Notes for climate dynamics course (EPS 231)

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1 Introduction

This evolving document will contains the reference material used for each lecture of the EPS 231 climate dynamics course. These notes as well as the materials/ images/ papers mentioned here are also posted on the course home page; please see there for the appropriate links.

2 Basics, energy balance, multiple climate equilibria

- Short and long-wave radiation, energy balance, greenhouse effect, (sections 2.3.2, 2.3.3, 2.4, 2.5, 2.6 Hartmann, 1994); albedo, ice, clouds, meridional heat transport;

- Multiple equilibria and climate stability: a 0d Budyko-Sellers model; *Version 1:* show the pdf `energy_balance_0d.pdf` with the graphical solutions of the steady state solution to the equation $CT_t = (Q/4)(1-\alpha(T) - \varepsilon\sigma T^4$ obtained using energy_balance_0d.m, and then the quicktime animation of the bifurcation behavior. [Possibly, *Version 2:* (p 92-94; eqns 1,2,3,4,7,8,13; and Figs. 1,2 from North et al., 1981) here;] Some nonlinear dynamics background: saddle node bifurcation ((p 45, Strogatz, 1994) or Applied Math 203 notes p 47), and then the energy balance model as two back to back saddle nodes and the resulting hysteresis as the insolation is varied; climate implications: (1) faint young sun paradox! (2) snowball (snowball obs from Ed Boyle's lecture);

- The Budyko and Sellers 1d models: (p 94-97; eqns 14, 15, 16, 17, 20, 21, 22, 23, 33; and Figs. 3, 4, 5, 6, 8 from North et al., 1981). Order of presentation:

  (A) derive the 1d energy balance model for the diffusive and Budyko versions (eqns 22 and 33): Assume the ice cap extends where the temperature is less than -10C, the ice-free areas have an absorption (one minus albedo) of $a(x) = a_f$, and in the ice-covered areas $a(x) = a_i$; approximate the latitudinal structure of the annual mean insolation as function of latitude using $S(x) = 1 + S_2P_2(x)$, with $P_2(x) = (3x^2-1)/3$ being the second Legendre polynomial (HW); add the transport term represented by diffusion $(r^2\cos\theta)^{-1}\partial/\partial\theta(D\cos\theta\partial T/\partial\theta)$, which, using $x = \sin\theta$, $1-x^2 = \cos^2\theta$ and $d/dx = (1/\cos\theta)d/d\theta$ gives the result in the final steady state 1D equation (22),

$$-\frac{d}{dx}D(1-x^2)\frac{dT(x)}{dx} + A + BT(x) = QS(x)a(x,x_0)$$

Instead, we could model the transport term a-la Budyko as $\gamma[T(x) - T_0]$ with T_0 being the temperature averaged over all latitudes, and the temperature then can be solved analytically (HW).

(B) Preliminary model results for $T(x)$: Fig 4 shows the results for an infinite transport case (flat line), actual observed profile, and the one with no transport (approximating the albedo via (18) $1-\alpha(x) \equiv a(x) = a_0 + a_2 P_2(x)$, and assuming no ice!!).

(C) Expanding the temperature in Legendre polynomials $T = \sum_{n\ (even)} T_n P_n(x)$ one could truncate the solution and Fig 5 shows the two vs four mode approximation with the observed profile. Note that in the two-mode approximation, the Sellers and Budyko models are equivalent (HW).

(D) Bifurcation results with diffusive meridional heat transport: next, calculate the ice edge latitude as function of the solar constant Q: Let $a(x_s) = (a_i + a_f)/2$ and $T(x_s) = -10$C; expand the temperature and Legendre polynomials and eliminate the derivatives in the diffusion term using the fact that these polynomials and eigenfunctions of the diffusion operator: $-\frac{d}{dx}(1-x^2)\frac{d}{dx}P_n(x) = n(n+1)P_n(x)$. Letting $x = x_s$ and using the above values for the albedo and temperature, one can find an analytic solution for $Q(x_s)$ (eq 37), shown in Fig. 8.

(E) Stability, small ice cap instability: Fig 8 generally has an S-shape like the 0d model above, with the implied stability and hysteresis. However, note that there is another small S-shape in the left upper-most part of Fig 8 in North et al. (1981). This represents another small unstable part which is referred to as the "small ice cap instability" because it occurs where the ice edge x_s is very close to the north pole. Physically this means that a very small ice cap (x_s larger than about 0.9) is unstable, while a larger one (x_s between 0.5 and 0.9) is stable. An even larger one (x_s between 0 and 0.5) is again unstable.

(F) Numerically calculated hysteresis in 1D Budyko and Sellers models: figures ebm1d-budyko.jpg and ebm1d-sellers.jpg obtained using ebm_1dm.m).

(G) One noteworthy difference between Budyko and Sellers is the transient behavior, with Budyko damping all scales at the same rate, and Sellers being scale-selective. (original references are Budyko, 1969; Sellers, 1969).

3 ENSO

3.1 ENSO background and delay oscillator models

Sources: Woods Hole (WH) notes (Cessi et al., 2001), lectures 0, 1, 2, here, plus the following: Gill's atmospheric model solution from Dijkstra (2000) technical box 7.2 p 347; recharge oscillator from Jin (1997) (section 2, possibly also section 3);

- The climatological background: easterlies, walker circulation, warm pool and cold tongue, thermocline slope (lecture 0 from WH notes).
- A heuristic derivation of the delayed oscillator mechanism: equatorial Rossby and Kelvin waves, thermocline slope, SST dynamics, atmospheric heating and wind response. The coupled feedback, and the heuristic delayed oscillator equation from section 2.1 in the WH notes. Next, the linearized stability analysis of the Schopf-Suarez delayed oscillator from the WH notes section 2.1.1. Show numerical solution of this model for values on both sides of the first bifurcation point using delay_Schopf_Suarez_1989.m Following this, and in order to explain the nonlinear damping term and more importantly the proximity of ENSO to the first bifurcation point beyond steady state, discuss Hopf bifurcation from non linear dynamics notes pp 25-28 here.
- A more quantitative derivation: starting from the shallow water equations, equatorial waves and all of section 1.2 in lecture 1 of WH notes, followed by lecture 2.
- Self-sustained vs damped and Hopf bifurcation: either from my nonlinear dynamics course notes or the excellent and accessible book by Strogatz (1994).

3.2 ENSO's irregularity

3.2.1 Chaos

- Phase locking: flows on a circle, synchronization/ phase locking, fireflies.
- Circle map and quasi-periodicity route to chaos:

  $$\theta_{n+1} = f(\theta_n) = \theta_n + \Omega - \frac{K}{2\pi}\sin 2\pi\theta_n, \quad \theta_n = mod(1)$$

  $K = 0$ and quasi-periodicity, winding number: $\lim_{n\to\infty}(f^n(\theta_0) - \theta_0)$, not taking θ_n as mod(1) for this calculation. $0 < K < 1$ and phase locking, Arnold tongues, Farey tree. $K = 1$ and the devil's staircase.
- Some generalities on identifying quasi-periodicity route to chaos in a complex system, including delay coordinate phase space reconstruction.
- Slides on transition to chaos in CZ model.
- References for phase locking: (Strogatz, 1994), for quasi-periodicity route to chaos: Schuster (1989). For both: course notes for applied math 203, pages 71,75,77-83 in lecture_bif1d2_eli.pdf. delay coordinate phase space reconstruction in lecture_bif2d3_eli.pdf.

3.2.2 Noise

- Non normal amplification from WH notes, plus: maximization of $\Psi(\tau)^T\Psi(\tau)$ instead of $d/dt\Psi(t=0)$ described in the WH notes. Eigenvalue is the amplification factor from the initial conditions to the amplified state.

- WWBs in observations: seem stochastic, seen to precede each El Nino event, affect Pacific by forcing of equatorial Kelvin waves. Show Hovmoller diagram with WWBs and SST from Yu et al. (2003), in jpg file; wind stress sequence showing WWB evolution from Vecchi and Harrison (1997) (all Figs at end of this report); effects of wind bursts on SST and thermocline depth (heat content) from Mcphaden and Yu (1999) Figs 1,2,3 (last one is model results); ocean-only model response to a strong WWB: Zhang and Rothstein (1998), Figs. 4, 5, showing the response to a wind burst after 10 days and after several months;

- Stochastic optimals: the derivation from Tziperman and Ioannou (2002): consider a stochastically forced linear system:
$$\dot{P} = AP + f(t)$$
solution is
$$P(\tau) = e^{A\tau}P(0) + \int_0^\tau ds\, e^{A(\tau-s)} f(s) = B(\tau,0)P(0) + \int_0^\tau ds\, B(\tau,s) f(s)$$
variance of the solution is given by
$$\begin{aligned} var(\|P\|) &= \langle P_i(\tau)P_i(\tau)\rangle - \langle P_i(\tau)\rangle\langle P_i(\tau)\rangle \\ &= \left\langle \int_0^\tau ds \int_0^\tau dt B_{il}(\tau,s) f_l(s) B_{in}(\tau,t) f_n(t) \right\rangle \\ &= \int_0^\tau ds \int_0^\tau dt B_{il}(\tau,s) B_{in}(\tau,t) \langle f_l(s) f_n(t)\rangle \end{aligned}$$

  Specifying the noise statistics as separable in space and time, with C_{ln} being the noise spatial correlation matrix and $D(t-s)$ the temporal correlation function (delta function for white noise),
$$\langle f_l(s) f_n(t)\rangle = C_{ln} D(t-s)$$
we have
$$\begin{aligned} var(\|P\|) &= \int_0^\tau ds \int_0^\tau dt B_{il}(\tau,s) B_{in}(\tau,t) C_{ln} D(t-s) \\ &= Tr\left(\int_0^\tau ds \int_0^\tau dt [B^T(\tau,s) B(\tau,t)] C\right) \\ &\equiv Tr(ZC) \end{aligned}$$

  This implies that the most efficient way to excite the variance is to make the noise spatial structure be the first eigenvector of Z. To show this, show that eigenvectors of Z maximize $J = Tr(CZ) = Z_{ij}C_{ji}$; assuming that the spatial noise structure is f_i, we have $C_{ij} = f_i f_j$ and we need to maximize $Z_{ik} f_j f_k + \lambda(1 - f_k f_k)$; differentiating wrt f_i we get that f_i is an eigenvector of Z; show picture of optimal modes from WH notes; discuss their model dependence; model dependence of the optimals from Fig. 11, 12 and 17 in Moore and Kleeman (2001).

- Are WWBs actually stochastic, or are their statistics a strong function of the SST, making the stochastic element less relevant?

3.3 Teleconnections

- Motivation for ENSO teleconnection: show pdf copy of impacts page from El Nino theme page.
- Further motivation: results of barotropic model runs from Hoskins and Karoly (1981) Figures 3,4,6,8,9, showing global propagation of waves due to tropical disturbances.
- (Optional: On the basics of ray tracing from http://www.ma.ic.ac.uk/~pdellar/M3A13/Sources/RayTheory.pdf (posted on course home page as RayTheory-Dellar-2005.pdf) or Lighthill (1978)). Discussion of stationary Rossby waves in an eastward flow (Hoskins and Karoly, 1981, section 5);
- Ray tracing: section 5b p 1190-1191. First the qualitative discussion after solution 5.23. Discuss trapping by jet via the calculation of the stationary wave number $K_s = (\beta_M/u_M)^{1/2} = k^2 + l^2$; Note that if $k > K_s$ then l must be imaginary, which implies trapping. Also note that baroclinic atmospheric waves are going to be trapped at the equator with a scale of the Rossby radius of deformation which is some 1000km or so (see discussion on page 1195 left column).
- Ray tracing theory based on Ray-tracing.pdf and Ray-Tracing-Rossby-Waves-Jeff.txt.
- Derivation of quantitative solution from the beginning of subsection 5b. Expanding on the WKB solution briefly mentioned just before equation 5.21 (Bender and Orszag (1978) section 10.1): consider $d^2P/dy^2 + l^2(\varepsilon y)P = 0$; to transform to standard WKB form, define $Y = \varepsilon y$ so that $\varepsilon^2 d^2P/dY^2 + l^2(Y)P = 0$; try a WKB solution corresponding to a wave-like exponential with a rapidly varying phase plus a slower correction $P = \exp(S_0(Y)/\delta + S_1(Y))$ to find
$$\varepsilon^2[(S_0'/\delta + S_1')^2 + (S_0''/\delta + S_1'')]P + l^2P = 0;$$
let $\delta = \varepsilon$ and then $O(1)$ equation is ${S_0'}^2 + l^2(Y) = 0$ so that $S_0 = \int l(Y)\,dy$; $O(\varepsilon)$ equation, after using the $O(1)$ equation, is $2S_0'S_1' + S_0'' = 0$ and the solution is $S_1' = -S_0''/(2S_0') = (dl/dY)/(2l) = d/dY(\ln l^{1/2})$ so that $S_1 = \ln l^{1/2}$ which means that the wave amplitude is $l^{1/2}$. This gives the solution in (Hoskins and Karoly, 1981) equation (5.21, 5.23), see further discussion there.

- Discuss the constant angular momentum flow solution (section 5c, page 1192, Fig. 12) and then the one using realistic zonal flows (Figs. 13, 14, 15, etc);
- Finally, mention that later works showed that stationary linear barotropic Rossby waves excite nonlinear eddy effects which may eventually dominate the teleconnection effects.

4 Thermohaline circulation

4.1 Preliminaries, scaling, energetics, Stommel box model

- Background, schematics of THC (Schmitz, 1995, all figures, but especially plates 9,10 – note switched captions); sections and profiles of T, S from EPS131/supporting-material/02-Temperature-Salinity; meridional mass and heat transport, climate relevance; measurements from RAPIS (Cunningham et al., 2007) and inverse methods (Ganachaud and Wunsch, 2000); anticipated response during global warming; MOC vs THC; mixed boundary conditions;
- Scaling for the amplitude and depth of the THC from Vallis (2005) chapter 15, section 15.1, showing that the THC amplitude is a function of the vertical mixing which in turn is due to turbulence; Energetics, Sandstrom theorem (eqn 15.23) and the "no turbulence theorem" in the absence of mechanical forcing by wind and tides (eqn 15.27) without mechanical mixing, and (15.30) with; hence the importance of mechanical energy/ mechanical forcing for the maintenance of turbulence and of the THC. Sections 15.2, 15.3 (much of this material is originally from Paparella and Young, 2002);
- Tidal energy as a source for mixing energy (Munk and Wunsch, 1998, Figures 4,5).

4.2 Stability and multiple equilibria

The Stommel-Taylor two box model from these notes, multiple equilibria, bifurcation and hysteresis. Qualitative discussion on proximity of present day THC to a stability threshold (Tziperman, 1997; Toggweiler et al., 1996). And again the magic(!) power of box models to predict GCM results (Fig. 2 from Rahmstorf (1995)). [See also Dijkstra (2000, section 3.1.1, 3.1.2, 3.1.3), and Aarnout van Delden's slides 1-3, 8-11].

4.3 Advective and convective feedbacks

- Advective feedback and convective feedback from sections 6.2.1, 6.2.2 in Dijkstra (2000) and from the original Lenderink and Haarsma (1994); saddle node bifurcation reminder from my nonlinear dynamics teaching notes or Strogatz (1994); Note that the middle line between L1 and L2 in Fig. 6.7 in Dijkstra's book should not be there; if the Heaviside function is replaced by a continuous function, it probably should be there. Hysteresis from

Fig. 8 in Lenderink and Haarsma (1994) and potentially convective regions in their GCM from Fig. 11.

- Use of eigenvectors for finding the instability mechanism: linearize, solve eigenvalue problem, substitute spatial structure of eigenvalue into equations and see which equations provide the positive/ negative feedbacks; results for THC problem (e.g., Tziperman et al., 1994, section 3): destabilizing role of $v'\nabla\bar{S}$ and stabilizing role of $\bar{v}\nabla S'$; difference in stability mechanism in upper ocean ($v'\nabla\bar{S}$) vs that of the deep ocean (where $\nabla\bar{S}=0$ and $\bar{v}\nabla S'$ is dominant); for temperature, also ($v'\nabla\bar{T}$ is more important, but from the eigenvectors one can see that v' is dominated by salinity effects; GCM verification and the distance of present-day THC from stability threshold: Figs 4,5,6 from Tziperman et al. (1994); Fig 3 from Toggweiler et al. (1996); Figs 1, 2, 3 from Tziperman (1997).

4.4 THC variability

A review of classes of THC oscillations: small amplitude/ large amplitude; linear and stochastically forced/ nonlinear self-sustained; loop oscillations due to advection around the THC path, or periodic switches between convective and non convective states; relaxation oscillations; noise induced switches between steady state, stochastic resonance;

Details of the major types of THC oscillations:

- Essentially linear Loop-oscillations due to advection around the circulation path: Stability regimes in a 4-box model: stable, stable oscillatory, [Hopf bifurcation], unstable oscillatory, unstable; Note changes from 2-box Stommel model: oscillatory behavior and change to the point of instability on the bifurcation diagram; Note the need of stochastic forcing to excite this type of variability.

- Convection only: Flip-flop oscillations (Welander1982_flip_flop.m) and loop oscillations (Dijkstra, 2000, sections 6.2.3, 6.2.4);

- Both convection and slow diffusion: Relaxation oscillations/ Thermohaline flushes/ "deep decoupling" oscillations (Winton, 1993, section IV); analysis of relaxation oscillations following Strogatz (1994) example 7.5.1 pages 212-213, or nonlinear dynamics course notes: slow phase and fast phase etc; relaxation oscillations and THC flushes in ocean GCMs will be postponed to the discussion DO and Heinrich events;

- THC variability that involve a wind-gyre element Delworth et al. (1993) which is also an example of a stochastically forced damped oscillatory mode (Griffies and Tziperman, 1995). More on this below.

4.4.1 Stochastic THC variability

- **First**, variability due to noise induced transitions between steady states Cessi (1994) in a Stommel 2 box model: section 2 with model derivation and in particular getting to eqn 2.9

with temperature fixed and salinity difference satisfying an equation of a particle on a double potential surface; section 3 with deterministic perturbation;

- (time permitting) As a preparation for the rest of this class: the derivation of diffusion equation for Brownian motion following Einstein's derivation from Gardiner (1983) section 1.2.1; next, justify the drift term heuristically; then, derivation Fokker-Plank equation from Rodean (1996), chapter 5; Note that equation 5.17 has a typo, where the LHS should be $\frac{\partial}{\partial t}T_\tau(y_3|y_1)$; Then, first passage time for homogeneous processes from Gardiner (1983) section 5.2.7 equations 5.2.139-5.2.150; 5.2.153-5.2.158; then the one absorbing boundary (section b) and explain the relation of this to the escape over the potential barrier, where the potential barrier is actually an absorbing boundary, with equations 5.2.162-5.2.165; Note that 5.2.165 from Gardiner (1983) is identical to equation 4.7 from Cessi (1994); Next, random telegraph processes are explained in Gardiner (1983) section 3.8.5, including the correlation function for such a process; Cessi (1994) takes the Fourier transform of these correlation functions to obtain the spectrum in the limit of large jumps, for which the double well potential problem is similar to the random telegraph problem.

  Next, back to Cessi (1994) section 4: equation 4.4 (Fokker-Planck), 4.6 and Fig. 6 (the stationary solution for the pdf); then the expressions for the mean escape time (4.7) and the rest of the equations all the way to end of section 4, including the random telegraph process and the steady probabilities for this process;

  Finally, from section 5 of Cessi (1994) with the solutions for the spectrum in the regime of small noise (linearized dynamics) and larger noise (random telegraph); For the solution in the small noise regime (equation 5.3), let $y' = y - y_a$ and then Fourier transform the equation to get $-i\omega\hat{y}' = -V_{yy}\hat{y} + \hat{p}'$ where hat stands for Fourier transform; then write the complex conjugate of this equation, multiply them together using the fact that the spectrum is $\mathcal{S}_a(w) = \hat{y}'\hat{y}'^{\dagger}$ to get equation 5.5; Show the fit to the numerical spectrum of the stochastically driven Stommel model, Figure 7;

- A GCM version of jumping between two equilibria under sufficiently strong stochastic forcing: Weaver and Hughes (1994).

- **Second**, an alternative mechanism for stochastic excitation of THC variability: exciting a damped oscillatory mode: first, Hasselmann's model with a red spectrum

$$\begin{aligned} \dot{x} + \gamma x &= \xi(t) \\ P(\omega) = |\hat{x}|^2 &= \xi_0^2/(\omega^2 + \gamma^2) \end{aligned}$$

  vs a damped oscillatory mode excited by noise that results in a spectral peak,

$$\begin{aligned} \ddot{x} + \gamma\dot{x} + \Omega^2 x &= \xi(t) \\ P(\omega) = |\hat{x}|^2 &= \xi_0^2/((\Omega^2 - \omega^2) + \gamma^2) \end{aligned}$$

- Next, the GCM study of Delworth et al. (1993); this paper also demonstrates the link between the variability of meridional density gradients and of the THC; Note the proposed role of changes to the gyre circulation in this paper, mention related mechanisms based on ocean mid-latitude Rossby wave propagation; then a box model fit to the GCM, showing that the horizontal gyre variability may not be critical and that the variability is due to the excitation of a damped oscillatory mode (Griffies and Tziperman, 1995); Useful and interesting analysis methods: composites (DMS Figs. 6,7), and regression analysis between scalar indices (Figs. 8,9) and between scalar indices and fields (Figs. 10, 11, 12).

- **Third**, stochastic forcing is expected to interact with non normal dynamics of the THC. Reminder of what is transient amplification; the 3-box model of Tziperman and Ioannou (2002): A more general issue that comes up in this application of transient amplification is the treatment of singular norm kernel (appendix) and infinite amplification; show and explain the first mechanism of amplification (Figure 2); note how limited the amplification may actually be in this mechanism; then the more interesting example of Zanna and Tziperman (2005), showing figures for the amplification and mechanism, taken from a talk on this subject (file nonormal_THC.pdf).

- **Fourth**, stochastic resonance between periodic FW forcing of the Stommel model and noise forcing (use Matlab code `Stommel_stochastic_resonance.m` from APM115, and jpeg figures with results: SRa.jpg, SRb.jpg, SRc.jpg);

- (time permitting) Zonally averaged models and closures to 2d models (Dijkstra, 2000, section 6.6.2, pages 282-286, including technical box 6.3); Atmospheric feedbacks Marotzke (1996)?

5 Equable climate

Dorian's lecture. and Matlab file.

6 Dansgaard-Oeschger, Heinrich events

Observed record of Heinrich and DO events: IRD, Greenland warming, possible relation between the two; synchronous collapses? or maybe not? Use Figures of obs from Heinrich_slides.pdf

THC flushes and DO events: DO explained by large amplitude THC changes (Ganopolski and Rahmstorf, 2001); from this paper, show hysteresis diagrams for modern and glacial climates, the ease of making a transition between the two THC states in glacial climate; time series of THC during DO events; these oscillations are basically the same as Winton's deep decoupling oscillations and flushes;

Alternatively, sea ice as an amplifier of THC variability: Preliminaries: sea ice albedo and insulating feedbacks; volume vs area in present-day climate (i.e. typical sea ice thickness in arctic and southern ocean); simple model equations for sea ice volume (Sayag et al., 2004, eqn numbers from): sea ice melting and formation (18), short wave induced melting (3rd term on rhs in 20), sea ice volume equation (20); climate feedbacks: insulating feedback (3), albedo feedback (19).

Sea ice and DO events: Figure of Camille's (Li et al., 2005) AGCM experiments (again Heinrich_slides.pdf). Sea ice as an albedo of *small* THC variability (Kaspi et al., 2004); Possible variants of the sea ice amplification idea: Stochastic excitation of THC+sea ice=DO like variability (Fig. 5 in Timmermann et al., 2003); self-sustained DO events with sea ice amplification (Vallis paper in Paleoceanography).

Precise clock behind DO events? Stochastic resonance? First, (Rahmstorf, 2003): clock error, triggering error and dating error; is it significant, or can we find a periodicity for which some "clock" might fit the time series? Next, stochastic resonance: (Alley et al., 2001): consider a histogram of waiting time between DO events and find that these are multiples of 1470, suggesting stochastic resonance as a possible explanation. The bad news: no clock, (Ditlevsen et al., 2007).

Heinrich events: binge-purge mechanism: following MacAyeal (1993a), and including argument for which external forcing is not likely (p 777) and heuristic argument for the time scale (p 782); then show equations and solution for the more detailed model of MacAyeal (1993b) from Heinrich_slides.pdf (or Kaspi et al., 2004);

7 Glacial cycles

Briefly: glacial cycle phenomenology

Milankovitch forcing

Basic feedbacks: lecture 8 from WH notes;

Supplement the discussion of the parabolic profile with the more accurate expression shown by the solid line in Fig 11.4 in Paterson (1994); First, rate of strain-stress relationship from Van-Der-Veen (1999): strain definition (section 2.1, p. 7-9); rate of strain is even better explained by Kundu and Cohen (2002) sections 3.6 and 3.7, pages 56-58; stress and deviatoric stress and stress-rate of strain relationship and Glenn's law from section 2.3, pages 13-15 of Van-Der-Veen (1999). Next, the ice sheet profile derivation: the following derivation is especially sloppy in dealing with the constants of integration, and very roughly follow Chapter 5, p 243 eqns 6-10 and p 251, eqns 18-22 from Paterson (1994):

$$\dot{\varepsilon}_{xz} = \frac{1}{2}\frac{du}{dz} = A\tau_{xz}^n = A(\rho g(h-z)\frac{dh}{dx})^n$$

integrate from $z=0$ to z, and use the b.c. $u(z=0)=u_b$,

$$u(z)-u_b = 2A(\rho g\frac{dh}{dx})^n\frac{(h-z)^{n+1}}{n+1} - 2A(\rho g\frac{dh}{dx})^n\frac{h^{n+1}}{n+1}$$

Let $u_b = 0$ (no sliding) and average the velocity in z,

$$\begin{aligned}\bar{u} &= (1/h)\int_0^h dz 2A\left(\rho g\frac{dh}{dx}\right)^n \frac{(h-z)^{n+1}}{n+1} - 2A\left(\rho g\frac{dh}{dx}\right)^n\left(\frac{h^{n+1}}{n+1}\right) \\ &= \frac{2A}{(n+1)}\left(\rho gh\frac{dh}{dx}\right)^n h\left(\frac{1}{n+2}-1\right) \\ &= -\frac{2A}{(n+2)}\left(\rho gh\frac{dh}{dx}\right)^n h. \end{aligned} \tag{1}$$

Next we use continuity, assuming a constant accumulation of ice at the surface, $d(h\bar{u})/dx = c$ which implies together with the last equation

$$cx = h\bar{u} = -\frac{2A}{(n+2)}\left(\rho gh\frac{dh}{dx}\right)^n h^2 = K_2\left(h\frac{dh}{dx}\right)^n h^2$$

where ablation is assumed to occur only at the edge of the ice sheet at $x = L$. The last eqn may be written as

$$K_3 x^{1/n} dx = h^{2/n+1} dh$$

and solved to obtain

$$(x/L)^{1+1/n} + (h/H)^{2/n+2} = 1.$$

Note that this satisfies the b.c. of $h(x=0) = H$ and $h(x=L) = 0$. This last equation provides the better fit to obs in Paterson Fig 11.4 (also shown in WH notes).

Simple glacial models: lecture 9 from WH notes;

More ice sheet dynamics: nonlinear diffusion equation…temperature equation with internal strain heating;

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