

Equatorial Kelvin waves

The derivation here follows Gill (1982). Consider first the case of an equatorial Kelvin wave, which is a special solution for the case of zero meridional velocity ($v = 0$), no forcing and no dissipation. In this case, these equations

$$\begin{aligned}\frac{\partial u}{\partial t} - fv &= -g' \frac{\partial h}{\partial x} \\ \frac{\partial v}{\partial t} + fu &= -g' \frac{\partial h}{\partial y} \\ \frac{\partial h}{\partial t} + H \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) &= 0\end{aligned}$$

with $f = f_0 + \beta y$ and $f_0 = 0$, reduce to,

$$\begin{aligned}\frac{\partial u}{\partial t} &= -g' \frac{\partial h}{\partial x} \\ \beta y u &= -g' \frac{\partial h}{\partial y} \\ \frac{\partial h}{\partial t} + H \frac{\partial u}{\partial x} &= 0.\end{aligned}$$

Note the geostrophic balance in the y -momentum equation. Substituting $e^{i(kx - \omega t)}$ dependence for all three variables, we get from the first that $u = (kg'/\omega)h$, so that the third one gives the dispersion relation

$$\omega^2 = (g'H)k^2$$

which is the dispersion relation of a simple shallow water gravity wave. The second equation then gives $\beta y \frac{kg'}{\omega} h = -g' \frac{\partial h}{\partial y}$, or

$$\frac{\partial h}{\partial y} = -\frac{\beta k}{\omega} y h.$$

We are searching for equatorial-trapped solutions, and we note that the solution for the y -structure decays away from the equator only when $k > 0$. This implies that the wave solution we have found must be eastward propagating! Using the dispersion relation, with

$$c \equiv \sqrt{g'H} \approx (9.8 \times 10^2 \times \text{cm sec}^{-2} \times 5 \times 10^{-3} \times 100 \times 10^2 \text{cm})^{1/2} \approx 2.2 \text{m/sec}$$

we finally have

$$h_{\text{Kelvin}}(x, y, t) \propto e^{-\frac{1}{2}(\beta/c)y^2} e^{i(kx - \omega t)}.$$

Note that the decay scale away from the equator is the equatorial Rossby radius of deformation defined as

$$L_{eq}^R \equiv \sqrt{c/2\beta} \approx (c/(2 \times 2.3 \times 10^{-11} \text{m}^{-1} \text{sec}^{-1}))^{1/2} \approx 220 \text{km}$$

References

Gill, Adrian E. (1982). *Atmosphere-Ocean Dynamics*. Academic Press, Inc, San Diego, CA, 662pp, p. 662.