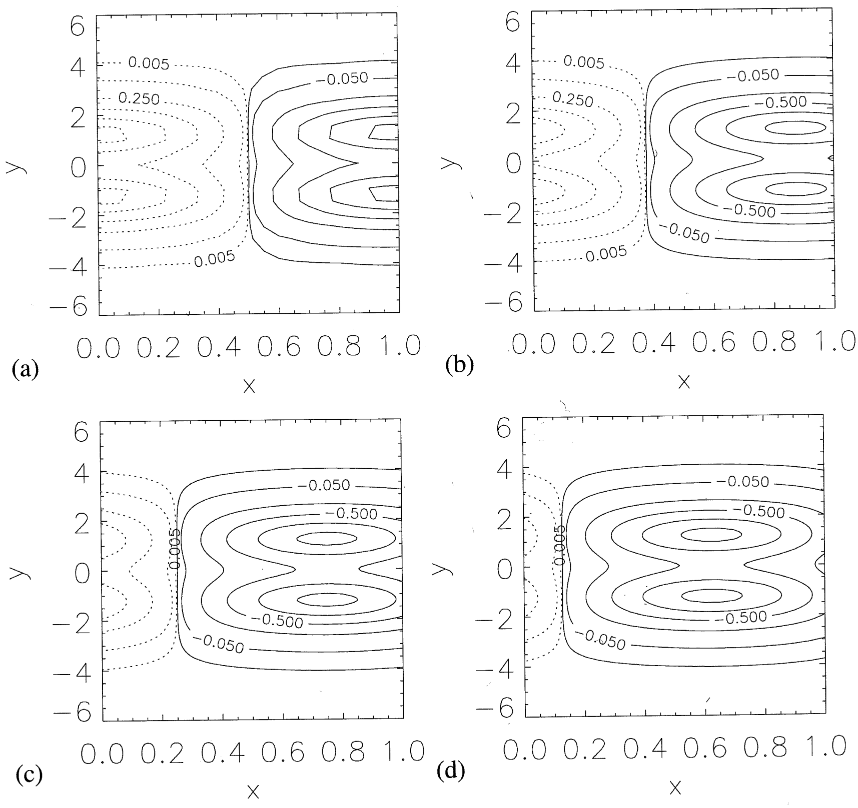

Figure 7.12. *Patterns of the thermocline field h of the $j = 1$ Rossby wave for four different times during one period $\mathcal{P} = 6$ of evolution (a) $t = 0$ (b) $t = 3/8$, (c) $t = 3/4$ and (d) $t = 9/8$. The wavenumber k is equal to π and plotted is $(\psi_0(y) + \psi_2(y)/\sqrt{2})cos(\pi(x - t)))/(2\sqrt{2})$, where ψ_0 and ψ_2 are Hermite functions as in (7.18).*

off-equatorial and at about $1.33 \times \lambda_o$ from the equator and this distance increases (Fig. 7.13) for higher Rossby waves, i.e. larger j. For c_o = 2 m/s, the dimensional

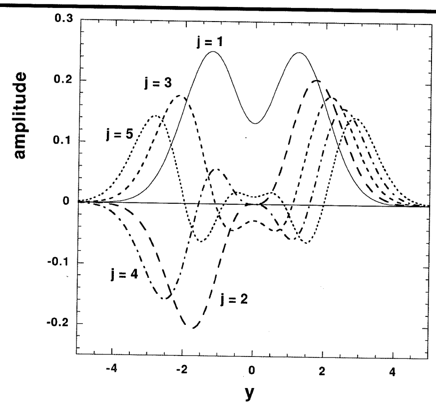

Figure 7.13. Meridional structure of the dimensionless thermocline field $\hat{h}_j$ in (7.31b) associated with the first 5 (long) free Rossby waves (j = 1, ..., 5).

values of the meridional locations at which Rossby wave thermocline amplitudes have their maximum are shown in Table 7.1 for the long waves with j = 1, ..., 5 together with dimensional crossing times for a basin of 15,000 km.

Wave type	y_{max}	θ_{max}	τ_c (days)
Kelvin	0.0	0.0	87
Rossby, $j = 1$	1.22	± 3.31	260
Rossby, $j = 2$	1.75	± 4.75	434
Rossby, $j = 3$	2.17	± 5.88	608
Rossby, $j = 4$	2.50	± 6.77	781
Rossby, $j = 5$	2.83	± 7.67	955

Table 7.1. Typical quantities of free equatorial waves for $c_o = 2\ m/s$, and $\beta_0 = 2.2 \times 10^{-11}(ms)^{-1}$, such that $\lambda_o = 301.5\ km$. The dimensionless quantity y_{max} is the position of the maximum amplitude of the thermocline depth as seen in Fig. 7.13; θ_{max} is the latitude of this position. The travel time is based on the time it takes for the wave to cross a basin of 15,000 km. The Kelvin wave travels from west to east whereas all Rossby waves travel from east to west.

7.2.4. Forced response in a basin

Using the model derived in the previous section, the changes in the ocean circulation in a finite basin due to the presence of a prescribed wind stress are considered next. Under limitations of small amplitude forced motion, the shallow water model can be linearized around a motionless reference state with constant thermocline depth H. Small amplitude zonal winds are assumed to be present, while the meridional component of the wind is neglected. A further simplification