Notes for climate dynamics course (EPS 231)

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Contents

1 Introduction

This is an evolving detailed syllabus of EPS 231, see course web page, all course materials are available under the downloads directory. If accessing from outside campus or via the university wireless network, you will need to connect via the Harvard VPN.

Homework assignments every 9–10 days are 50% of the final grade, and a final course project constitutes the remaining 50%. There is an option to take this course as a pass/fail with the instructor's approval during the first week of classes. The subject of the final project would be discussed in a couple of individual meetings with students during the semester, and would ideally be related to either climate subjects, modeling approaches, nonlinear dynamics methods or data analysis covered in class, and may be related to the research project of the student. The length of the final report should be some 6–10 pages including a few figures, 12pt, in pdf format, and the expected effort is some 6–8 days of work. Please include an abstract, introduction with the background/ motivation, and methods section including precise details of data sources or model versions/ configuration with relevant links, results, and discussion/ conclusions.

Collaboration policy. We **strongly** encourage you to discuss and work on homework problems with other students and with the teaching staff. Of course, after discussions with peers, you need to work through the problems yourself and ensure that any answers you submit for evaluation are the result of your own efforts, reflect your own understanding and are written in your own words. In the case of assignments requiring programming, you need to write and use your own code. Please appropriately cite any books, articles, websites, lectures, etc that have helped you with your work.

2 Basics, energy balance, multiple climate equilibria

Downloads available here.

2.1 Multiple equilibria, climate stability, greenhouse

1. `energy_balance_0d.pdf` with the graphical solutions of the steady state solution to the equation $CT_t = (Q/4)(1 - \alpha(T)) - \epsilon\sigma T^4$ obtained using energy_balance_0d.m, and then the Quicktime animation of the bifurcation behavior.

2. Some nonlinear dynamics background: saddle-node bifurcation (Strogatz (p 45, 1994) or notes), and then the energy balance model as two back-to-back saddle nodes and the resulting hysteresis as the insolation is varied;

3. Climate implications: (1) faint young sun paradox! (2) snowball (snowball obs from Ed Boyle's lecture);

4. The 2-level greenhouse model (notes) and see slides on the mechanism for warming due to the anthropogenic CO_2 increase, based on a change in the emission level.

5. More 1d bifurcations: transcritical and pitchfork (notes).

2.2 (Time permitting:) Small ice cap instability

1. Introduction: The Budyko and Sellers 1d models. Simple diffusive energy balance models produce an abrupt disappearance of polar ice as the global climate gradually warms, and a corresponding hysteresis. The SICI eliminates polar ice caps smaller than a critical size (18 deg from the pole) determined by heat diffusion and radiative damping parameters. Below this size, the ice cap is incapable of determining its own climate which then becomes dominated, instead, by heat transport from surrounding regions. (Above wording from Winton (2006), teaching notes below based on North et al. (1981)).

2. Derive the 1d energy balance model for the diffusive and Budyko versions (Eqns 22 and 33): Assume the ice cap extends where the temperature is less than -10C, the ice-free areas have an absorption (one minus albedo) of $a(x) = a_f$, and in the ice-covered areas $a(x) = a_i$; approximate the latitudinal structure of the annual mean insolation as a function of latitude using $S(x) = 1 + S_2 P_2(x)$, with $P_2(x) = (3x^2 - 1)/3$ being the second Legendre polynomial (HW); add the transport term represented by diffusion $(r^2 \cos\theta)^{-1}\partial/\partial\theta(D\cos\theta\partial T/\partial\theta)$, which, using $x = \sin\theta$, $1 - x^2 = \cos^2\theta$ and $d/dx = (1/\cos\theta)d/d\theta$ gives the result in the final steady state 1D equation (22),

$$-\frac{d}{dx}D(1-x^2)\frac{dT(x)}{dx} + A + BT(x) = QS(x)a(x, x_0).$$

as a boundary condition, use the symmetry condition that $dT/dx = 0$ at the equator, leading to only the even Legendre polynomials.

3. (Time permitting:) Alternatively to the above, we could model the transport term a-la Budyko as $\gamma[T(x) - T_0]$ with T_0 being the temperature averaged over all latitudes, and the temperature can then be solved analytically (HW).

4. How the steady-state solution is calculated: eqns 22-29; then 15 and 37, the equation just before 37 and the 4 lines in the paragraph before these equations. The result is Fig. 8: plot of Q (solar intensity) as a function of x_s (edge of ice cap). Analysis of the results: see highlighted Fig. 8 in the Sources directory, unstable small ice cap (which cannot sustain its own climate against heat diffusion from mid-latitudes), unstable very-large ice cap (which is too efficient at creating its own cold global climate and grows to a snowball), and stable mid-size cap (where we are now).

5. Heuristic explanation of SICI: Compare Figs 6 and 8 in North et al. (1981), SICI appears only when diffusion is present. It is therefore due to the above-mentioned mechanism: competition between diffusion and radiation. To find the scale of the small-cap: it survives as long as the radiative effect dominates diffusion: $BT \sim DT/L^2$, implying that $L \sim \sqrt{D/B}$. Units: $[D]$ =watts/(degree K×m^2) (page 96), $[B]$ =m^2/sec (p 93), so that $[L]$ is non dimensional. That's fine because it is in units of sine of the latitude. Size comes out around 20 degrees from the pole, roughly the size of present-day sea ice!

6. The important lesson(!): p 95 in North et al. (1981), left column second paragraph, apologizing for the (correct...) prediction of a snowball state.

7. This is a complex PDE (infinite number of degrees of freedom), displaying a simple bifurcation structure. In such a case we are guaranteed by the central manifold and normal form theorems that it can be transformed to the normal form of a saddle-node bifurcation near the appropriate place in parameter space. First, transform to the center manifold and get an equation independent of stable and unstable manifolds (first page of lecture_04_cntr_mnfld.pdf); next, transform to normal form within the center manifold (lecture_03_bif1d2.pdf, p 89).

8. (Time permitting:) Numerically calculated hysteresis in 1D Budyko and Sellers models: figures obtained using the code here.

9. (Time permitting:) One noteworthy difference between Budyko and Sellers is the transient behavior, with Budyko damping all scales at the same rate, and Sellers being scale-selective. (original references are Budyko, 1969; Sellers, 1969).

3 ENSO

Downloads available here.

3.1 ENSO background and equatorial waves

Sources: Woods Hole (WH) notes (Cessi et al., 2001), lectures 0, 1 here.

3.2 Delay oscillator model

Sources: Woods Hole (WH) notes (Cessi et al., 2001), lecture 2, here, plus the following: Gill's atmospheric model solution from Dijkstra (2000) technical box 7.2 p 347; recharge oscillator from Jin (1997) (section 2, possibly also section 3);

- The climatological background: easterlies, walker circulation, warm pool and cold tongue, thermocline slope (slides, and lecture 1 from WH notes).
- Dynamical basics (WH lecture 1): reduced gravity equations on an equatorial beta plane. Equatorial Rossby and Kelvin waves, thermocline slope is set by a balance between wind stress and pressure gradient, SST dynamics, atmospheric heating and wind response to SST from Gill's model. The coupled ocean-atmosphere feedback.
- The heuristic delayed oscillator equation from section 2.1 in WH notes. One detail to note regarding how do we transition from $+\hat{b}h_{\text{off}-\text{eq}}(t-[\frac{1}{2}\tau_R+\tau_K])$ to $-\bar{b}\tau_{eq}(t-[\frac{1}{2}\tau_R+\tau_K])$ and then to $-bT(t-[\frac{1}{2}\tau_R+\tau_K])$: $h_{\text{off}-\text{eq}}$ depends on the Ekman pumping off the equator. In the northern hemisphere, if the wind curl is positive, the Ekman pumping is positive, upward ($w_E = \text{curl}(\vec{\tau}/f)/\rho$), and the induced thermocline depth anomaly is, therefore, negative (a shallowing signal). The wind curl may be approximated in terms of the equatorial wind only (larger than the off-equatorial wind), consider the northern hemisphere:

  $$h_{\text{off}-\text{eq}} \propto -w^{Ekman}_{\text{off}-\text{eq}} \propto -\text{curl}(\tau_{\text{off}-\text{eq}}) \approx \partial_y \tau^{(x)}_{\text{off}-\text{eq}} \approx (\tau^{(x)}_{\text{off}-\text{eq}} - \tau^{(x)}_{\text{eq}})/L \propto -\tau^{(x)}_{\text{eq}}.$$

  Finally, as the east Pacific temperature is increasing, the wind anomaly in the central Pacific is westerly (positive), leading to the minus sign in front of the T term,

  $$-\tau^{(x)}_{\text{eq}}(t-[\frac{1}{2}\tau_R+\tau_K])/L \propto -T(t-[\frac{1}{2}\tau_R+\tau_K]).$$

- Next, the linearized stability analysis of the Schopf-Suarez delayed oscillator from the WH notes section 2.1.1. The dispersion relation in the WH notes is $\sigma = 1 - 3\bar{T}^2 - \alpha\exp(-\sigma\delta)$. Its real part is $0 = \sigma_R - (1 - 3\bar{T}^2 - \alpha\exp(-\sigma_R\delta)\cos(-\sigma_I\delta))$, and its imaginary part is $0 = \sigma_I - (-\alpha\exp(-\sigma_R\delta)\sin(-\sigma_I\delta))$. The slides shows the Suarez and Schopf (1988) analysis, with an example with two unstable roots and many stable ones. Also shown are time series of a numerical solution of this model for values on both sides of the first bifurcation with damped self-sustained variability point, obtained using delay_Schopf_Suarez_1989.m.

- Self-sustained vs damped: nonlinear damping term and the proximity of ENSO to the first (Hopf) bifurcation point beyond steady-state, discuss Hopf bifurcation using notes or Strogatz (1994).
- (Time permitting:) A more quantitative derivation of delay oscillator, starting from the shallow water equations and using Jin's two-strip approximation (WH notes, section 2.2).

3.3 ENSO's irregularity

3.3.1 Chaos

- ENSO phase locking to seasonal cycle: flows on a circle, synchronization/ phase locking, Huygens clocks, firefly and flashlight example from Strogatz. Connection to ENSO and the seasonal cycle.
- ENSO irregularity as chaos driven by the seasonal cycle: circle map and quasi-periodicity route to chaos:

$$\theta_{n+1} = f(\theta_n) = \theta_n + \Omega - \frac{K}{2\pi}\sin 2\pi\theta_n, \quad \theta_n = mod(1)$$

  $K = 0$ and quasi-periodicity, $0 < K < 1$ and phase locking, Arnold tongues; $K = 1$ and the devil's staircase. $K > 1$, overlapping of resonances and chaos. A generalization from the circle map to a pendulum and to the delayed oscillator driven by the seasonal cycle. (Time permitting:) winding number: $\lim_{n\to\infty}(f^n(\theta_0) - \theta_0)$, not taking θ_n as mod(1) for this calculation, Farey tree

- (Time permitting:) Some generalities on identifying the quasi-periodicity route to chaos in a complex system, including delay coordinate phase space reconstruction.
- References for phase locking: Strogatz (1994), for quasi-periodicity route to chaos: Schuster (1989). For both: course notes for Applied Math 203. Delay coordinate phase space reconstruction in here.
- (Time permitting:) Slides on the transition to chaos in the CZ model.

3.3.2 Noise

- Non-normal amplification and optimal i.c.; Stochastic optimals: mathematical details are covered below in the context of AMOC/THC variability. Here, introduce the concept without derivation.

- WWBs in observations: seem stochastic, seen to precede each El Niño event, affect Pacific by forcing of equatorial Kelvin waves. Show Hovmoller diagram with WWBs and SST from Yu et al. (2003); wind stress sequence showing WWB evolution from Vecchi and Harrison (1997); effects of wind bursts on SST and thermocline depth (heat content) (Mcphaden and Yu, 1999); ocean-only model response to a strong WWB: Zhang and Rothstein (1998);

- Discuss: Do WWBs look like stochastic optimals/optimal initial conditions? (Moore and Kleeman, 2001). Then: Are WWBs actually stochastic, or are their statistics a strong function of the SST so that they are triggered by warmer conditions and therefore only amplify already initiated El Niño events, rather than initiating them (Tziperman and Yu, 2007).

3.4 ENSO teleconnections

1. Motivation for ENSO teleconnection: worldwide weather effects.

2. Further motivation: results of barotropic model runs from Hoskins and Karoly (1981), showing global propagation of waves due to tropical disturbances.

3. General ray tracing theory based on notes-ray-tracing.pdf. Note that this is a non-rigorous derivation, not using multiple-scale analysis.

4. Dispersion relation Rossby wave in presence of a mean zonal flow. In Cartesian coordinates, then mention HK1981 paper which uses the same equation, but in spherical coordinates using Mercator projection.

5. Qualitative discussion of Rossby ray propagation based on dispersion relation (Hoskins and Karoly, 1981).

6. Why the ray is reflected rather than being trapped at the critical latitude

7. (Time permitting:) Amplitude of the propagating Rossby waves using WKB solution (Bender and Orszag (1978) section 10.1; Hoskins and Karoly (1981) equation (5.21, 5.23).

8. Show rays for constant angular momentum flow

9. Finally, later works showed that stationary linear barotropic Rossby waves excite non-linear eddy effects which may eventually dominate the teleconnection effects.

10. Baroclinic atmospheric waves are not effective at producing ENSO teleconnections because they are more easily trapped at the equator within a scale of the Rossby radius of deformation, 1000 km or so.

3.5 ENSO diversity

See downloads folder here.

4 Meridional overturning circulation

Downloads available here.

4.1 Phenomenology, Stommel box model

- Background, schematics of AMOC, animations of CFCs in the ocean, sections and profiles of T, S from here, meridional mass and heat transport, climate relevance; RAPID measurements (Cunningham et al., 2007); anticipated response during global warming; MOC vs THC.

- Stommel model: mixed boundary conditions, Stommel-Taylor model notes; Qualitative discussion on proximity of present-day AMOC to a stability threshold (Tziperman, 1997; Toggweiler et al., 1996). And again the surprising ability of simple models to predict/ explain GCM results (Fig. 2 from Rahmstorf, 1995). See also (Dijkstra, 2000, section 3.1.1, 3.1.2, 3.1.3).

4.2 Scaling and energetics

1. (Time permitting:) Scaling for the amplitude and depth of the AMOC from Vallis (2006) chapter 15, section 15.1, showing that the AMOC amplitude is a function of the vertical mixing which in turn is due to turbulence. Mention only briefly the issue of the "no turbulence theorem" and the importance of mechanical forcing (details in the next section, not to be covered explicitly).

2. (Time permitting:) Energetics, Sandstrom theorem stating that "heating must occur, on average, at a lower level than the cooling, in order that a steady circulation may be maintained against the regarding effects of friction" (Eqn 15.23 Vallis, 2006). The "no turbulence theorem" in the absence of mechanical forcing by wind and tides (Eqn 15.27) without mechanical mixing, and (15.30) with; hence the importance of mechanical energy/ mechanical forcing for the maintenance of turbulence and of the AMOC. Sections 15.2, 15.3 (much of this material is originally from Paparella and Young, 2002);

3. (Time permitting:) Tidal energy as a source for mixing energy (Munk and Wunsch, 1998, Figures 4,5).

4.3 Convective oscillations

- Advective feedback and convective feedback from sections 6.2.1, 6.2.2 in Dijkstra (2000), both show why we expect temperature and salinity to play independent roles.

- Flip-flop oscillations (Welander1982_flip_flop.m) and loop oscillations (Dijkstra, 2000, section 6.2.3);

- (Time permitting:) Multiple convective equilibria from Lenderink and Haarsma (1994), hysteresis from Fig. 8 in this paper, and "potentially convective" regions in their GCM from Fig. 11. When put in the regime with no steady states, this model shows flip-flop oscillations like Welander's, without the need to prescribe an artificial convection threshold.

- Analysis of relaxation oscillations following Strogatz (1994) example 7.5.1 pages 212-213, or nonlinear dynamics course notes: slow phase and fast phases.

- (Note:) Winton's deep decoupling oscillations are covered below as part of DO events. They are similar to the Welander flip-flop oscillations, except that the convection also affects the overturning circulation leading to strong MOC variability/ relaxation oscillations.

4.4 Stochastically driven MOC variability

A review of classes of AMOC oscillations: small amplitude/ large amplitude; linear and stochastically forced/ nonlinear self-sustained; loop oscillations due to advection around the AMOC path, or periodic switches between convective and non-convective states; relaxation oscillations; noise-induced switches between steady-state, stochastic resonance;

Details of noise-driven AMOC variability:

1. Linear Loop-oscillations due to advection around the circulation path:

   (a) Linearized stability analysis (write equations, explain linearization, writing of linearized equations in matrix form) and bifurcations (section 6.2.4 in Dijkstra (2000) or Figs 3, 4 from Tziperman et al. (1994)); Stability regimes in a 4-box model: stable, stable oscillatory, [Hopf bifurcation], unstable oscillatory, unstable; Note changes from 2-box Stommel model: oscillatory behavior and change to the point of instability on the bifurcation diagram; Stochastic forcing can excite this damped oscillatory variability.

   (b) The GCM study of Delworth et al. (1993) (DMS), Figs. 4, 5, 6, 8; this paper also demonstrates the link between the variability of meridional density gradients and of the AMOC; Note the proposed role of changes to the gyre circulation in this paper, mention related mechanisms based on ocean mid-latitude Rossby wave propagation;

   (c) A box model fit to the DMS GCM, showing that the horizontal gyre variability may not be critical and that the variability is due to the excitation of a damped oscillatory mode (Fig. 6, Griffies and Tziperman, 1995);

   (d) Useful and interesting analysis methods in DMS: composites (Figs. 6,7), and regression analysis between scalar indices (Figs. 8,9) and between scalar indices and fields (Figs. 10, 11, 12).

2. A complementary view of the above stochastic excitation of damped AMOC oscillatory mode: first, Hasselmann's model which is driven by white noise and results in a red spectrum response (to derive the spectrum of the response, apply Fourier transform to the first equation, and multiply the transformed equation by its complex conjugate).

$$\begin{aligned}\dot{x}+\gamma x &= \xi(t)\\ -i\omega\hat{x}+\gamma\hat{x} &= \xi_0\\ P(\omega)=\hat{x}\hat{x}^*=|\hat{x}|^2 &= \xi_0^2/(\omega^2+\gamma^2).\end{aligned}$$

Compare this to a damped oscillatory mode excited by stochastic forcing (noise) that results in a spectral peak, using the following derivation of the spectral response,

$$\begin{aligned}&\ddot{x}+\gamma\dot{x}+\Omega^2x=\xi(t)\\ &-\omega^2\hat{x}-i\gamma\omega\hat{x}+\Omega^2\hat{x}=\xi_0\\ &\hat{x}=\xi_0/(\Omega^2-\omega^2-i\gamma\omega)=\xi_0(\Omega^2-\omega^2+i\gamma\omega)/((\Omega^2-\omega^2)^2+\gamma^2\omega^2)\\ &\hat{x}^*=\xi_0(\Omega^2-\omega^2-i\gamma\omega)/((\Omega^2-\omega^2)^2+\gamma^2\omega^2)\\ P(\omega)=|\hat{x}|^2=\hat{x}\hat{x}^*=&\frac{\xi_0^2}{(\Omega^2-\omega^2)^2+\gamma^2\omega^2}.\end{aligned}$$

3. Stochastic variability due to noise-induced transitions between steady states. First, the GCM study that shows jumps between three very different MOC equilibria under sufficiently strong stochastic forcing: Figs. 2, 4 from Weaver and Hughes (1994). Then Fig. 5 from Cessi (1994) with a careful analysis of the same type of variability in the Stommel box model.

4. (Time permitting:) details of Cessi analysis: Stommel 2 box model from section 2 with model derivation and in particular getting to equation 2.9 with temperature fixed and salinity difference satisfying an equation of a particle on a double potential surface; section 3 with deterministic perturbation;

5. Stochastic resonance: periodic FW forcing plus noise. Matlab code `Stommel_stochastic_resonance.m`, and jpeg figures with results: `Stochastic_Resonance_a,b,c.jpg`;

6. (Time permitting:) AMOC oscillations due to "Thermal" Rossby waves analyzed by Te Raa and Dijkstra (2002), using equations 8–10 and Figure 7 of Zanna et al. (2011).

7. Stochastic forcing, non-normal AMOC dynamics, transient amplification; first the basic derivation: Stochastic optimals: the derivation from Tziperman and Ioannou (2002): consider a stochastically forced linear system:

$$\dot{\mathbf{P}}=A\mathbf{P}+\mathbf{f}(t)$$

The solution is,

$$\mathbf{P}(\tau) = e^{A\tau}\mathbf{P}(0) + \int_0^\tau ds\, e^{A(\tau-s)}\mathbf{f}(s) = B(\tau,0)\mathbf{P}(0) + \int_0^\tau ds\, B(\tau,s)\mathbf{f}(s)$$

where the first term is a response to the initial conditions which decays in time and can therefore be ignored, and the second is the response to the stochastic forcing. Assuming for simplicity a zero-mean state variable $\mathbf{P}(t)$, the variance is given by the following expression (summation convention assumed),

$$\begin{aligned} var(\|\mathbf{P}\|) &= \langle P_i(\tau)P_i(\tau)\rangle \\ &= \left\langle \int_0^\tau ds \int_0^\tau dt B_{il}(\tau,s) f_l(s) B_{in}(\tau,t) f_n(t) \right\rangle \\ &= \int_0^\tau ds \int_0^\tau dt B_{il}(\tau,s) B_{in}(\tau,t) \langle f_l(s) f_n(t)\rangle \end{aligned}$$

Specifying the noise statistics as separable in space and time, letting the ith component of the noise be, say, $f_i\nu(t)$, and with $C_{ln} = f_l f_n$ being the noise spatial correlation matrix and $D(t-s) = \langle \nu(t)\nu(s)\rangle$ the temporal correlation function (delta function for white noise),

$$\langle f_l(s) f_n(t)\rangle = C_{ln} D(t-s)$$

we have

$$\begin{aligned} var(\|P\|) &= \int_0^\tau ds \int_0^\tau dt B_{il}(\tau,s) B_{in}(\tau,t) C_{ln} D(t-s) \\ &= Tr\left(\int_0^\tau ds \int_0^\tau dt [B^T(\tau,s) B(\tau,t) D(t-s)] C \right) \\ &\equiv Tr(ZC) \end{aligned}$$

This implies that the most efficient way to excite the variance is to set the noise spatial structure to be equal to the first eigenvector of Z. To show this, show that eigenvectors of Z maximize $J = Tr(CZ) = \sum_{ij} Z_{ij} C_{ji}$; given the above expression for $C_{ij} = f_i f_j$ and we need to maximize $Z_{ij} f_i f_j + \lambda(1 - f_k f_k)$; differentiating with respect to f_n we get that $\mathbf{f} = \{f_n\}$ is an eigenvector of the matrix Z. The vector that maximizes the variance is, as usual, the one corresponding to the largest eigenvalue of Z.

Then more specifically to AMOC: The 3-box model of Tziperman and Ioannou (2002), or the spatially resolved 2d model of Zanna and Tziperman (2005)). Figures for the amplification and mechanism from file non-normal_AMOC.pdf.

8. (Time permitting:) A more general issue that comes up in this application of transient amplification is the treatment of singular norm kernel (appendix) and infinite amplification; show and explain the first mechanism of amplification (Figure 2); note how

limited the amplification may be in this mechanism.

9. (Time permitting:) Use of eigenvectors for finding the instability mechanism: linearize, solve eigenvalue problem, substitute spatial structure of eigenvalue into equations and see which equations provide the positive/ negative feedbacks; results for AMOC problem (e.g., Tziperman et al., 1994, section 3): destabilizing role of $v'\nabla\bar{S}$ and stabilizing role of $\bar{v}\nabla S'$; difference in stability mechanism in upper ocean ($v'\nabla\bar{S}$) vs that of the deep ocean (where $\nabla\bar{S} = 0$ and $\bar{v}\nabla S'$ is dominant); for temperature, also ($v'\nabla\bar{T}$ is more important, but from the eigenvectors one can see that v' is dominated by salinity effects; GCM verification and the distance of present-day AMOC from stability threshold: Figs 4,5,6 from Tziperman et al. (1994); Fig 3 from Toggweiler et al. (1996); Figs 1, 2, 3 from Tziperman (1997).

4.5 (Time permitting:) More on stochastic variability

- As a preparation for the rest of this class: the derivation of diffusion equation for Brownian motion following Einstein's derivation from Gardiner (1983) section 1.2.1; next, justify the drift term heuristically; then, derivation Fokker-Plank equation from Rodean (1996), chapter 5; Note that equation 5.17 has a typo, where the LHS should be $\frac{\partial}{\partial t}T_\tau(y_3|y_1)$; Then, first passage time for homogeneous processes from Gardiner (1983) section 5.2.7 equations 5.2.139-5.2.150; 5.2.153-5.2.158; then the one absorbing boundary (section b) and explain the relation of this to the escape over the potential barrier, where the potential barrier is actually an absorbing boundary, with equations 5.2.162-5.2.165; Note that 5.2.165 from Gardiner (1983) is identical to equation 4.7 from Cessi (1994); Next, random telegraph processes are explained in Gardiner (1983) section 3.8.5, including the correlation function for such a process; Cessi (1994) takes the Fourier transform of these correlation functions to obtain the spectrum in the limit of large jumps, for which the double well potential problem is similar to the random telegraph problem.

  Next, back to Cessi (1994) section 4: equation 4.4 (Fokker-Planck), 4.6 and Fig. 6 (the stationary solution for the pdf); then the expressions for the mean escape time (4.7) and the rest of the equations all the way to end of section 4, including the random telegraph process and the steady probabilities for this process;

  Finally, from section 5 of Cessi (1994) with the solutions for the spectrum in the regime of small noise (linearized dynamics) and larger noise (random telegraph); For the solution in the small noise regime (equation 5.3), let $y' = y - y_a$ and then Fourier transform the equation to get $-i\omega\hat{y}' = -V_{yy}\hat{y} + \hat{p}'$ where hat stands for Fourier transform; then write the complex conjugate of this equation, multiply them together using the fact that the spectrum is $\mathcal{S}_a(w) = \hat{y}'\hat{y}'^{\dagger}$ to get equation 5.5; Show the fit to the numerical spectrum of the stochastically driven Stommel model, Figure 7;

- Zonally averaged models and closures to 2d models (Dijkstra, 2000, section 6.6.2, pages 282-286, including technical box 6.3); Atmospheric feedbacks (Marotzke, 1996)?

5 Dansgaard-Oeschger and Heinrich events

Downloads here.

6 Glacial cycles

Downloads here.

6.1 Overview and basics

First, briefly: glacial cycle phenomenology, main questions.

Next, basic ingredients that will later be incorporated into different glacial theories (lecture 8 from WH notes, unless otherwise noted),

1. energy balance and albedo feedback
2. ice sheet dynamics and Glenn's law (section 6.2 below)
3. parabolic ice sheet profile (section 6.2.2 below)
4. accumulation and ablation (mass balance) as a function of ice sheet height, equilibrium line
5. ice streams, calving
6. dust loading and enhanced ablation
7. temperature-precipitation feedback
8. shallow ice approximation (section 3.1, p 222, Schoof and Hewitt, 2013, and section 6.2.3 below)
9. isostatic adjustment
10. Milankovitch forcing (section 6.3 below)
11. geothermal heating

6.2 Ice dynamics preliminaries

6.2.1 Intro to ice dynamics

(Time permitting:) Rate of strain definition from Kundu and Cohen (2002) sections 3.6 and 3.7, pages 56–58; stress and deviatoric stress and stress-rate of strain relationship and Glenn's law from section 2.3, pages 13-15 of Van-Der-Veen (1999).

6.2.2 Parabolic profile of ice sheets

Plastic ice sheet. Consider first a simple version assuming ice is a plastic material (WH notes p 101). Consider a balance of forces for a glacier that is symmetric in longitude x. The glacier height as a function of latitude is $h(y)$. The balance of forces on a slice of the glacier between latitudes $(y, y+dy)$ is between hydrostatic pressure integrated along the face of the slice, and stress applied by bottom friction

$$\int_0^{h(y+dy)} \rho_{ice} g z dz - \int_0^{h(y)} \rho_{ice} g z dz = \tau(y, z=0) dy$$

or simply

$$h(y)\frac{dh}{dy}\rho_{ice}g = \tau(y, z = 0) = \tau_0$$

where we assume that the bottom is at the yield stress τ_0 (i.e., glacier in a "critical" state). In other words, we assume perfect plasticity: the glacier yields to the hydrostatic-induced stress at the above critical stress. The solution to the last equation gives the desired parabolic profile that is not a bad fit to observations,

$$\frac{1}{2}h(y)^2 = \frac{\tau_0}{g\rho_{ice}}(y - y_0).$$

Viscous ice sheet. (Time permitting:) Now the more accurate expression shown by the solid line in Fig 11.4 in Paterson (1994): the following derivation roughly follows Chapter 5, p 243 Eqns 6–10 and p 251, Eqns 18–22 from Paterson (1994),

$$\dot{\epsilon}_{xz} = \frac{1}{2}\frac{du}{dz} = A\tau_{xz}^n = A\left(\rho g(h - z)\frac{dh}{dx}\right)^n.$$

Integrate from $z = 0$ to z, and use the b.c. $u(z = 0) = u_b$,

$$u(z) - u_b = -2A\left(\rho g\frac{dh}{dx}\right)^n \frac{(h - z)^{n+1}}{n + 1} + 2A\left(\rho g\frac{dh}{dx}\right)^n \frac{h^{n+1}}{n + 1}.$$

Let $u_b = 0$ (no sliding) and average the velocity in z,

$$\begin{aligned}
\bar{u} &= (1/h)\int_0^h dz 2A\left(\rho g\frac{dh}{dx}\right)^n \frac{(h - z)^{n+1}}{n + 1} - 2A\left(\rho g\frac{dh}{dx}\right)^n \left(\frac{h^{n+1}}{n + 1}\right) \\
&= \frac{2A}{(n + 1)}\left(\rho gh\frac{dh}{dx}\right)^n h\left(\frac{1}{n + 2} - 1\right) \\
&= \frac{2A}{(n + 2)}\left(-\rho gh\frac{dh}{dx}\right)^n h. \qquad (1)
\end{aligned}$$

Next, we use continuity, assuming a constant accumulation of ice at the surface, $d(h\bar{u})/dx = c$ which implies together with the last equation

$$cx + K_1 = h\bar{u} = \frac{2A}{(n + 2)}\left(-\rho gh\frac{dh}{dx}\right)^n h^2 = K_2\left(h\frac{dh}{dx}\right)^n h^2,$$

where ablation is assumed to occur only at the edge of the ice sheet at $x = L$. The last equation may be written as

$$(K_3x + K_4)^{1/n}dx = h^{2/n+1}dh$$

and solved using boundary conditions of $h(x=0)=H$ and $h(x=L)=0$ to obtain

$$(x/L)^{1+1/n}+(h/H)^{2/n+2}=1.$$

This last equation provides a better fit to obs as shown in Paterson Fig 11.4 (also shown in WH notes).

6.2.3 Shallow ice approximation

(Time permitting:) Let (s,b) be the ice surface and ice bottom heights. Then, the momentum equations, Glenn's law, the top and bottom boundary condition, and mass conservation equations are,

$$\begin{aligned}
0 &= -p_x+\frac{\partial \tau_{xz}}{\partial z}, \quad p_z=-\rho g, \\
u_z &= 2A|\tau_{xz}|^{n-1}\tau_{xz} \\
u(z=b) &= 0, \quad \tau_{xz}(s)=0, \quad p(s)=0 \\
s_t+\partial_x q &= a, \quad \text{where} \quad q=\int_b^s u\,dz
\end{aligned}$$

Note that in the x-momentum equation, the time rate of change, nonlinear advection terms and the Coriolis term are all negligible. The stress term on the RHS depends on the velocity via the dependence of the stress on the range of strain. Integrating the hydrostatic equation from the surface, one finds

$$p(x,z)=(s(x)-z)\rho g.$$

Substitute in the momentum equation to find,

$$\frac{\partial \tau_{xz}}{\partial z}=\rho g s_x$$

Integrate using the zero stress condition at the top, $z=s$,

$$\tau_{xz}=\rho g(z-s)s_x$$

substitute in Glenn's law,

$$u_z=2A(\rho g)^n(z-s)^n|s_x|^{n-1}s_x.$$

Integrate from the bottom to z to find u, using b.c. of zero velocity at the bottom,

$$u(z)=\frac{2A}{n+1}(\rho g)^n[(s-z)^{n+1}-(s-b)^{n+1}]|s_x|^{n-1}s_x,$$

and then again from bottom to top to find q,

$$\begin{aligned} q &= \frac{2A}{n+1}(\rho g)^n[\frac{1}{n+2}(s-b)^{n+2} - (s-b)^{n+2}]|s_x|^{n-1}s_x \\ &= -\frac{2A}{(n+2)}(\rho g)^n(s-b)^{n+2}|s_x|^{n-1}s_x. \end{aligned}$$

The mass conservation equation can therefore be written as a nonlinear diffusion equation for the ice surface elevation, s, with a diffusion coefficient D that depends on s,

$$\begin{aligned} s_t - \partial_x(Ds_x) = a, \\ D = \frac{2A}{(n+2)}(\rho g)^n(s-b)^{n+2}|s_x|^{n-1}. \end{aligned}$$

Note that the shallow ice approximation and the shallow water approximation are making very different assumptions and reflect very different physical balances despite the similar names…

6.3 Milankovitch

- Basics from Paillard (2001) and from Muller and MacDonald (2002) Chapter 2.
- Precession effect is anti-symmetric with respect to seasons and hemispheres, and the annual average of precession vanishes at each latitude; precession has no effect when eccentricity is zero (circular orbit).
- Obliquity does affect the annual mean at a given latitude, but not the global average. Larger obliquity leads to more radiation at the poles in summers, but still none in winter, so more generally high latitude annual insolation depends on obliquity.
- Paillard (2001) Figs. 3,4,5
- Animations and images in a webpage by Peter Huybers, https://phuybers.sites.fas.harvard.edu/Inso/index.html
- Integrated insolation: this is left to the end of the glacial cycle mechanism discussion.

6.4 Glacial cycle mechanisms

General outline:

1. Temperature-precipitation feedback (Ghil 94)
2. Elevation-desert effect (Ghil 94)
3. Adhemar's model, Croll's model (Fig. 1, Paillard, 2001),

4. Milankovitch: a correct calculation of orbital parameters and insolation, and the realization that it's the summers that matter.
5. Calder's model (Eqn on p. 332 and Fig. 9 of Paillard, 2001),
6. Imbrie and Imbrie (Eqn on p. 333 and Fig. 10 of Paillard, 2001),
7. Paillard's model (sec 3.3, Eqn on p. 339 and Figs 12 and 13 of Paillard, 2001),
8. Le-Treut and Ghil (1983) and the 100 cycle as a difference-tone of insolation's 19k and 23k frequencies, due to nonlinear glacial dynamics. Then Rial (1999) with taking the idea even further, (both from WH notes).
9. Stochastic resonance
10. 100 kyr from a collapse of the ice sheets due to geothermal heat build-up and induced basal melting (Huybers and Tziperman, 2008)
11. Saltzman "Earth system" models
12. Discussion: (A) we are trying to merely fit the record or explain the mechanism? (B) a reminder that a useful glacial theory must be falsifiable...
13. The shallow ice approximation plus orbital forcing Pollard (1982)
14. Sea ice switch
15. Phase locking, and a discussion of how a good fit to ice volume does not imply a correct physical mechanism; the difference between locking to periodic and quasi-periodic forcing
16. Huybers' integrated insolation: The 41kyr problem, positive degree days as a motivation for integrating insolation beyond a threshold corresponding to the melting point; cancellation of precession because summer intensity and duration are exactly out of phase (Kepler Laws). This is further explained in Huybers-integrated-insolation.pdf slides 5,7,9,10.

6.5 CO_2

1. Show glacial CO2 time series, and then use sections 1–4 of the notes to explain the basics of the carbonate system and its response to various inputs/ changes.
2. The 3 box model of Toggweiler (1999) showing how CO_2 is determined by the Southern Ocean biology and can change in response to change in mixing between surface water and NADW below (notes again, following Toggweiler's 1999 paper)

7 Pliocene climate

Downloads here.

Phenomenology (Molnar and Cane, 2007; Dowsett et al., 2010) and relevant proxies for temperature, productivity, upwelling, CO_2, etc. The Pliocene is considered the nearest past analog of future warming (although only for the equilibrium response). Observations of global temperature, permanent El Niño and warming of mid-latitude upwelling sites.

Possible mechanisms for permanent El Niño: FW flux causing a collapse of the meridional density gradient and therefore meridional ocean heat flux, which leads to a deepening of the thermocline; Hurricanes and tropical mixing; the opening of central American seaway; movement of new Guinea and changes in Pacific-Indian water mass exchange; atmospheric superrotation, possibly driven by strengthening MJO. Mechanisms for the warming of upwelling sites.

8 Equable climate

Downloads here.

Earth's Climate was exceptionally warm, and the equator-to-pole temperature difference (EPTD) was exceptionally small, during the Eocene (55Myr ago), when continental configuration was not dramatically different from present-day. Many explanations have been proposed, and we will briefly survey some.

First, Phenomenology and relevant proxies from slides.

8.1 Equator-to-pole Hadley cell

(Farrell, 1990). The idea briefly: Angular momentum conservation leads to large zonally-averaged zonal velocity $\bar{u}$ in the upper branch of the Hadley cell: The angular momentum equation is derived from the zonal momentum equation in spherical coordinates, where, in the absence of forcing or friction we have

$$\frac{DM}{dt} = \frac{1}{\rho}\frac{\partial p}{\partial \lambda}$$
$$M = (\bar{u} + \Omega r \cos\theta) r \cos\theta.$$

so that in an axisymmetric system the angular momentum is conserved. Consider a parcel of air that starts at the surface near the equator with $\bar{u}(\text{equator}) = 0$ and then rises and travels poleward in the Hadley cell. We find from angular momentum conservation that

$$\begin{aligned} M(equator) &= M(\theta), \\ \Omega r^2 &= (\bar{u} + \Omega r \cos\theta) r \cos\theta \\ \bar{u}(\theta) &= \Omega r\left(\frac{1}{\cos\theta} - \cos\theta\right) = \Omega r \frac{\sin^2\theta}{\cos\theta} \\ \bar{u}(30) &= (6,300,000 \times 2\pi/(24 \times 3600)) \times (1/\cos(30) - \cos(30)) = 132 m/sec. \end{aligned}$$

The resulting large $\bar{u}_z$ is balanced via thermal wind by strong T_y, leading to a large EPTD (eqn 1.5 in Farrell, 1990). To break this constraint, can dissipate some angular momentum, reduce f (as on Venus), or increase the tropopause height H which appears in the solution for the edge of the Hadley cell.

- Start with the frictionless theory from Vallis (2006), highlighted parts of sections 11.2.2-11.2.3.
- Then cover highlighted parts of Farrell (1990), which is based on an extension of Held and Hou (1980) to include dissipation, based on Hou (1984).

8.2 Polar stratospheric clouds (PSCs)

PSCs are formed in the lower stratosphere (15-25 km) at temperatures below -78°C. They are optically thick and high and therefore have a significant greenhouse effect. Water ice, which is a major ingredient of PSCs, is formed in the stratosphere via methane (CH_4) oxidation. Methane, unlike water vapor, is able to get past the tropopause "cold trap" given its freezing point of -182°C. Sloan et al. (1992) proposed that PSCs may have contributed to equable climate conditions, and Kirk-davidoff et al. (2002) suggested a positive feedback that would enhance the formation of PSCs in a warm climate. Details follow.

- **Preliminaries 1.** QG potential vorticity, EP fluxes, transformed Eulerian mean equations (Vallis).

- **Preliminaries 2.** Planetary wave forcing of the stratosphere: Topographically forced vertically propagating planetary waves, conditions on the mean zonal flow that lead to vertical propagation (Vallis).

- **Zonal stratospheric circulation and seasonal limits on upward wave propagation.** (Vallis) SW absorption near the summer pole leads to a reversed temperature gradient in the summer hemisphere: i.e., $T_y < 0$ in the southern hemisphere during Jan, and $T_y > 0$ in the northern hemisphere during July. There is no short-wave radiation in the polar areas in the winter hemisphere (northern Jan, southern July), and the temperature gradient is not reversed. During the southern winter (July), therefore, $T_y > 0$ in the southern hemisphere, and therefore $u_z \propto \rho_y / f_0 \propto - T_y / f_0 > 0$. Using $u = 0$ at top of the stratosphere, we get $u < 0$ (easterlies) during summer (Jan) in the southern hemisphere stratosphere. Similarly, $u < 0$ (easterlies) during summer (July) in the northern hemisphere. During winter the stratospheric zonal winds are westerlies. Note that stationary Rossby waves cannot propagate from the troposphere into easterlies, and they can therefore only reach the stratosphere in the winter hemisphere.

- **Brewer-Dobson stratospheric circulation:** (Vallis) The zonally averaged Transformed Eulerian Mean (TEM) momentum balance derived above is $-f_0 \overline{v}^* = \overline{v'q'}$. Assuming the potential vorticity flux is down-gradient (which is equatorward in the northern hemisphere because the gradient is dominated by β), the RHS is negative, so that the mean flow $\overline{v}^* > 0$ is poleward. In the southern hemisphere, the RHS is positive f_0 is negative, so $\overline{v}^* < 0$ is poleward again. The B-D circulation warms the pole and cools the equator in the stratosphere (Vallis Eqn 13.89): $N^2 \overline{w}^* = \frac{\theta_E - \theta}{\tau}$, together with positive w^* in tropics and negative in polar areas forced by poleward B-D meridional flow. This leads to $\theta < \theta_E$ (cooling!) at the equator (where $\overline{w}^* > 0$) and $\theta > \theta_E$ (warming!) at the pole (where $\overline{w}^* < 0$).

- **Feedback between EPTD, vertically propagating planetary waves, Brewer-Dobson stratospheric circulation and PSCs:** (Kirk-davidoff et al., 2002) warmer climate means weaker tropospheric EPTD, this leads to weaker mean tropospheric zonal winds and weaker synoptic-scale motions (which are, in turn, created via baroclinic instability of the mean winds and meridional temperature gradient). Both of these factors weaken the production of vertically propagating Rossby waves (forced by mean zonal winds interacting with topography, and by synoptic motions). As a result, there is a weaker Eliassen-Palm flux EP into the stratosphere, a weaker convergence $\nabla\cdot EP = \overline{v'q'}$, and therefore a weaker B-D circulation, and as a result a colder pole and warmer equator. The colder pole allows more PSCs to develop. This, in turn, further weakens the EPTD in the troposphere, providing a positive feedback.

- **However:** Korty and Emanuel (2007) show that the BD circulation could strengthen because additional zonal wavenumbers can penetrate into the stratosphere in a warmer climate with a weaker EPTD. Furthermore, nearly all studies of future climate scenarios project a stronger BD circulation, probably because of an upward shift of the critical layer and therefore of the wave-breaking altitude resulting from the eastward acceleration and upward movement of the subtropical jets. This allows both the resolved (planetary) and parameterized (internal) waves to penetrate higher, and drive a stronger BD circulation (Butchart, 2014).

8.3 Hurricanes and ocean mixing

The idea: Hurricanes are a heat engine, when described as such they can be shown to get stronger in warmer climates. This leads to stronger ocean mixing, and therefore to stronger ocean MOC. The stronger meridional heat flux by the ocean MOC reduces the equator-to-pole temperature gradient and explains a major feature of equable climates.

- **Potential intensity** (notes, based on Kerry's web page)
- Entropy reminder: consider a container with fluid, divided into two equal parts with temperatures $T_H > T_C$. Removing the divider, the temperature will eventually be homogenized to $(T_C + T_H)/2$. During the process, the infinitesimal change in entropy due to the transfer of an infinitesimal amount of heat $dQ > 0$ between the two systems leads to a gain dQ for the cold system and a loss of dQ for the hot system (gain of $-dQ$), thus the entropy change is

$$dS = \frac{dQ}{T_C} + \frac{-dQ}{T_H} = dQ\frac{T_H - T_C}{T_H T_C} > 0.$$

  so the increase in entropy is because temperature flows from the hot reservoir to the cold one.

- **Carnot cycle:** three first pages from wikipedia article, including description of four steps, calculation of efficiency. See also Carnot animation in the slides from here.
- **Hurricane as a Carnot cycle:** (Figure 1 in Emanuel, 1991). (1) Equivalent to isothermal expansion: air acquires heat (in the form of moisture from evaporation) as it

flows along the surface toward the center at a temperature T_H. (2) Adiabatic expansion as the parcel goes up in the slanted eyewall, doing work on the environment. (3) Isothermal compression: hurricane releases the heat to the environment (via radiation) while mixing out of the convective plumes at the top of the storm and descending and therefore compressing at a temperature T_C. (4) adiabatic compression while descending back to the surface.

- **Calculating the efficiency ϵ of a hurricane:** The *efficiency* of work done (kinetic energy generation) in a Carnot cycle in terms of the temperatures of the warm and cold reservoirs involved is $\epsilon = (T_H - T_C)/T_H$. For Hurricanes, $T_H = SST$ is the temperature of the heat source (the ocean surface). T_C is the average temperature at which heat is lost via radiation by the air parcels at the top of the storm. The taller a hurricane is, the lower the temperature T_C at its top and thus, the greater the thermodynamic efficiency. For a typical hurricane, $\epsilon \approx 1/3$.

- **Connection to ocean mixing and equable climate:** Finally, assuming that stronger hurricanes lead to stronger ocean mixing, and this to stronger MOC (Bryan, 1987). Stronger MOC means warmer poles (Emanuel, 2002).

- **Unfortunately,** consequences on SST and on EPTD of strongly enhanced tropical ocean mixing in an idealized coupled model seem small (Figs. 4, 11, 12 of Korty et al., 2008).

8.4 Breakup of subtropical marine stratocumulus clouds

Schneider et al. (2019)

8.5 Suppression of Arctic air events by low clouds over continents

Cronin and Tziperman (2015)

8.6 Convective cloud feedback

The idea: tropospheric clouds over the Arctic during the polar night should have a very strong greenhouse effect, but no cooling albedo effect. Ice-free Arctic Ocean can support atmospheric convection and convective clouds, and these clouds then protect the surface from cooling too rapidly and maintain the ice-free Arctic during the polar night. This is a positive feedback between the convection and the surface temperature. With the Arctic ice-free, high-latitude continents are more likely to remain above freezing, as observed. The feedback also dramatically reduces the equator-to-pole temperature gradient, especially during winter.

- **Importance of clouds for Arctic energy balance:** back-of-the-envelope calculation for the Arctic, comparing options for warming the Arctic, from Powerpoint.

- **Results of box model:** hysteresis and multiple equilibria, from Powerpoint. Our objective is to derive a simple analytic model describing the feedback and resulting hysteresis.

- **Atmospheric convection preliminaries:** Derivation of the conservation of Moist Static Energy (and, optionally, of the moist adiabatic lapse rate which is needed in the followings), beginning of section 6.1 from my notes based on Marshall and Plumb (2008). Then using MSE to diagnose convective stability, section 7.3 from my notes.

- **Analytic two-level model:** section 2 of Abbot and Tziperman (2009), or the single slide from the presentation, demonstrating the multiple equilibria and hysteresis analytically.

- **Supporting results from more complex models:** SCAM, IPCC, SP-CESM, from Powerpoint presentation (e.g., Abbot and Tziperman, 2008; Abbot et al., 2009).

9 Data analysis tools

For observations and model output (hands-on practice in sections). Supporting material.

- EOFs
- SVD analysis of two co-varying fields
- Composite analysis
- Linear inverse models, POPs
- Spectral analysis

10 Review of nonlinear dynamics concepts covered in the course

1. One-parameter bifurcations:

   (a) 1d: saddle-node,

   $$\dot{x} = \mu - x^2$$

   transcritical,

   $$\dot{x} = \mu x - x^2$$

   pitchfork: super-critical and sub-critical.

   $$\begin{aligned} \dot{x} &= \mu x - x^3 \\ \dot{x} &= \mu x + x^3 - x^5 \end{aligned}$$

   (b) Hysteresis in the case of two back-to-back saddle-node bifurcations, and in the case of a subcritical pitchfork.

   (c) 1d bifurcations in higher-dimensional systems

   (d) 2d: Hopf, super-critical

   $$\begin{aligned} \dot{r} &= \mu r - r^3 \\ \dot{\theta} &= \omega + br^2 \end{aligned}$$

   and sub-critical.

   $$\begin{aligned} \dot{r} &= \mu r + r^3 - r^5 \\ \dot{\theta} &= \omega + br^2 \end{aligned}$$

   (e) Hysteresis in subcritical Hopf.

2. Nonlinear phase-locking via analysis of flows on a circle, firefly and flashlight.

3. A two-parameter bifurcation: quasi-periodicity route to chaos in the circle map.

$$\theta_{n+1} = \theta_n + \Omega - \frac{K}{2\pi} \sin(2\pi\theta_n)$$

4. Relaxation oscillation, analysis via nullclines of the Van der Pol oscillator,

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0$$

5. Saddle node bifurcation of cycles.

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