Chapter 1
Tides: A Tutorial

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Abstract These notes, from a course I gave at a CNRS school in Cargèse in March 2009, have the aim of quickly letting non-experts pick up a physical intuition and a sense of orders of magnitude in the theory of tides. 'Tides' include ocean tides as well as tidal effects in astronomy. The theory is illustrated by a variety of back-of-the-envelope problems, some of them surprising, all of them simple.

1.1 What These Notes Do

The reader is asked to refer to, and sooner or later to memorize, the data listed in Section 1.2. These data allow performing order-of-magnitude estimates in all the illustrative

Problems, which are boxed against a grey background

... and whose solutions are proposed under the line.

Section 1.3 is a review of elementary material on gravitation. I tried to archive a sampling of neat factoids from the classical literature that are no longer always reproduced in the modern. The theory of tides proper is in 1.4 and 1.5, emphasizing ocean tides. 1.6 explores applications to astronomy.

However, the attitude adopted in these notes is an applied mathematician's, rather than an oceanographer's or an astronomer's: we want to form an *intuition* for the *principles* and to estimate *orders of magnitude* on *toy problems*. Predictions of day-to-day ocean tides subject to accidental features of sea floors and coastlines are outside our program: nor Laplace's tidal equations (1776), nor mapping of co-tidal lines (Whewell, –1836), nor harmonic analysis of tidal records (Kelvin, 1867–)

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are touched upon,[1] let alone progress in the last half-century thanks to large-scale computing and satellite technology. Tout cela est prodigieusement conté dans les chapitres suivants de ce livre. . .

Technical terms are underlined on their first appearance.

Throughout the notes, an indicator (`PIC1` ►) means please look at the picture marked `PIC1` on one of the plates on a later page, (◄ `PIC∞`) at `PIC∞` on an earlier page.

1.2 Reference Data

	Earth ♁	Moon ☽	Sun $\odot$
radius	$R_{♁} = \dfrac{4\times 10^7}{2\pi}$ m	$R_{☽} \approx \dfrac{1}{4} R_{♁}$	$R_{\odot} \approx 100\, R_{♁}$
mass	$M_{♁} \approx 6\times 10^{24}$ kg	$M_{☽} \approx \dfrac{1}{80} M_{♁}$	$M_{\odot} \approx \dfrac{1}{3}\times 10^6\, M_{♁}$
density	$\rho_{♁} \approx 5.5\, \rho_{\text{water}}$	$\rho_{☽} \approx 3.3\, \rho_{\text{water}}$	$\rho_{\odot} \approx 1.4\, \rho_{\text{water}}$

distance Earth-Moon	$D_{☽}$	$\approx$	$60\, R_{♁}$
distance Earth-Sun (1 A.U.)	$D_{\odot}$	$\approx$	$\dfrac{1}{4}\times 10^5\, R_{♁}$
density of water	ρ_{water}	$=$	10^3 kg/m^3
gravitational constant		$\approx$	$\dfrac{2}{3}\times 10^{-10}$ N m^2/kg^2
gravitational acceleration at sea level	$g = \dfrac{GM_{♁}}{R_{♁}^2}$	$\approx$	10 m/sec^2
weight of a small apple		$\approx$	1 N
speed of light in vacuo	c	$\approx$	3×10^8 m/sec
1 year		$\approx$	$\pi \times 10^7$ sec, with $\lvert\text{error}\rvert < 0.4\,\%$
	(alternatively	$\approx$	$10^{7.5}$ sec, with $\lvert\text{error}\rvert < 0.25\,\%$)

From now on, we shall use the reference data all the time, everywhere.

[1]Except here.

Problem 1.21 Which looks wider to a terrestrial observer, the Sun or the full Moon?

We use the reference data to estimate their apparent angular diameters:

$$\frac{2R_{\odot}}{D_{\odot}} \approx \frac{2 \cdot 100}{\frac{1}{4} \times 10^5} = 8 \times 10^{-3} \text{ radian, or just under } \frac{1}{2} \text{ degree}$$

$$\frac{2R_{☽}}{D_{☽}} \approx \frac{2 \cdot \frac{1}{4}}{60} = \text{also just under } \frac{1}{2} \text{ degree}$$

which is a memorable round number (1 degree is the width of a finger at the end of an outstretched arm). This coincidence of apparent diameters is responsible for the occurrence of total eclipses.

Problem 1.22 How strong is the gravitational attraction between the Earth and the Moon?

We use the reference data to estimate

$$\frac{GM_{♁}M_{☽}}{D_{☽}^2} = \frac{GM_{♁}}{60^2 R_{♁}^2} \cdot \frac{1}{80} M_{♁} = g \cdot \frac{1}{60^2 \cdot 80} \cdot M_{♁}$$

$$\approx 10 \cdot \frac{1}{48 \cdot 6 \times 10^3} \cdot 6 \times 10^{24} \approx 2 \times 10^{20} \text{ N},$$

yet another memorable round number. Sometimes the trick of rewriting with the aid of g spares us parades of decimals.

1.3 Gravitation

1.3.1 Why $1/r^2$?

Why does the gravitational attraction $F(r)$ vary like $1/r^2$?

Imagine a point mass, which generates a vector field of gravitational force in the space surrounding it. Let us consider the flux of this field through a sphere of radius r centered at the mass (`PIC1` ►).

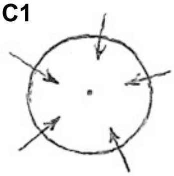
If we take any two concentric spheres of different radii, then the fluxes through these spheres must be the same, since we are assuming no other source/sink of gravitation in the vacuum between the spheres (`PIC2` ►). So

$$F(r) \cdot 4\pi r^2 = \text{const} \quad \Longrightarrow \quad F(r) \propto r^{-2} \quad \text{in } \mathbb{R}^3.$$

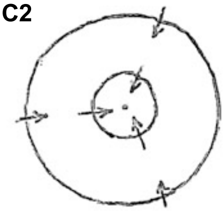
The same argument works in any dimension.[2]

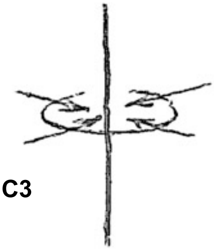
[2]Mathematically we have rediscovered the Green's function for the Laplacian in $\mathbb{R}^n$.

PIC1



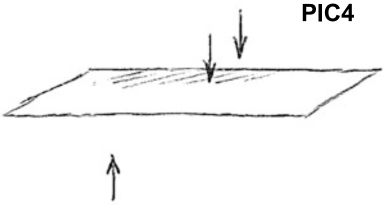
PIC2



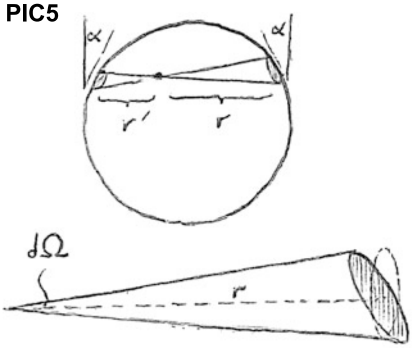


PIC3

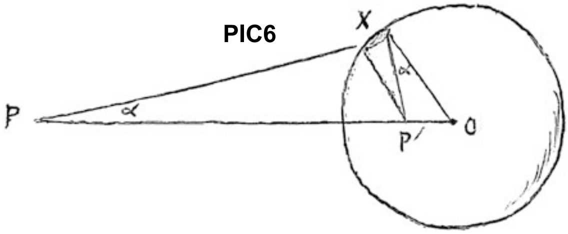
PIC4



PIC5



PIC6



Theorem 1.3.11 *In $\mathbb{R}^n$, the force of gravitational attraction due to a point mass varies like $F(r) \propto r^{1-n}$. The potential varies like r^{2-n} in dimension $n \neq 2$, like* $\log r$ *in dimension $n = 2$.*

Problem 1.3.12 An infinite line on which mass is distributed uniformly (◄ `PIC3`). How does the attraction F by the line depend on the distance r from the line? The same problem for an infinite uniform plane (◄ `PIC4`).

For the line, we are solving the attraction problem effectively in dimension $n = 2$, so $F(r) \propto r^{-1}$. For the plane, effectively $n = 1$ and $F(r) \propto \pm\,\mathrm{const}$, i.e. the attraction does not depend on how distant we are from the plane, though of course it changes sign from one side of the plane to the other.

1.3.2 Attraction by a Spherical Shell

Theorem 1.3.21 *Inside a spherical shell on which mass is distributed uniformly, the force of gravitational attraction is zero.*[3]

Proof (◄ `PIC5`) The attraction toward right is

$$\frac{r^{n-1}\,\mathrm{d}\Omega}{\cos\alpha} \cdot r^{1-n} = \frac{\mathrm{d}\Omega}{\cos\alpha}.$$

Likewise, the attraction toward left is $\mathrm{d}\Omega / \cos\alpha$. These cancel each other, and such a cancelation occurs in every direction. □

Theorem 1.3.22 *Outside a spherical shell, the attraction is as if the shell's entire mass were concentrated at its center.*[4]

Proof (◄ `PIC6`) Take P' to be 'inverse' of P with respect to the sphere, such that $OP' \cdot OP = \text{radius}^2 = OX^2$. By similar triangles OPX and OXP', we have $P'X/PX = OX/OP$. Since by symmetry the overall attraction acts along OP only, we may consider

$$\begin{aligned} &(\text{mass element}) \cdot (\text{attraction per unit mass}) \cdot (\text{component along } OP) \\ &= \frac{P'X^{n-1}\,\mathrm{d}\Omega}{\cos\alpha} \cdot \frac{1}{PX^{n-1}} \cdot \cos\alpha = OX^{n-1}\,\mathrm{d}\Omega \cdot \frac{1}{OP^{n-1}}. \end{aligned}$$

But $OX^{n-1} \int \mathrm{d}\Omega$ is the mass of the sphere. □

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[3] The zero-gravity conclusion is equally valid for the inside of a uniform ellipsoidal shell; by 'shell' is meant a region bounded between similar concentric (not confocal) ellipsoids.

[4] Outside an ellipsoidal shell the result is more complicated.

Fig. 1.1
Newton (1643–1727)

These are propositions LXX, LXXI in book I of Newton's *Principia*.

Remark 1.3.23

(1) A similar idea proves that the attractions by the two shaded slices (`PIC7` ►) are equal.
(2) These are theorems in potential theory, about *any* field whose potential u is harmonic, $\nabla^2 u = 0$.

Digression 1.3.24 What do you think of the position of the Sun on this British pound note?

The Sun would be at the center if $F(r) \propto r$ (harmonic oscillator).

Problem 1.3.25 An infinitely long uniform cylindrical shell (`PIC8` ►). How does the attraction F by the cylinder depend on the distance r from the cylinder's axis?

The effective dimension is $n = 2$. $F(r) = 0$ inside, $F(r) \propto r^{-1}$ outside.

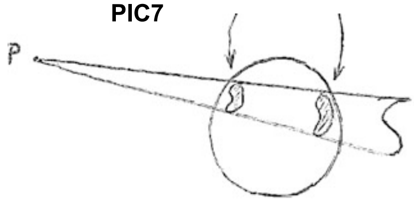
Beware: it is *not* the case that the attraction by a body is always directed toward its center of mass.

Problem 1.3.26 Along which direction does a uniform rod AB attract a given point P?

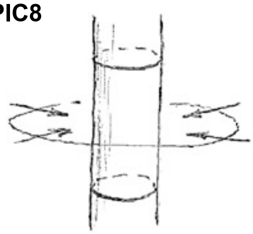
(`PIC9` ►) We have $x = h \tan\theta$, $\mathrm{d}x = h \sec^2\theta \, \mathrm{d}\theta$, while $h \sec\theta = r$. The attraction by $\mathrm{d}x$ is $\propto \mathrm{d}x/r^2 = \mathrm{d}\theta/h$, i.e. the contribution to the attraction is distributed uniformly in the angle θ. Hence the attraction on P is along the bisector of the angle APB subtended by the rod. This bisector does not pass through the center of mass unless P happens to lie on the perpendicular bisector of AB.

In the next problem, a nice property of conic sections allows us to describe the levels surfaces of the potential, equipotentials.

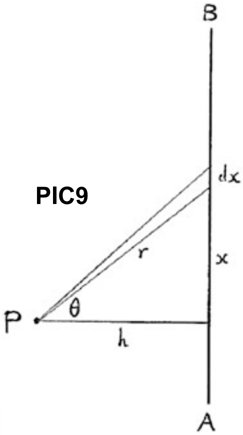
PIC7



PIC8

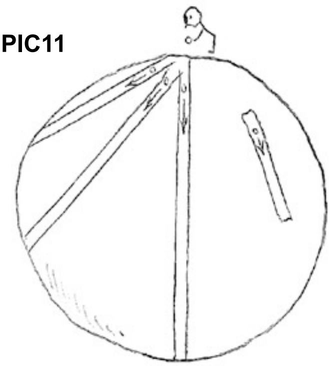


PIC9

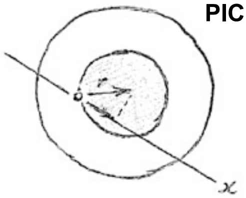


PIC10

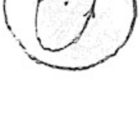
PIC11



PIC12

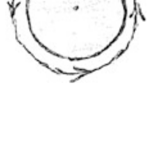


PIC13



PIC14

PIC15



Problem 1.3.27 What are the equipotentials of the attraction by the rod?

Through P draw an ellipse of foci A, B. By the focal property of the ellipse, a light ray from A to P reflects and goes to B. Hence, at P, the angle bisector of APB is perpendicular to the ellipse. The equipotentials, which at every point are perpendicular to the attraction, are ellipsoids of revolution all having A, B as foci.

In case the attraction by a body *is* always directed toward some fixed point, we can prove the little-known converse to theorems above, using the expansion which will be introduced in Section 1.3.5:

Theorem 1.3.28 *Suppose there exists a point O such that, at every location outside the body, the body's attraction on that location is directed toward O.*[5] *Then O is the body's center of mass, and the moments of inertia about all axes through O are equal, i.e. the body is 'inertially spherical' around O.*

1.3.3 Attraction by a Solid Ball

This can be treated by 'onionifying' the solid ball as a layered assembly of spherical shells (◄ `PIC10`).[6]

In particular, as far as the gravitational field outside is concerned, solid balls with rotationally symmetric mass distribution can be replaced by points at their centers. This is why celestial mechanics started off as such a clean subject.

Problem 1.3.31 (◄ `PIC11`) Frictionless tunnels are dug in various directions through a planet of uniform density ρ. Drop stones in the tunnels. How does the stone's period of oscillation depend on the direction of the tunnel? What would the period be if the planet had the average density of the Earth?

Write V = volume of the unit ball. At the instant depicted in (◄ `PIC12`), only the inner ball attracts the stone. For every axis x,

$$x\text{-component of attraction} = \frac{G\rho V r^n}{r^{n-1}} \cdot \frac{x}{r} = G\rho V x,$$

so along this axis

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} x = -G\rho V x,$$

which represents a harmonic oscillator. Since $G\rho V$ is independent of the tunnel, the period $2\pi/\sqrt{G\rho V}$ ($= \sqrt{3\pi/G\rho} \approx 84$ min for the Earth, $n = 3$) is the same for all tunnels and all amplitudes (◄ `PIC13,14,15`).

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[5] A weak hypothesis, only about the line of attraction passing through O, nothing about the *size* of attraction.

[6] *Principia* book I, proposition LXXIV.

Problem 1.3.32 Check directly that the caressing orbit (◄ PIC15) has the claimed period.

Balancing the centrifugal force and the attraction,

$$\frac{v^2}{r} = \frac{G\rho V r^n}{r^{n-1}} \implies \text{period} = \frac{2\pi r}{v} = 2\pi/\sqrt{G\rho V}.$$

1.3.4 Legendre Polynomials

In many problems in potential theory, the expression

$$\frac{1}{\sqrt{D^2 - 2Dd\cos\theta + d^2}} = \frac{1}{D}\left[1 - 2\frac{d}{D}\cos\theta + \left(\frac{d}{D}\right)^2\right]^{-1/2}$$

arises, cf. Section 1.3.5. The parameters d and D will be shown to have natural interpretations that make $d \ll D$, so we are led to expand the expression in powers of d/D. We define the coefficients by

$$[\cdots]^{-1/2} = \sum_{n\geqslant 0} P_n(\cos\theta)\left(\frac{d}{D}\right)^n$$

and call them Legendre polynomials.[7] The memorable, and the most important, low-degree Legendre polynomials are

$$P_0(z) = 1, \quad P_1(z) = z, \quad P_2(z) = \frac{3z^2 - 1}{2}, \quad P_3(z) = \frac{5z^3 - 3z}{2}, \quad \ldots.$$

(Alas, the memorable pattern does not continue.) It can be shown that in general

$$P_n(z) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(z^2 - 1\right)^n \quad \forall n \geqslant 0,$$

deg $P_n(z) = n$, $P_n(1) = 1$. Please familiarize yourself with their graphs (PIC16 ►).

Fig. 1.2
Legendre (1752–1833)

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[7]Traditionally they are defined as solutions to a certain ODE that crops up when we try to separate $\nabla^2 u = 0$ in spherical polar coordinates. The definition chosen here is equivalent to the traditional one, but it is better motivated and easier to use for us.

The portrait of Adrien Legendre shown above is a famous one reproduced in many books. It recently came to light that this was a portrait of another, *Louis*, Legendre, cf. *Notices of the AMS* **56** (2009) 1440–1443.

1.3.5 Approximation Formulae for Bodies of Arbitrary Shape

As we saw in Sections 1.3.2 and 1.3.3, when the body is rotationally symmetric, its attraction is the same as that by a point mass. MacCullagh's formulae below give next-order corrections to the attraction when the body is no longer rotationally symmetric.

(i) The potential of a body in the far field.
(PIC17 ►) Notation: M mass of the body; I_1, I_2, I_3 its principal moments of inertia around the center of mass O; I its moment of inertia around the axis OP. Then *minus* the potential at P divided by G, acting on a unit mass, is

$$\int \frac{\mathrm{d}M}{XP} = \int \frac{\mathrm{d}M}{D}[\cdots]^{-1/2} = \frac{1}{D}\int \mathrm{d}M \left\{1 + \cos\theta\frac{d}{D} + \frac{3\cos^2\theta - 1}{2}\left(\frac{d}{D}\right)^2 + \cdots\right\}$$

(cf. Section 1.3.4 for the expansion of $[\cdots]^{-1/2}$). The term $\cos\theta d/D$ gives 0 on being integrated. On the other hand,

$$\frac{3\cos^2\theta - 1}{2} = \frac{3(1-\sin^2\theta) - 1}{2} = 1 - \frac{3}{2}\sin^2\theta,$$

so the above integral gives

$$\frac{M}{D} + \frac{I_1 + I_2 + I_3 - 3I}{2D^3} + \cdots.$$

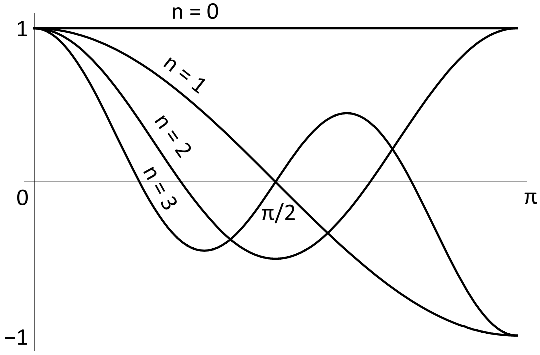
In the 'inertially spherical' case $I_1 + I_2 + I_3 - 3I = 0$, and all the higher-order terms vanish, too.

(ii) The potential between two far bodies.
Notation as in (PIC18 ►).

$$-\frac{\text{potential}}{G} = \frac{MM'}{D} + \frac{M(I_1' + I_2' + I_3' - 3I')}{2D^3} + \frac{M'\big(I_1 + I_2 + I_3 - 3I\big)}{2D^3} + \cdots.$$

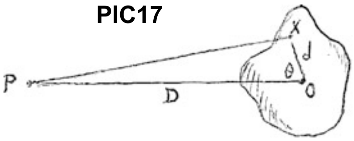
Fig. 1.3
MacCullagh (1809–1847)

PIC16



graphs of $P_n(\cos\theta)$

PIC17



PIC18

PIC19

PIC20

1.4 Tides—Static Picture

1.4.1 Plan

Why do tides exist?

Imagine two coconuts floating together near an attracting body (◄ PIC19). They feel a slight difference in the force of attraction, because one of them is slightly closer to, and the other one is slightly farther from, the attracting body. If we subtract the average attraction, then we see (◄ PIC20).

This, in a nutshell, is what the tidal effect is: difference, or *derivative*, in the attraction. The attraction varies like inverse square in the distance from the body, therefore the tidal effect, its derivative, varies like *inverse cube*. Qualitatively, the tidal effect is present as soon as the graph of the force as function of the distance is *concave*; the specific form $F(r) \propto -1/r^2$ is sufficient but not at all necessary.

Still, there are many other things we need to understand, for dynamic responses of a system to tidal effects can be tricky. For example: surely, as do the majority of textbooks since Newton (1687), we guess the (exaggerated) shape of the ocean to look like (PIC21 ►)?... Well, that guess is *wrong*.

The right prediction looks rather like (PIC22 ►).[8] This 'paradox' is one example among many of the characteristics about tides we try to understand in these notes.

Here is the plan we shall follow:

Theory of tides

- generating force for tides—static picture } neglecting dissipation
- response of the ocean to this force—dynamic picture } neglecting dissipation
- effects of dissipation, astronomical applications, etc.

Pictorial Convention In all the pictures, the body that is exerting the attraction, called

primary,

will be depicted on the right, while the body that is subjected to the attraction, called

secondary,

will be depicted on the left. In reality, everybody is attracting everybody all at once; 'primary' and 'secondary' are mere labels to clarify whose tidal influence on whom we are studying. Confusingly, primary and secondary can swap from one problem to the next, e.g. for ocean tides (1.4.3, 1.6.3) the Moon is primary and the Earth secondary, whereas for tidal locking (1.6.2) it is the other way around.

A good complementary reading is J. Lighthill, *Ocean Tides from Newton to Pekeris*, Israel Academy of Sciences and Humanities, 1995.

[8]It goes without saying that the observed tides of the real ocean are enormously complicated and do not resemble either of these pictures. But we are saying that, if we take the simplest model, of a spherical Earth covered by a sheet of ideal fluid, subjected to the dynamics of the Earth and the Moon, then the picture is (PIC22 ►) rather than (PIC21 ►).

1.4.2 Tidal Potential and Tidal Force

Let us figure out the tidal potential, then the tidal force, by making the picture of Section 1.4.1 quantitative.

We write the potential as

$$U_{\text{tide}} = U_{\text{cf}} + U_{\text{pr}}$$

where the three Us account for tidal force, centrifugal force, and attraction by the primary, in energy per unit mass. Since we are calculating the tidal effect of the primary at a given location in space like the black dot • of (PIC23 ►), the attraction by the secondary is irrelevant to us. Let us erase the secondary (PIC24 ►). Then our • is in orbit around the primary, so

$$\text{orbital centrifugal acceleration} \approx -\frac{GM}{D^2} \quad \Longrightarrow \quad U_{\text{cf}} = \frac{GM}{D^2} d \cos\theta.$$

Next,

$$U_{\text{pr}} = -\frac{GM}{D}[\cdots]^{-1/2} = -\frac{GM}{D}\left\{1 + \cos\theta\frac{d}{D} + \frac{3\cos^2\theta - 1}{2}\left(\frac{d}{D}\right)^2 + \cdots\right\}$$

(cf. 1.3.4 for the meaning of $[\cdots]^{-1/2}$ and its expansion in terms of Legendre polynomials). In the last sum { }, the constant 1 is immaterial for the potential and $\cos\theta d/D$ is canceled by U_{cf}. Altogether

$$U_{\text{tide}} \approx -\frac{GM}{D^3} d^2 \frac{3\cos^2\theta - 1}{2},$$

whence a formula often quoted in the literature for the representative tidal force per unit mass

$$\boxed{\mathcal{F} = -\frac{\partial}{\partial d} U_{\text{tide}}\Big|_{d=r,\theta=0} \approx 2\frac{GM}{D^3} r.}$$

$\mathcal{F}$ varies like D^{-3} in the distance D from the primary. A quicker way to derive this formula is that the attraction varies like inverse square, while the tidal force is the small difference in the attraction over a displacement $\Delta D \approx -r$, i.e. it arises essentially as the *derivative* of D^{-2}:

$$\mathcal{F} \approx \frac{\partial}{\partial D}\frac{GM}{D^2} \cdot \Delta D \approx 2\frac{GM}{D^3} r.$$

(Recall the 'nutshell' comment in Section 1.4.1.)

Now

$$\frac{3\cos^2\theta - 1}{2} = \frac{1}{2}\left(3\frac{1 + \cos 2\theta}{2} - 1\right) = \frac{3}{4}\cos 2\theta + \frac{1}{4}.$$

The last term 1/4, independent of θ, cannot deform the sphere. Rewriting U_{tide} with the aid of $g = Gm/r^2$ (gravitational acceleration on the surface of the secondary), we finally obtain the part u_{tide} of the tidal potential responsible for deforming the sphere, as

$$u_{\text{tide}} = -\frac{3}{4}\frac{M}{m}\left(\frac{r}{D}\right)^3 \frac{d^2}{r} g \cos 2\theta$$

in energy per unit mass. Hence the tidal force f per unit mass that deforms the sphere has components (PIC25 ►)

$$f_{\text{vert}} = -\frac{\partial}{\partial d} u_{\text{tide}}\Big|_{d=r} = \frac{3}{2}\frac{M}{m}\left(\frac{r}{D}\right)^3 g \cos 2\theta,$$

$$f_{\text{horiz}} = \frac{1}{d}\frac{\partial}{\partial \theta} u_{\text{tide}}\Big|_{d=r} = -\frac{3}{2}\frac{M}{m}\left(\frac{r}{D}\right)^3 g \sin 2\theta,$$

which over the surface of the secondary give the picture (PIC26 ►). We see that the effect of the tidal force is to stretch the secondary in the primary's direction and to squeeze it in the transverse directions, in the shape of a rugby ball. The direction of f varies as a function of θ whereas its magnitude $\sqrt{f_{\text{vert}}^2 + f_{\text{horiz}}^2}$ does not. We name and retain for future use the key ratio

$$\boxed{\frac{f}{g} = \frac{3}{2}\frac{M}{m}\left(\frac{r}{D}\right)^3.}$$

Problem 1.4.21 Estimate f/g for the Earth (secondary) under the influence of the Moon (primary).

$$\frac{f}{g} = \frac{3}{2}\frac{M_{☽}}{M_{♁}}\left(\frac{R_{♁}}{D_{☽}}\right)^3 \approx \frac{3}{2} \cdot \frac{1}{80} \cdot \left(\frac{1}{60}\right)^3 \approx 8.6 \times 10^{-8},$$

which is tiny.

If the ellipticity of the Moon's orbit (PIC27 ►) is taken into account, it turns out that f/g varies between 7.5×10^{-8} at the apogee and 10^{-7} at the perigee. Nevertheless, this tiny ratio produces the majestic ocean tides that wash the Earth.

1.4.3 Shape of the Ocean

Imagine an ocean that covers the secondary (PIC28 ►).

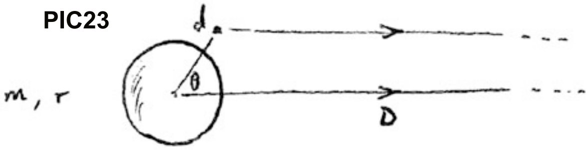
$$-u_{\text{tide}}|_{d=r} = gh \quad \Longrightarrow \quad h(\theta) = \frac{3}{4}\frac{M}{m}\left(\frac{r}{D}\right)^3 r \cos 2\theta = \frac{1}{2}\frac{f}{g} r \cos 2\theta.$$

Spinning this about the line directed toward the primary (rightward in the pictures), we obtain the rugby-ball shape of the ocean as deformed by the primary's tidal force

PIC21

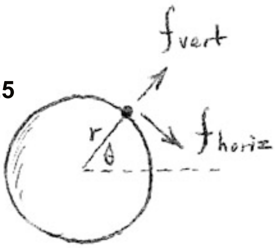
PIC22

PIC23

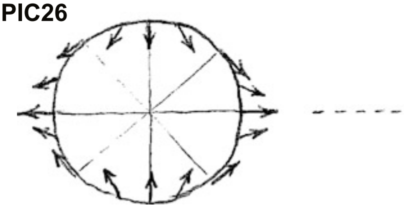


PIC24

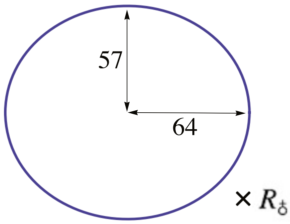
PIC25



PIC26



PIC27



(PIC29 ►). The reader is reminded that θ is the angle measured from this line, *not* colatitude from the North Pole. But on the side of the secondary facing the primary, θ happens to represent latitude. (PIC30 ►) shows that, as the secondary rotates on its axis, a point on a given latitude θ traces a circle and attains h of the low tide at $\times$. This h also equals h at $\bullet$. The upshot is that h has the common value of

$$\min h = \frac{1}{2}\frac{f}{g}r\cos\left(2\cdot\frac{\pi}{2}\right) = -\frac{1}{2}\frac{f}{g}r$$

for all latitudes θ. Clearly $\Delta h(\theta) = h(\theta) - \min h$, and we have[9] $\cos 2\theta - 1 = 2\cos^2\theta$. The daily amplitude of the tide $\Delta h(\theta)$ as a function of the latitude θ is

$$\Delta h(\theta) = \frac{f}{g}r\cos^2\theta.$$

It is proportional to the key ratio f/g.

Problem 1.4.31 At what latitude on the Earth does the daily amplitude of the tide attain its maximum? minimum? Estimate these amplitudes.

Using f/g from Problem 1.4.21,

$$\max_\theta \Delta h(\theta) = \frac{f}{g}R_♁ \approx 8.6\times10^{-8}\cdot\frac{4\times10^7}{2\pi} \approx 0.5\text{ m} \quad \text{at the equator} \quad (\theta = 0,\ \pi),$$

$$\min_\theta \Delta h(\theta) = 0 \qquad \text{at the poles} \quad (\theta = \pi/2).$$

So far we have pretended that the Earth's axis of rotation was perpendicular to the plane of the Moon's orbit. In reality the axis is *tilted* (PIC31 ►): this produces two unequal high tides, 'small' high tide and 'big' high tide, and brings the low tides nearer the 'small' high tide (PIC32 ►).

The axial tilt β varies between 17° and 29° owing to the precession of the Moon's orbit. Since the lunar revolution (period $\approx 27 + 1/3$ days) goes in the same direction as the terrestrial rotation (period = 24 hours), at a given location on the Earth a high tide arrives every

$$\frac{1}{2}\left(24 + \frac{24}{27+1/3}\right) \approx 12 \text{ hours } 26 \text{ minutes},$$

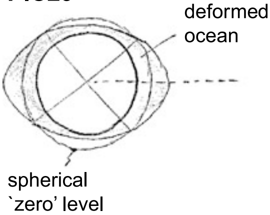
and this arrival gets delayed by 52 minutes (= twice the above number − 24 hours) per day.

The behavior of a real ocean tides depends sensitively on local geography. The largest Δh in the world is observed in the Bay of Fundy (Canada), where it attains 17 m.

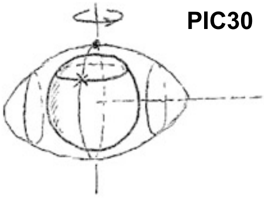
[9] Undoing an earlier trigonometric transformation of Section 1.4.2.

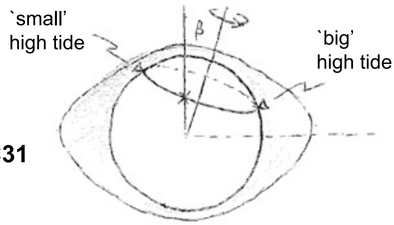
PIC28

PIC29



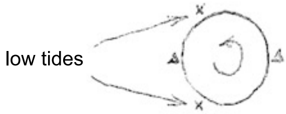
PIC30



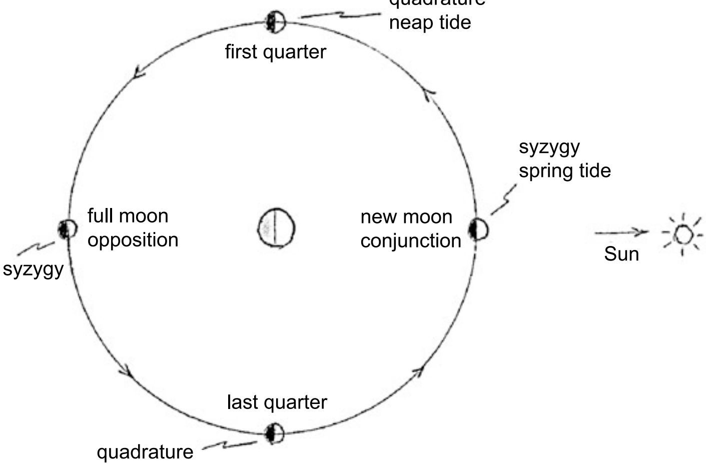


PIC31

PIC32



PIC33



1.4.4 What About the Sun?

Problems 1.4.21 and 1.4.31 examined the Moon's tidal influence. Let us estimate how the Sun's influence compares with it. In view of the expression for the key ratio f/g in Section 1.4.2,

$$\frac{\Delta h_{☽}}{\Delta h_{\odot}} = \frac{f_{☽}}{f_{\odot}} = \frac{M_{☽}}{M_{\odot}}\left(\frac{D_{\odot}}{D_{☽}}\right)^3 \approx \frac{\frac{1}{80}}{\frac{1}{3}\times 10^6}\cdot 400^3 \approx 2.4.$$

Thus, the influence of the Sun is modest but far from negligible.

(◄ PIC33) When the three bodies (Sun, Earth, Moon) become aligned, the lunar and solar tides strengthen each other; this is spring tide, and the configuration is called syzygy.[10] In contrast, they weaken each other when the three bodies form a right angle; this is neap tide, and the configuration is called quadrature.

1.5 Tides—Dynamic Picture

1.5.1 Forced Oscillator

It is time to return to the 'paradox' of Section 1.4.1 (PIC34 ►).

A preparatory discussion on the relative motion between the Moon (primary) and the Earth (secondary). Normally, over a time-scale of a day, we think of the Moon as stationary and of the Earth as rotating on its axis. The ocean as a whole rotates with the Earth:[11] after all, if instead the ocean were stationary and the solid Earth rotated underneath it, the sea floor would be swept by the water at a mad speed of 4×10^7 m/day ≈ 1666 km/hour. In the (non-inertial) frame in which the Earth and the ocean are together stationary, it is the Moon that runs around them—once a day, and retrograde. Until the end of Section 1.5 we shall work in this frame.[12] With the preparation out of the way, back now to the 'paradox'.

What will happen if, in the absence of the revolving primary, the ocean on the stationary secondary is put in the state (PIC35 ►) and released? It will oscillate as in (PIC36 ►), with some period T_{free}. In the presence of the primary, the tidal effect exerts a *periodic external forcing*, with some period T_{ext}, and the oscillating ocean responds to it by modifying its behavior.

Theorem 1.5.11

$T_{\text{ext}} > T_{\text{free}} \Longrightarrow$	*oscillator's response*	*in phase*	*with external forcing*.
$T_{\text{ext}} < T_{\text{free}} \Longrightarrow$	···	*out of phase*	···

---

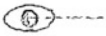
[10]Etymology: *syzygy* < Greek σύζυγος (spouse) < ζυγός (yoke), cf. *conjugate* < Latin *jugum*.

[11]We are modeling a spherical Earth covered by a sheet of ideal fluid, cf. footnote 8 in 1.4.1.

[12]As does the human society, which insists that the Moon and the Sun rise in the east and set in the west. Actually most of this motion is caused by us spinning from west to east.

PIC34

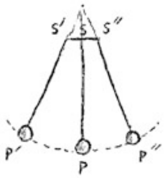
rather than



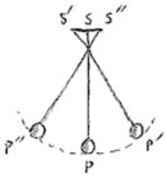
PIC35

PIC36

PIC37



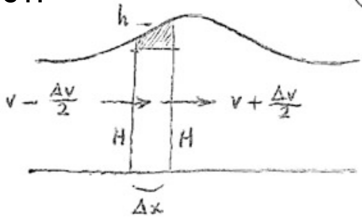
PIC38



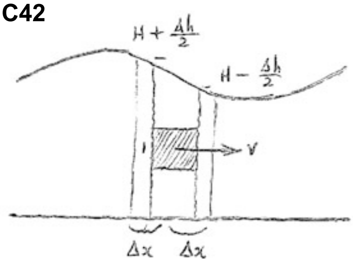
PIC39

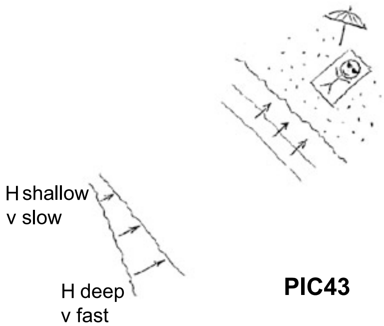
PIC40

PIC41



PIC42





PIC43

The conclusions can be checked by a home experiment. Hang a pendulum from a pivot, and wriggle the pivot horizontally. If this external wriggling is *slow* compared with the free oscillation of the pendulum ($T_{\text{ext}} > T_{\text{free}}$), then we observe the in-phase response (◄ PIC37). If the external wriggling is *fast* ($T_{\text{ext}} < T_{\text{free}}$), then we observe the out-of-phase response (◄ PIC38).

Proof The governing ODE

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}x + \left(\frac{2\pi}{T_{\text{free}}}\right)^2 x = f\exp\left(i\frac{2\pi}{T_{\text{ext}}}t\right)$$

is solved by

$$x(t) = \frac{f}{\left(\frac{2\pi}{T_{\text{free}}}\right)^2 - \left(\frac{2\pi}{T_{\text{ext}}}\right)^2}\exp\left(i\frac{2\pi}{T_{\text{ext}}}t\right).$$

The coefficient in front of exp has the same sign as f if $T_{\text{ext}} > T_{\text{free}}$, the opposite sign if $T_{\text{ext}} > T_{\text{free}}$. □

1.5.2 Free Oscillation of the Ocean

To apply Theorem 1.5.11 to the tide, we estimate T_{free} for the free oscillation of the ocean, following Airy's canal theory (1845).

Imagine digging a canal of depth H all the way along the equator (◄ PIC39). Let a hump of water, collapsing under its own weight, propagate as a wave along this canal, as in (◄ PIC40).

Theorem 1.5.21 *The speed of propagation of this wave is* $\sqrt{gH}$.

Proof (◄ PIC41) Within a narrow slab of width Δx, the conservation of volume says

$$\frac{\partial}{\partial t}h\Delta x = H\cdot\left(v + \frac{\Delta v}{2}\right) - H\cdot\left(v - \frac{\Delta v}{2}\right)$$
$$\Longrightarrow \quad \frac{\partial h}{\partial t} = H\frac{\partial v}{\partial x}.$$

(◄ PIC42) The equation of momentum per unit mass for a block of water of height 1 says

$$\frac{\partial}{\partial t}1\cdot\Delta x\cdot v = \frac{g\left(H + \frac{\Delta h}{2}\right)\Delta x - g\left(H - \frac{\Delta h}{2}\right)\Delta x}{\Delta x}$$
$$\Longrightarrow \quad \frac{\partial v}{\partial t} = g\frac{\partial h}{\partial x}.$$

Out drops $\partial^2 h/\partial t^2 = gH\partial^2 h/\partial x^2$, a wave equation with the propagation speed $\sqrt{gH}$. □

Fig. 1.4
Airy (1801–1892)

Problem 1.5.22 Why do waves arrive with their fronts parallel to the beach?

Because the dependence: speed $= \sqrt{gH}$ causes the wave front to turn (◄ PIC43).

Problem 1.5.23 Why do waves get steeper as they approach a beach and eventually break (PIC44 ►)?

Because the profile of a wave strains as the depth of water varies. Say a wave passes over an underwater 'step'. In (PIC45 ►) the front of the wave moves faster than the rear $\Longrightarrow$ the wave gets stretched and flatter. In (PIC46 ►) the rear of the wave moves faster than the front $\Longrightarrow$ the wave gets squeezed and steeper.

For H = average depth of the ocean ≈ 4 km, we find

$$\sqrt{gH} \approx \sqrt{\frac{1}{100} \cdot 4 \text{ km/sec}} \approx 700 \text{ km/hour}$$

(only a bit slower than a jet plane). At this speed the wave tours half of the canal, i.e. half-circumference of the Earth, and the water humps due to the tide exchange their positions, in

$$\frac{\frac{1}{2} \cdot 40000}{700} \approx 30 \text{ hours} = T_{\text{free}}.$$

As regards T_{ext}, it is easy: the Moon runs from one side of the Earth to the other side, retrograde, in $T_{\text{ext}} \approx 12$ hours.

$T_{\text{ext}} < T_{\text{free}}$ implies, by Theorem 1.5.11, that the response of the ocean must be *out of phase* with the tidal force, which means (PIC47 ►).[13]

[13] In order to have the in-phase picture (PIC48 ►), we would require a deeper ocean $H > 20$ km.

Fig. 1.5
Roche (1820–1883)

Remark 1.5.24

(1) T_{free} of the oscillation of a *spherical sheet* of water of depth 4 km, which is more realistic than that in a canal, turns out to be ≈24 hours.[14] Thus the out-of-phase inequality $T_{\text{ext}} < T_{\text{free}}$ is satisfied with margin to spare.
(2) Our estimate of T_{free} is sensible: when a major earthquake strikes Chile, Japan receives a tsunami approximately 24 hours later (and vice versa), the tsunami having crossed the Pacific, which extends about half-way around the Earth.
(3) The oceanographers are interested in the ocean tides, but the astronomers are more interested in the tidal response of a solid crust. The latter response is (`PIC49` ►) rather than (`PIC50` ►) because, of the elastic waves in the crust, even the slowest[15] travels at ≈1 km/sec, which implies the in-phase inequality $T_{\text{free}} \approx 5.5$ hours $< T_{\text{ext}}$. The amplitude of such a tide is of the order of 0.5 m on the equator.

1.6 Astronomical Applications

1.6.1 Tidal Tearing

What holds us on the ground (`PIC51` ►)? It is g. If the tidal force f per unit mass of the primary becomes 'a few times' g of the secondary (`PIC52` ►), then the particles on the secondary can no longer hold together. Whereupon the secondary begins to be torn apart by the tidal force of the primary. . . .

The Roche limit (1848) is the proximity within the primary at which this tearing begins. Roughly speaking, there exists a critical distance D_{Roche} such that

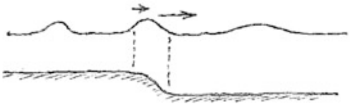
$$\frac{f}{g} = \frac{3}{2}\frac{M}{m}\left(\frac{r}{D_{\text{Roche}}}\right)^3 \approx \text{`a few times'}$$
$$\Longrightarrow \quad D_{\text{Roche}} \approx \text{`a few times'}\left(\frac{M}{m}\right)^{1/3} r = \text{`a few times'}\left(\frac{\mathrm{P}}{\rho}\right)^{1/3} R.$$

In the last equality we used $m \sim \rho r^3$, $M \sim \mathrm{P} R^3$.

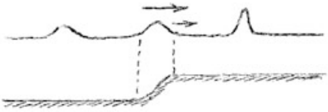
[14] A hump spreads and propagates as a ring and meets as a new hump on the antipodes.
[15] Rayleigh wave (1885).

PIC44

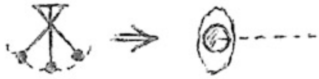
PIC45



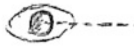
PIC46



PIC47

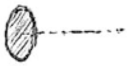


PIC48

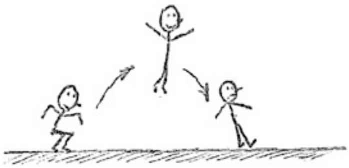


PIC49

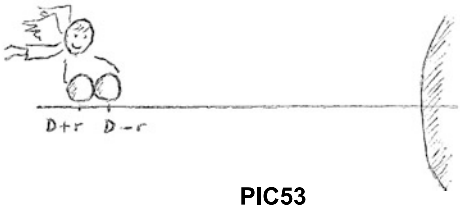
PIC50



PIC51



PIC52



PIC53

Alternative interpretation: D_{Roche} is attained when the amplitude of the tide Δh of Section 1.4.3 exceeds the radius r of the secondary.

Let us estimate what 'a few times' should mean. (◄ PIC53) An angel deposits a pair of coconuts in contact (each with parameters m, r, ρ) at the distances $D \pm r$ from the primary. Will the coconuts detach themselves?

The gravitational cohesion between the coconuts is $Gm^2/(2r)^2$. Competing against this, the force of detachment due to the tide is

$$\begin{aligned}\text{attraction}|_{D-r} - \text{attraction}|_{D+r} &\approx \frac{\partial}{\partial D}\left(\frac{GMm}{D^2}\right)\cdot(-2r)\\ &= \frac{4GMm}{D^3}r,\end{aligned}$$

which is $2m$ times the representative tidal force $\mathcal{F}$ per unit mass of Section 1.4.2. Therefore the detachment begins at

$$D_{\text{Roche}} = \left(16\frac{M}{m}\right)^{1/3} r \approx 2.5\left(\frac{M}{m}\right)^{1/3} r.$$

Thus, 'a few times' should be between $2\times$ and $3\times$.

Example 1.6.11 Instances of tidal tearing include: the formation of planetary rings and, in more recent history, the fragmentation of the comet Shoemaker-Levy 9 as it approached Jupiter (July 1992).

1.6.2 Tidal Locking

Why does the Moon always show the same face to us?

Imagine a barbell (secondary) in circular orbit around M (primary), at angular frequency $\omega = \dot{\varphi}$ (PIC54 ►). ω is determined by the condition that, in a circular orbit, the centrifugal force and the attraction balance:

$$\frac{(D\omega)^2}{D} = \frac{GM}{D^2} \quad \Longrightarrow \quad \omega = \sqrt{\frac{GM}{D^3}}.$$

The Lagrangian may be written down in terms of the parameters given in (PIC54 ►). We have

$$\begin{aligned}\text{kinetic energy} &= \frac{1}{2}m\big(\dot{D}_+^2 + (D_+\dot{\varphi}_+)^2 + \dot{D}_-^2 + (D_-\dot{\varphi}_-)^2\big),\\ \text{potential energy} &= -GMm\left(\frac{1}{D_+} + \frac{1}{D_-}\right)\end{aligned}$$

and from geometry

$$\begin{aligned} D_\pm &\approx D \pm \ell\cos\psi & &\quad & \dot{D}_\pm &\approx \mp\ell\dot{\psi}\sin\psi\\ \varphi_\pm &\approx \varphi \pm \frac{\ell}{D}\sin\psi & &\Longrightarrow & \dot{\varphi}_\pm &\approx \omega \pm \frac{\ell}{D}\dot{\psi}\cos\psi.\end{aligned}$$

$(\ell/D)^2$ being neglected as high order, the approximate Lagrangian comes out to be

$$L = \text{kin} - \text{pot} \approx 3mD^2\omega^2 + m\ell^2\big(\dot{\psi}^2 - 4\omega\dot{\psi}\cos^2\psi + 3\omega^2\cos^2\psi\big).$$

The Euler-Lagrange equation

$$\frac{\partial L}{\partial \psi} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\psi}} = 0$$

reads

$$(2\ddot{\psi}) + 3\omega^2 \sin(2\psi) = 0,$$

which is the equation of pendulum motion. The frequency of small oscillations of this pendulum is $\sqrt{3}$ times the frequency of the orbital revolution. With dissipation, any motion of the barbell asymptotes to the unique equilibrium $\psi = 0$.

Now suppose a secondary (e.g. Moon) revolves around a primary (e.g. Earth). Typically the secondary is *not* 'inertially spherical'. We can think of a barbell as its toy model. Then on the orbital revolution a tidal oscillation gets superposed (`PIC55` ►). With dissipation, the oscillation asymptotes to $\psi = 0$, i.e. settles in the radial direction to the primary, and so the secondary ends up getting *locked* with its face toward the primary.

This *is* a tidal effect. Indeed, in constant gravity (zero derivative in the attraction) there is no torque restoring the barbell to face the primary (`PIC56` ►).

Examples 1.6.21 Instances of tidal locking include: mutual locking of Pluto and Charon and, in more recent history, stabilization of artificial satellites (to keep them facing the Earth), notably Gemini 11 and 12 (1966).

1.6.3 Tidal Dissipation

As discussed in Section 1.5.1, in the absence of any tidal force, the ocean would rotate together with the solid Earth underneath as a single rigid body (`PIC57` ►) $\Longrightarrow$ no dissipation.

In the presence of the Moon and its tidal force (`PIC58` ►), the ocean is 'held in place' and *rubs* the solid Earth, which is rotating underneath $\Longrightarrow$ dissipation, another effect of the tide.

Let us estimate the rate of tidal dissipation.

We distinguish two kinds of angular momentum involved in this phenomenon: the one carried by the secondary (e.g. Earth)'s rotation about its own axis, which we call spin angular momentum, and the other carried by the revolution of the primary (e.g. Moon) around the secondary,[16] which we call orbital angular momentum. The spin a.m. and the orbital a.m. are respectively

$$I\dot{\psi} \quad \text{and} \quad L = D \times m_{\text{red}} D\dot{\phi}$$

where we define $m_{\text{red}} = Mm/(M+m)$, the so-called reduced mass.

The tidal torque τ due to the primary acts on the secondary's spin a.m. and does work at the rate

$$\dot{E} = \tau\left(\dot{\psi} - \dot{\phi}\right) = I\ddot{\psi}\left(\dot{\psi} - \dot{\phi}\right).$$

[16] More precisely, the revolution of the primary and the secondary around each other.

The details of the dissipation are so complex that it is impracticable to estimate $\dot{E}$ from first principles. But we *can* estimate $\dot{E}$ once we measure observationally the values of various parameters on the right-hand side of this equation. These stand for: $\dot{\psi}$ = how fast the secondary rotates about its axis, $\ddot{\psi}$ = how this rotation is being decelerated,[17] and $\dot{\phi}$ = how fast the primary revolves around the secondary. It will be helpful to note in addition that, since a month lasts approximately 30 days,[18] we have for the Earth-Moon pair

$$\frac{\dot{\phi}}{\dot{\psi}} \approx \frac{1}{30}.$$

Problem 1.6.31 Estimate the tidal dissipation on the Earth by the Moon.

$$I \approx \frac{2}{5} M_{\oplus} R_{\oplus}^2 \approx \frac{2}{5} \cdot 6 \times 10^{24} \cdot \left(\frac{4 \times 10^7}{2\pi}\right)^2 \approx 10^{38}\ \mathrm{kg\,m^2}$$
$$\dot{\psi} = \frac{2\pi}{24 \cdot 60 \cdot 60} \approx 7.3 \times 10^{-5}\ \mathrm{sec}^{-1} \quad \text{(childhood knowledge)}$$
$$\ddot{\psi} \approx -4.6 \times 10^{-22}\ \mathrm{sec}^{-2} \quad \text{(observational data)}.$$

Neglecting $\dot{\phi}/\dot{\psi} \ll 1$,

$$\dot{E} \approx I \ddot{\psi} \dot{\psi} \approx -3.7 \times 10^{12}\ \mathrm{J\,sec}^{-1} \approx -1.2 \times 10^{20}\ \mathrm{J\,year}^{-1}.$$

This is double the consumption of electricity in the world $\approx -6.1 \times 10^{19}$ J year^{-1}, according to the CIA data of 2005.[19]

I have been told that an astonishing 1/3, or some such fraction, of this dissipation takes place in the Bering Sea and the Sea of Okhotsk.

Digression 1.6.32 Sometimes we hear that the Coriolis force makes water spin one way or the other as it drains down a sink. Estimate how significant the Coriolis force is.

The Coriolis force per unit mass of water is $2\dot{\psi} \times v$. A natural standard for comparison is g, which the water also feels. From a sink of depth h water drains at speed $v = \sqrt{2gh}$ by Torricelli's law (1643). Hence, taking $h \approx 0.1$ m,

$$\frac{2\dot{\psi} \times v}{g} \approx \sqrt{\frac{8h}{g}}\,\dot{\psi} \approx 2 \times 10^{-5},$$

which is utterly invisible. The phenomenon is dominated by irregularities in the build of the sink and by random initial conditions of the water.

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[17] $\ddot{\psi} < 0$ because the friction on the sea floor by the 'tidally held' ocean slows down the secondary's rotation.

[18] Lest the astronomers complain: anomalistic, draconic, sidereal, synodic.... For our approximate purposes here it does not matter which, they are all a little under 30 days.

[19] *Julius Caesar* IV. iii. 218–219 may come to some people's mind.

We take up another consequence of tidal dissipation that affects the fate of a system consisting of a primary revolving around a secondary.

Tidal dissipation brakes the secondary's spin a.m. $I\dot{\psi}$. Meanwhile, the total a.m. of the system $I\dot{\psi} + L$ is conserved. To compensate, the orbital a.m. L grows. As it orbits around faster,[20] the primary must *drift away* from the secondary (`PIC59` ►), along a spiral.

Let us estimate the rate at which the primary drifts away from the secondary. Refer again to (`PIC58` ►).

In quasi-circular orbit,

$$\frac{m_{\text{red}}\left(D\dot{\phi}\right)^2}{D} \approx \frac{GMm}{D^2} \quad \Longrightarrow \quad D \approx \frac{L^2}{GMmm_{\text{red}}}.$$

On account of the conservation of total a.m. we have $\dot{L} = -I\ddot{\psi}$, which implies

$$\dot{D} = -\frac{2D^2\dot{\phi}I\ddot{\psi}}{GMm} \approx -2\frac{\dot{\phi}}{\dot{\psi}}\frac{D^2}{GMm}\dot{E} = -2\frac{\dot{\phi}}{\dot{\psi}}\frac{\dot{E}}{\text{mutual attraction}}.$$

$\dot{E} < 0$ tells us that $\dot{D} > 0$.

Problem 1.6.33 Estimate $\dot{D}_{☽}$ for the Earth-Moon pair. What are we led to conclude if we extrapolate naively into the past?

Using $\dot{E}$ from Problem 1.6.31 and the size of the mutual attraction from Problem 1.22,

$$\dot{D}_{☽} \approx -2 \cdot \frac{1}{30} \cdot \frac{-1.2 \times 10^{20}}{2 \times 10^{20}} \approx 0.04 \text{ m year}^{-1} = 4 \text{ cm year}^{-1}.$$

This is how fast the Moon is drifting away from the Earth. So

$$-\frac{D_{☽}}{\dot{D}_{☽}} \approx \frac{60 \cdot \frac{4\times10^7}{2\pi}}{0.04} \approx 10^{10} \text{ years}$$

in the past, the Moon must have been in contact with the Earth.

Our estimate of $\dot{D}_{☽}$ is consistent with observational data, yet our naive extrapolation leads to double the geological estimate of the age of the Earth-Moon pair. The error is imputable to our linear extrapolation: the tidal dissipation was more efficient when the Moon was nearer the Earth.

For controversies surrounding other methods of estimating the age, I recommend T. W. Körner, *Fourier Analysis,* Cambridge UP, 1988, chapters 56, 57, 58.

Examples 1.6.34 Where does the energy dissipated by the tide go? It heats up the secondary. Tidal heating is dramatic when a tidal force periodically kneads a small

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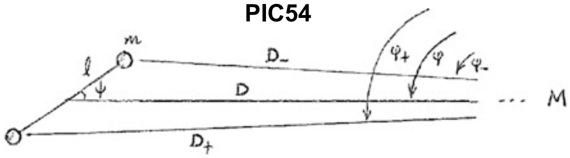
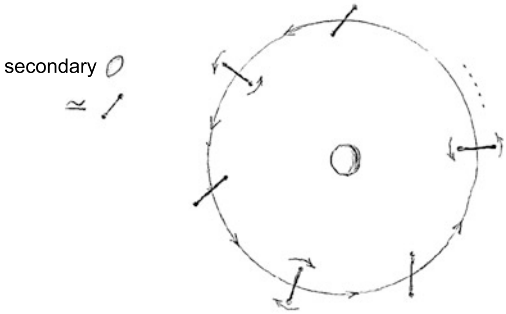
[20] And since simultaneously the Earth is spinning slower, we terrestrials have the impression that the Moon is orbiting all the faster. Halley was the first to notice this (1695).

PIC54



PIC55

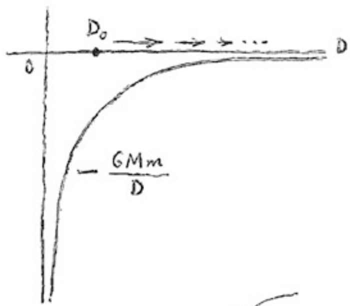


PIC57

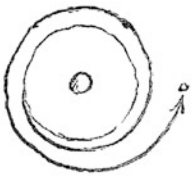
PIC56

PIC58

PIC60



PIC59



PIC∞

secondary in resonance, e.g. Io : Europa : Ganymede (4 : 2 : 1 resonance) around Jupiter, driving on Io the most violent volcanism in the solar system.

1.6.4 Visit to the Horizon

Can we visit the horizon of a black hole without being torn apart by its tidal force? To answer this question, we prepare one concept.

(◀ PIC60) Imagine an astro-tourist in the gravitational field of a black hole. Initially she is at a distance D_0 from the black hole and launches herself with an outward speed v_0. Will she be able to escape to infinity? At the start, her energy was

$$\frac{1}{2}mv_0^2 - \frac{GMm}{D_0}.$$

At the end, if she manages to escape to infinity with no residual speed to spare, then her energy will be

$$\frac{1}{2}m0^2 - \frac{GMm}{\infty} = 0.$$

It follows that the escape requires the inequality

$$D_0 \geqslant \frac{2GM}{v_0^2}.$$

In other words, if the astro-tourist starts with $D_0 < 2GM/v_0^2$, she will exhaust her momentum before reaching infinity and will fall back toward the black hole. But v_0 available is at most the speed of light c. Therefore

$$D_{\text{Sch}} = \frac{2GM}{c^2} = \frac{2GM_\odot}{c^2}\frac{M}{M_\odot}$$

$$\approx \frac{2 \cdot \frac{2}{3} \times 10^{-10} \cdot 2 \times 10^{30}}{(3 \times 10^8)^2} \approx 3 \times 10^3 \frac{M}{M_\odot} \text{ in meters,}$$

called the Schwarzschild radius (1915),[21] represents the size of the horizon, from the interior of which nothing, not even light, can escape.[22]

Fig. 1.6
Schwarzschild (1873–1916)

[21] He wrote this paper while serving on the Russian front in WWI, a year before he died. The other paper he wrote in the same year supplied a quantum explanation of the Stark effect.

[22] A rigorous calculation using general relativity yields the same expression $2GM/c^2$ for D_{Sch}.

Problem 1.6.41 What are the Schwarzschild radii of the Earth, the Moon, the Sun?

Approximately 1 cm, 0.1 mm, 3 km.

It remains to relate the Schwarzschild radius to the Roche limit. To visit the horizon without being hurt, the astro-tourist would like the representative tidal force $\mathcal{F}$ per unit mass of Section 1.4.2 to be $Ng_{♁}$, with $N = 2$ or 3 at most. At the horizon,

$$\frac{\mathcal{F}}{Ng_{♁}} = \frac{2\frac{GM}{D_{\text{Sch}}^3}d}{Ng_{♁}} \quad \text{where } d = \text{your diameter}$$
$$= \frac{2GM_{\odot}}{(3\times10^3)^3 g_{♁}}\frac{d}{N}\left(\frac{M_{\odot}}{M}\right)^2 \approx \frac{2\cdot\frac{2}{3}\times10^{-10}\cdot2\times10^{30}}{27\times10^9\cdot10}\frac{d}{N}\left(\frac{M_{\odot}}{M}\right)^2$$
$$\approx 10^9\frac{d}{N}\left(\frac{M_{\odot}}{M}\right)^2.$$

This ratio is maintained within $\leqslant 1$ provided

$$M \geqslant \sqrt{10^9\frac{d}{N}}M_{\odot} \approx 3\times10^4 M_{\odot}$$

for the choice $d \approx 2$ m, $N = 2$. A relatively painless visit to the horizon is possible provided the black hole is massive enough.

The center of our Galaxy is said to have mass $\approx 4\times10^6 M_{\odot}$. Its horizon, at $D_{\text{Sch}} \approx 0.01$ A.U., is a possible tourist attraction.

(◄ PIC∞)