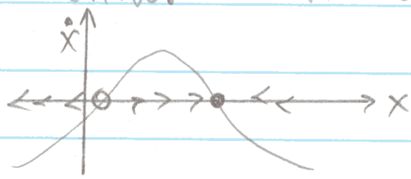
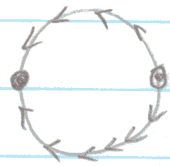
Flows on the circle

* impossibility of oscillations in 1d systems:
In a system in 1d $\dot{x}=f(x)$, the flows are either approaching a f.p. or ∞. cannot overshoot a f.p & thus cannot have oscillations around the f.p



The simplest systems in which oscillations are possible are flows on a circle $\dot{\theta}=f(\theta)$

example: $\dot{\theta}=\sin(\theta)$

fixed pts are: $\theta^*=\pi$  $\theta^*=0$

example: uniform oscillator: $\dot{\theta}=\omega$

$$\Rightarrow \theta(t)=\omega(t)+\theta_0$$

*→ clearly $f(\theta)$ must be 2π periodic or $\dot{\theta}$ is not uniquely defined (e.g $\dot{\theta}=\theta$ is wrong)

phase difference, beat phenomena:

consider $\dot{\theta}_1=\omega_1$, $\dot{\theta}_2=\omega_2$ } two oscillators $\omega_1=2\pi/T_1$, $\omega_2=2\pi/T_2$

phase difference is $\phi=\theta_1-\theta_2$

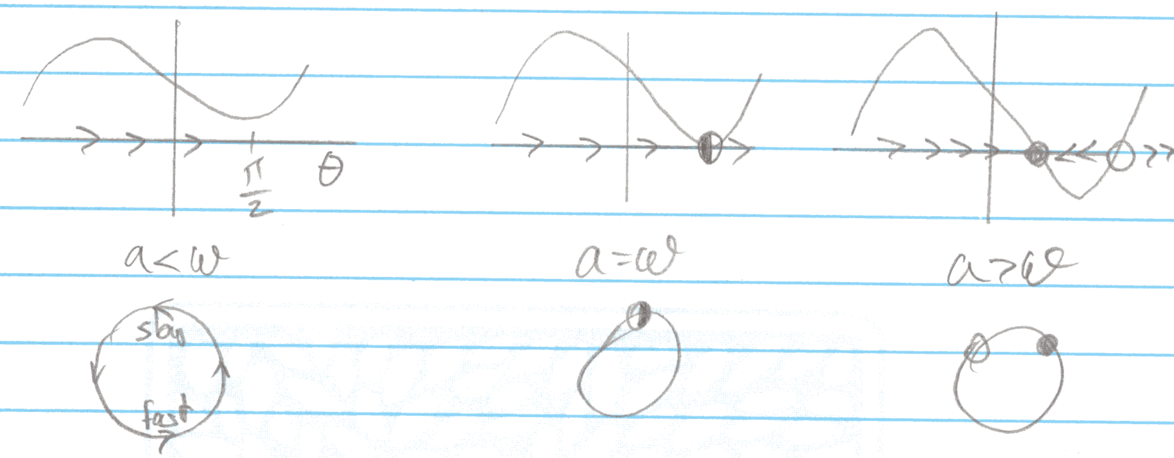
$\Rightarrow \dot{\phi}=\omega_1-\omega_2$, phases increases by 2π after a time $2\pi/(\omega_1-\omega_2)=\left(\frac{1}{T_1}-\frac{1}{T_2}\right)^{-1}$

so the closer are the periods, the longer is the beat period. this example corresponds to two non interacting oscillators. it gets more interesting when they do interact.

nonuniform oscillator:

$$\cancel{\ddot\theta} \qquad \dot\theta = \omega - a\sin\theta$$

neglected acceleration — friction — constant torque — gravity

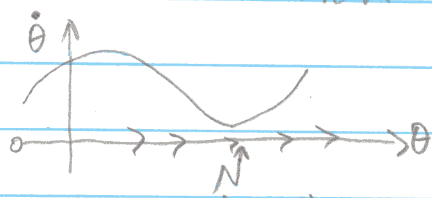


period: $T = \int_{\text{period}} dt = \int_0^{2\pi} \frac{dt}{d\theta}\, d\theta = \int_0^{2\pi} \frac{d\theta}{\omega - a\sin\theta} = \dots = \frac{2\pi}{\sqrt{\omega^2 - a^2}}$

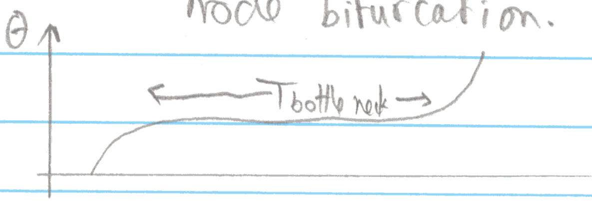
$$\Rightarrow T = \frac{2\pi}{\sqrt{\omega + a}\,\sqrt{\omega - a}} \underset{\text{for } \omega \approx a}{\approx} \frac{2\pi}{\sqrt{2\omega}} \frac{1}{\sqrt{\omega - a}}$$

= square root scaling law

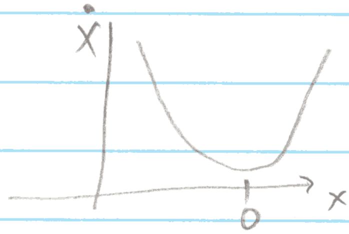
square root scaling is typical of systems near a saddle-node bifurcation: in a case like this:



slow flow, bottle neck, "ghost" of saddle node bifurcation.



consider $\dot{x} = r + x^2$



the time to pass via the bottle neck is $\int_{-\infty}^{\infty} \frac{dx}{r+x^2} = \frac{\pi}{\sqrt{r}}$ again square root law.

synchronization, nonlinear phase locking:

example: Fireflies (Strogatz)

fireflies tend to flash on & of together. why? how? Also, given an external periodic light, fireflies try to follow it, & may not be able to do it if it's too fast & then try again...

a specific model: let Θ be the phase of an external stimulation (flash light) of a constant rate:

$\dot{\Theta} = \Omega$. (flash occurs at $\Theta = 0$).

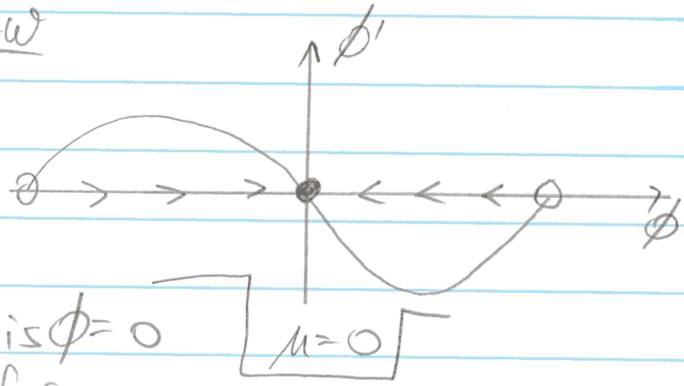
The fire fly increases/decreases it's rate of flashing depending on phase difference from external source:

$$\dot{\theta} = \underbrace{\omega}_{\text{natural rate for fireflies}} + \underbrace{A\sin(\Theta - \theta)}_{\text{adjustment to stimulus.}}$$

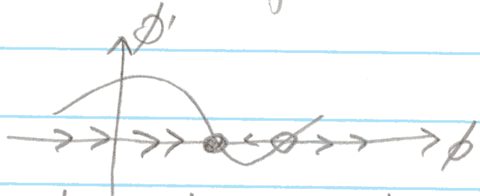
The phase diference between the stimulus & the firefly satisfies $\dot{\phi} = \dot{\Theta} - \dot{\theta} = \Omega - \omega - A\sin\phi$

let $\tau = At$, $\mu = \frac{\Omega - \omega}{A}$

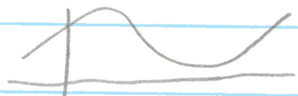
$\Rightarrow \phi' \equiv \frac{d\phi}{d\tau} = \mu - \sin\phi$



* $\mu = 0$ $\Rightarrow$ fixed pt is $\phi = 0$
$\Rightarrow$ no phase difference between flashlight & firefly. ($\mu = 0$ means natural firefly freq $\equiv$ external freq $\Omega = \omega$).



* $0 < \mu < 1$
$\Rightarrow$ a constant phase lag, fireflies are "phase locked" to flashlight.



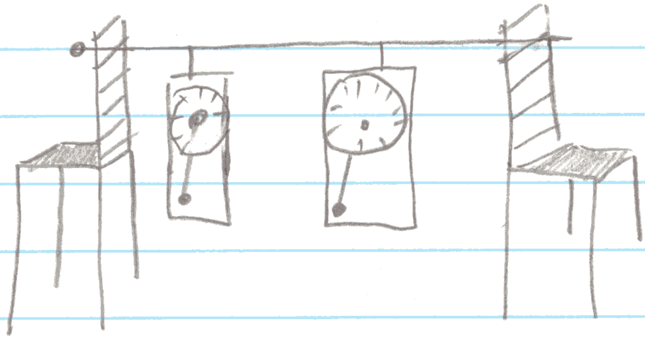
* $1 < \mu$
$\Rightarrow$ phase difference $\phi = \Theta - \theta$ varies.
$\Rightarrow$ "phase drift."

note that phase locking: (1) occurs because the nonlinear firefly oscillator can change it's period to fit the external one (Ω); (2) occurs over a specific range of parameters: $\omega - A < \Omega < \omega + A$
= "range of entrainment".
(a linear oscillator cannot change its period!!)

DATE

Another example of mode locking (= phase locking = nonlinear resonance)

(1) 17th century Dutch physicist Huyghens:
synchronisation between clocks:



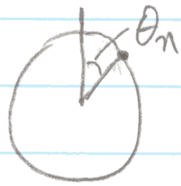
(2) Glacial cycles & Milankovitch forcing

...

Another view at phase locking: The circle map

[Schuster, chapter 6.2]

$$\theta_{n+1} = f(\theta_n) = \theta_n + \Omega + \frac{K}{2\pi}\sin(2\pi\theta_n) \quad \underline{[\text{mod } 1]}!$$



$\Rightarrow$ a simple model for a periodically forced oscillator:

Ω = forcing

$\frac{K}{2\pi}\sin(2\pi\theta_n)$ "=" gravity, nonlinear

$$\left[mr\ddot{\theta} = -b\dot{\theta} + mg\sin\theta + c\sin(\Omega \cdot t)\right]$$

when $K=0$, θ_n rotates uniformly;

* if Ω is rational, θ_n is periodic

e.g. $\Omega = \frac{1}{2} \Rightarrow$ period 2 $\theta_n = 0, \frac{1}{2}, 0, \frac{1}{2} \ldots$

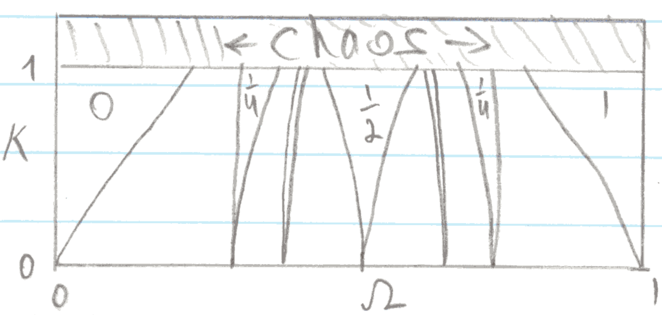
$\Omega = \frac{3}{4} \Rightarrow \theta_n = 0, \frac{3}{4}, 1\frac{1}{2}(=\frac{1}{2}), 1\frac{1}{4}(=\frac{1}{4}), 1(=0)$.

$\Rightarrow$ period 4, 3 rotations per period.

* if Ω is irrational, θ_n never repeats, & eventually covers all points on the circle.

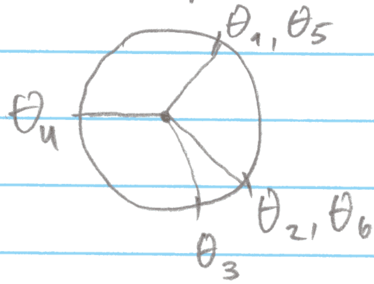
$\Rightarrow$ "quasi-periodic"

when $K>0$ (nonlinear regime), can have periodic solutions even for irrational Ω



Arnolds tongues

If we choose K, Ω values within the $\frac{1}{4}$ tongue, for example, then any initial conditions θ_0 will eventually converge to a specific 4-period solution, such as

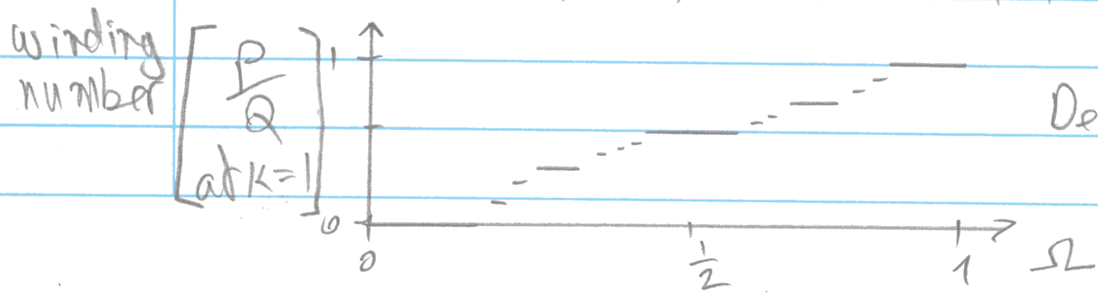


$\Rightarrow$ unequal jumps
(due to nonlinearity), but still 4-periodic.

* There are ∞ Arnold tongues, each corresponding to a ^nonlinear^ resonance between the forcing Ω & the nonlinear oscillator, each for a different rational P/Q ["Winding #" $\equiv \frac{\theta^n - \theta_0}{n}$ $(n \to \infty)$]
* [describe nonlinear resonance for an actual pendulum forced by periodic forcing, vs a linear resonance.]

<u>Devil's staircase</u>

At $K=1$, the tongues of periodic solutions cover the entire $\Omega = [0,1]$ interval, besides a fractal set of dimension <1 $\Rightarrow$ zero total length.



Devil's staircase!

Farey tree for the Devil's staircase

* note that width of steps is smaller if the denominator in the winding # of each step is larger.
* also, given two steps $\frac{p}{q}, \frac{p'}{q'}$ the largest step in between is $(p+p')/(q+q')$. this is because this is the rational # with the largest denominator between p/q, p'/q'.

[examples: $0/1 < 1/2 < 1/1$
$1/2 < 2/3 < 1/1$
$1/2 < 3/5 < 2/3$]

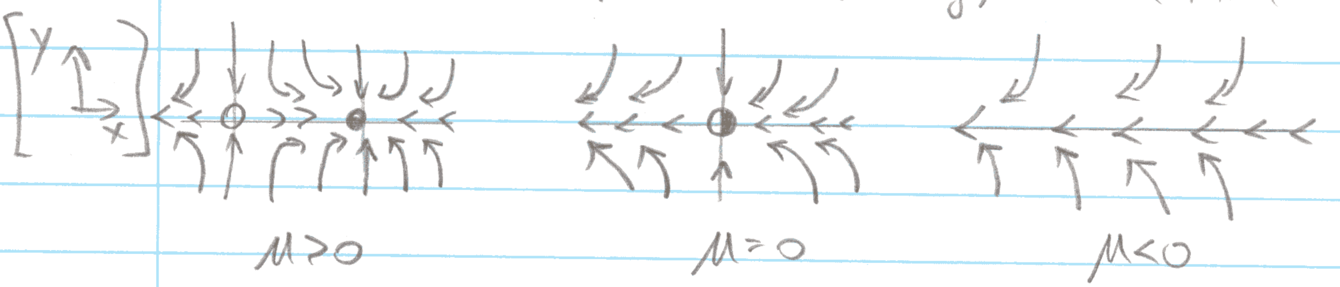
$\Rightarrow$ can order the steps using a Farey tree, which orders rationals $\frac{p}{q}$ by denominator q:

Center manifold theory [time permitting...]

It's important to understand that the saddle-node, transcritical & pitchfork bif's can occure in larger dimensional dynamical systems
$\dot{x} = f(x,\mu)$, $x \in \mathbb{R}^n$. when this occurs, there is a systematic theory for how to reduce the n-dim system into an equivalen 1-d system via a nonlinear change of coordinates. to get a feeling for this: consider a 2-d system:

$$\dot{x} = \mu - x^2 \quad (*)$$
$$\dot{y} = -y$$

behavior in y is very simple: $y(t) \to 0$ as $t \to \infty$.
behavior in x depends on μ; e.g. $\mu > 0 \Rightarrow (x^*, y^*) = (\pm\sqrt{\mu}, 0)$

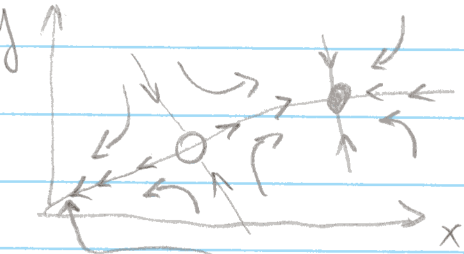


* the x-axis is the "Center manifold" where all the interesting stuff happens. note that it is "invariant", a solution with $y(t_0) = 0$ will remain on the x-axis for all t. y is a "stable manifold". there can also be "unstable manifold".
* Center manifold theory is a systematic method for finding such an invariant center manifold on which the interesting things occur...

example (strogatz p.243)

$$\dot{x} = -ax + y$$
$$\dot{y} = \frac{x^2}{1+x^2} - by \qquad (\#\#)$$

can show that phase space behavior is like this



[and this picture varies like on page 85 when a,b are varied...]

so that along the line, behavior is like on x-axis in simpler previous example. thus this line is the "center manifold."

* mathematically, center manifold is parallel to eigenvectors of Jacobian with zero real part of eigenvalue. more on this later...
* stable manifold is spanned by eigenvectors with negative real part of eigenvalues
* unstable manifold: by ... positive real part...

⇒ the simpler 3 1d normal forms we have discussed last time are generic & occur in larger dimensional systems as well

* center manifold theorem provides the recipie for transforming (##) to a different coordinate system in which one get a 1d eq'n on the center manifold, equivalent to * p.85 via nonlin. transfor.

normal forms:

once we have reduced the dynamics to the center manifold (say a 1d center manifold if bifurcation is one of the 3 we have discussed) the eq'n may still be complicated. normal form theorem gives us a recepie for transforming (again, a nonlinear transformation) to the standard normal forms we have discussed.

example (strogatz p. 52)

$$\dot{x} = r \ln x + x - 1$$

$x=1$ is fixed pt. shift problem by letting $u \equiv x-1$

$$\Rightarrow \dot{u} = r \ln(1+u) + u$$
$$= r\cdot\{u - \tfrac{1}{2}u^2 + O(u^3)\} + u$$
$$= \underbrace{(r+1)u - \tfrac{1}{2}ru^2}_{\text{looks like transcritical bif.}} + O(u^3)$$

let $u = a\cdot v$ where v is the new variable, subst in eq'n for u:

$$\dot{v} = (r+1)v - (\tfrac{1}{2}ra)v^2 + O(v^3)$$

let $R = r+1$, $a = 2/r \Rightarrow \dot{v} = Rv - v^2 + O(v^3)$

$\Rightarrow$ indeed transcritical.

more normal form: getting rid of $O(v^3)$

normal form theorem tells us we can transform to normal form up to arbitrary accuracy, not just to $O(v^3)$. to see how example, strogatz p. 80-81

* spse we have $\left\{ \dot{X} = RX - X^2 + a_n X^n + O(X^{n+1}) \right.$ $(n \geq 3)$, (0) & we want to eliminate the X^n term to improve accuracy of this approximation to the normal form $\dot{x} = rx - x^2$.

* use "near identity" transformation:

$$x = X + \underbrace{b_n X^n}_{\text{small correction}} + O(X^{n+1}) \qquad (1)$$

[note: this is not correct. The transformation here leads to an expression that is singular at R=0. need to include terms such as R*X etc. see notes and link in the lectures directory

* invert transformation: write $X = x + c_n x^n + O(x^{n+1})$ (2),

subst (2) in (1) & find that

$$x = [x + c_n x^n] + b_n [x + c_n x^n]^n + O(x^{n+1})$$

$$= x + (c_n + b_n) x^n + O(x^{n+1})$$

$$\Rightarrow c_n = -b_n$$

$$\Rightarrow X \approx x - b_n x^n$$

* Let now $\left[\frac{d}{dt} \text{ of } (1)\right] \Rightarrow \dot{x} = \dot{X} + n \cdot b_n \cdot X^{n-1} \cdot \dot{X}$

$$\Rightarrow \dot{x} = \dot{X} \cdot (1 + n \cdot b_n X^{n-1})$$

$$= (RX - X^2 + a_n X^n)(1 + n b_n X^{n-1})$$

$$= (R[x - b_n x^n] - [x - b_n x^n]^2 + a_n [x - b_n x^n]^n) * (1 + n b_n [x - b_n x^n]^{n-1})$$

* collect all terms in x^n, & choose b_n such that these terms vanish & we have

$$\dot{x} = Rx - x^2 + O(x^{n+1})$$

$\Rightarrow$ improved approx to normal form by one power... can proceed same again...