EPS131, Introduction to Physical Oceanography and Climate

Section 8: Western boundary currents, wind-driven great ocean gyres, abyssal circulation

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Contents

Our purpose now is to understand the large-scale wind-driven circulation, including the presence and location of western boundary currents such as the Gulf Stream and the Kuroshio.

1 Momentum equations for the wind-driven circulation, the beta plane

Consider now the following momentum balance,

$$\text{accelaration} + \text{coriolils} = \text{pressure} + \text{friction}$$

$$\frac{\partial u}{\partial t} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} - ru \quad (1)$$

$$\frac{\partial v}{\partial t} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} - rv \quad (2)$$

The Coriolis force varies with latitude, and we need to approximate it around a latitude θ_0 (e.g., for the North Atlantic, $\theta_0 \approx 40N$) using a Taylor expansion,

$$\begin{aligned} f &= 2\Omega \sin\theta \approx 2\Omega \sin\theta_0 + 2\Omega \cos\theta_0\,(\theta - \theta_0) \\ &= 2\Omega \sin\theta_0 + \frac{2\Omega}{R}\cos\theta_0\, R(\theta - \theta_0) \\ &= f_0 + \beta y \end{aligned}$$

where R is the Earth radius, $\beta = (2\Omega/R)\cos\theta_0$, and $y = R(\theta - \theta_0)$ is the distance from the reference latitude. Note that based on the above, $\beta = df/dy$.

2 Vorticity, planetary vorticity

Understanding the wind-driven circulation and western boundary currents requires using a vorticity rather than momentum budget. The vorticity vector (denoted bold face zeta, $\boldsymbol{\zeta}$) is defined as the curl of the velocity field,

$$\boldsymbol{\zeta} \equiv \nabla \times \mathbf{u} = \text{curl}(\mathbf{u}) = \begin{pmatrix} \dfrac{\partial w}{\partial y} - \dfrac{\partial v}{\partial z} \\[2ex] \dfrac{\partial u}{\partial z} - \dfrac{\partial w}{\partial x} \\[2ex] \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y} \end{pmatrix}$$

we note that the vertical component that is of interest to us in the present context is denoted by a non-bold zeta,

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.$$

To obtain an intuition of what vorticity represents, consider a "solid body rotation" of water in a bucket rotating at an angular velocity ω. In cylindrical coordinates, the velocity is,

$$v^{(r)} = 0, \quad v^{(\theta)} = \omega r.$$

The vorticity is, therefore,

$$\zeta = \text{curl}(v^{(r)}, v^{(\theta)}) = \frac{1}{r}\frac{\partial}{\partial r}[r v^{(\theta)}] - \frac{1}{r}\frac{\partial v^{(r)}}{\partial \theta} = \frac{1}{r}\frac{\partial}{\partial r}[r(\omega r)] = 2\omega,$$

or

$$\text{vorticity} = 2 \times \text{rotation rate}.$$

Planetary vorticity. If the ocean is at rest, it still has vorticity due to the Earth's rotation, which is a solid-body rotation for a resting ocean. At the north pole, $\omega = \Omega$ and, therefore, the planetary vorticity is 2Ω. At the south pole, it is -2Ω as the rotation is in the opposite direction relative to the local vertical coordinate. In between,

$$\text{planetary vorticity} = 2\Omega \sin\theta = f!$$

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3 Vorticity equation

We can finally derive an equation for the vorticity as follows. Take the curl of the momentum equations as

$$\frac{\partial(\text{eqn } 2)}{\partial x} - \frac{\partial(\text{eqn } 1)}{\partial y},$$

to find,

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \beta v = -r\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right).$$

Use the continuity equation,

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) - f\frac{\partial w}{\partial z} + \beta v = -r\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right). \tag{3}$$

Integrate from the ocean bottom to the base of the mixed layer, $\int_{-H}^{-50\ m} dz$ assuming that u, v are depth independent and using the boundary condition that w vanishes at the bottom, to find,

$$\frac{\partial}{\partial t}\left(\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}\right) - fw(-50\ m) + \beta V = -r\left(\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}\right),$$

where upper case letters represent the integrated velocities. Defining $\zeta = \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}$, and using the expression for the Ekman pumping velocity derived above, we find the *vorticity equation*,

$$\underbrace{\frac{\partial \zeta}{\partial t}}_{\text{(a)}} + \underbrace{\beta V}_{\text{(b)}} = \underbrace{-r\zeta}_{\text{(c)}} + \underbrace{f \operatorname{curl}\left(\frac{\boldsymbol{\tau}}{\rho_0 f}\right)}_{\text{(d)}}. \quad (4)$$

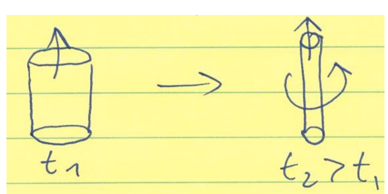
Consider the interpretation of each term here.

(a) Rate of change of the vorticity.

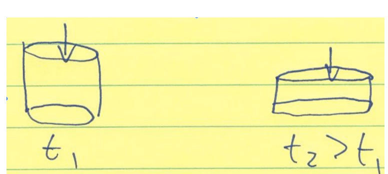
(b) $\beta V = V\, df/dy$, meridional advection of planetary vorticity.

(c) $-r\zeta$ dissipation of friction due to friction. This term leads to an exponential decay of the vorticity in the absence of other forcings.

(d) Wind curl changing the vorticity via Ekman pumping. To understand this, consider a fluid column extending from the bottom to the base of the Ekman layer. The column spins due to either (or both) the relative vorticity ζ and the planetary vorticity $f = 2\Omega \sin\theta$. Now, if the Ekman pumping is upward, the fluid column is being stretched and, therefore, its rotation (vorticity) should increase,



while if the Ekman pumping is downward, the fluid column is being compressed and, therefore, its rotation (vorticity) should decrease,



4 Vortex decay

Simplest vorticity balance: time rate of change and friction. The vorticity equation is,

$$\frac{\partial \zeta}{\partial t} = -r\zeta$$

and the solution is

$$\zeta(t) = \zeta_0 e^{-rt},$$

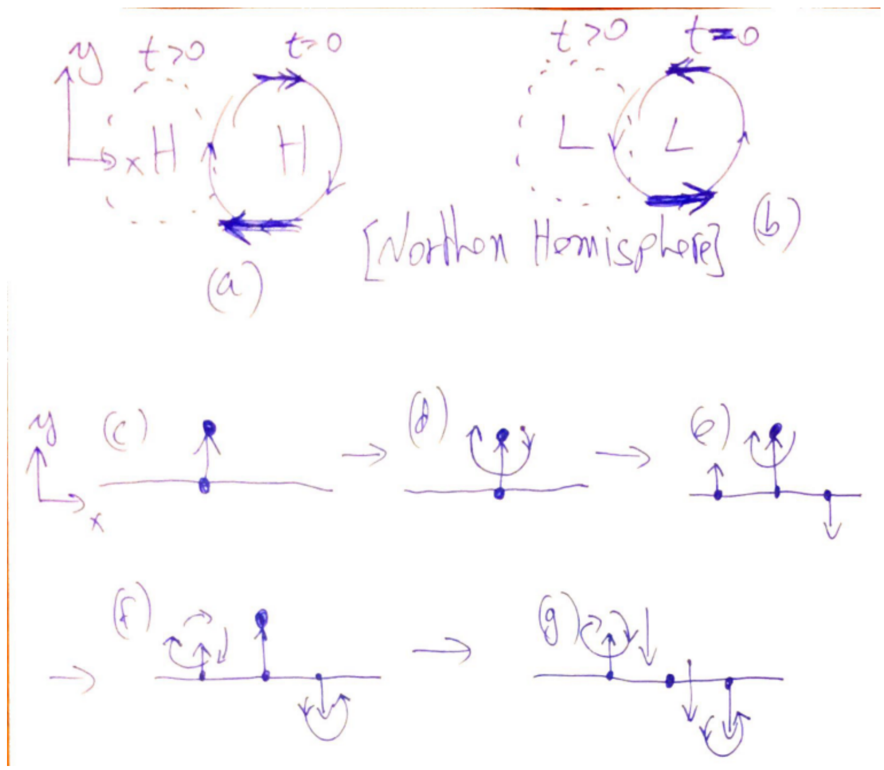
explaining the decay of vortices

5 Rossby waves

We can derive the solution for Rossby waves in a shallow-water ocean. We assume the ocean is shallow, in which case it can be shown that the horizontal velocities are depth-independent. Our starting point is the vorticity equation plus geostrophy. We ignore winds and friction for now in the vorticity budget, so we are left with a vorticity balance in which the time rate of change balances the meridional advection of planetary vorticity. Consider, first, two heuristic explanations for Rossby waves.

Panels a and b show that a high pressure or a low pressure would propagate westward because of the difference in the zonal transport from the geostrophic balance, $u = -\frac{1}{f\rho_0}\frac{\partial p}{\partial y}$, at the south vs north edge of the high/ low.

Panels (c–g) show a fluid element being displaced northward and being restored to its original location by the response of its neighboring columns. This shows that the planetary vorticity gradient (beta effect) is the restoring force for these waves.



Next, derive the dispersion relation. Because we ignore winds and friction for now, our above vorticity balance becomes,

$$\zeta_t + \beta V = 0.$$

We also use the geostrophic equations for a shallow-water ocean $-f_0 v = -g\eta_x$ and $f_0 u = -g\eta_y$. Integrating these two equations in depth from $z = -H$ to $z = 0$ and remembering that the velocities and surface elevation are both depth-independent, the equations become $-f_0 V = -gH\eta_x$ and $f_0 U = -gH\eta_y$. Plugging these integrated geostrophic equations into the vorticity equation, we find

$$\partial_t(\eta_{xx} + \eta_{yy}) + \beta\eta_x = 0.$$

Looking for a wave solution $\eta = \eta_0 \cos(kx + ly - \omega t)$, we find

$$\omega = \frac{-\beta k}{(k^2 + l^2)}.$$

Figure 1 shows a schematic of this dispersion relation. Because the signs of the frequency and wavenumbers are arbitrary, and only their ratio matters, we choose the convention that ω is positive. Under this convention, Rossby waves only involve negative wavenumbers.

The dispersion relation implies a westward phase propagation ($\omega/k < 0$) for Rossby

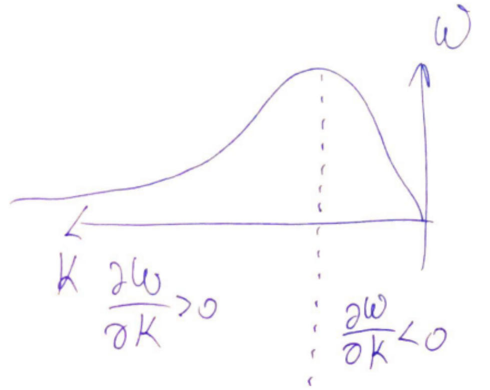


Figure 1: The Rossby wave dispersion relation.

waves. The group propagation velocities in the (x, y) directions are given by,

$$C_g^{(x)} = \frac{\partial \omega}{\partial k} = \beta \frac{k^2 - l^2}{(k^2 + l^2)^2}$$
$$C_g^{(y)} = \frac{\partial \omega}{\partial l} = \frac{2\beta k l}{(k^2 + l^2)^2}.$$

Based on the slope of the dispersion relation as a function of k, we see that the group velocity in the zonal direction (x) is eastward for short waves and westward for long waves.

6 The equatorial beta plane and equatorial Kelvin waves

These equatorial waves are related to the coastal Kelvin waves discussed previously and play an important role in El Niño. Remembering that we assumed the velocity perpendicular to the coast to vanish everywhere for coastal Kelvin Waves, we now consider the case of zero meridional velocity ($v = 0$), as well as no forcing and no dissipation. The Coriolis force at the equator, where $f_0 = 0$ is $f = \beta y$, and the two momentum equations and mass-conservation equation reduce to

$$\begin{aligned}
\frac{\partial u}{\partial t} &= -g' \frac{\partial h}{\partial x} \\
\beta y u &= -g' \frac{\partial h}{\partial y} \\
\frac{\partial h}{\partial t} + H \frac{\partial u}{\partial x} &= 0
\end{aligned}$$

Note the geostrophic balance in the y-momentum equation. Substituting $e^{i(kx-\omega t)}$ dependence for all three variables, we get from the first that $u = (kg'/\omega)h$, so that the third one gives the dispersion relation

$$\omega^2 = (g'H)k^2$$

which is the dispersion relation of a simple shallow-water gravity wave. The second equation then gives $\beta y \frac{kg'}{\omega} h = -g' \frac{\partial h}{\partial y}$, or

$$\frac{\partial h}{\partial y} = -\frac{\beta k}{\omega} yh.$$

We are searching for equatorial-trapped solutions, and we note that the solution for the y-structure decays away from the equator only when $k > 0$. This implies that the wave solution we have found must be eastward propagating! Using the dispersion relation with

$$c \equiv \sqrt{g' \times H} \approx \left((9.8\ m\, s^{-2} \times 5 \times 10^{-3}) \times 100\right)^{1/2} \approx 2.2\, m/s$$

we finally have

$$h_{Kelvin}(x, y, t) \propto e^{-\frac{1}{2}(\beta/c)y^2} e^{i(kx-\omega t)}.$$

Note that the decay scale away from the equator is the equatorial Rossby radius of deformation defined as

$$L^R_{eq} \equiv \sqrt{c/2\beta} \approx \left(c/(2 \times 2.3 \times 10^{-11} m^{-1} s^{-1})\right)^{1/2} \approx 220\ km$$

7 Wind-driven circulation away from western boundaries: Sverdrup balance

Consider first the vorticity balance in a steady state away from horizontal boundaries. In that case, the time rate of change term in (4) vanishes, and the friction term may be assumed small so that the vorticity equation reduces to the *Sverdrup balance*,

$$\beta V = f \operatorname{curl}\left(\frac{\boldsymbol{\tau}}{\rho_0 f}\right) \approx \operatorname{curl}\left(\frac{\boldsymbol{\tau}}{\rho_0}\right). \tag{5}$$

We can also write this transport by the wind-driven circulation in terms of the Ekman pumping that drives it. Given that $w_E = \operatorname{curl}(\boldsymbol{\tau}/\rho_0 f)$, the equation for the transport may be written as,

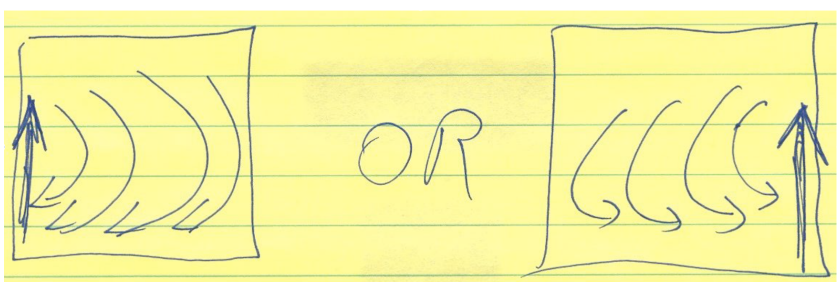
$$\beta V = f w_E, \tag{6}$$

which we can now use to interpret this balance.

In the Northern Hemisphere subtropical gyres (e.g., North Atlantic, 20–50°N), the Ekman pumping is downward, and the interior circulation is southward. This may be interpreted as vortex compression applied to fluid columns by the Ekman pumping, which decreases their vorticity. Given that the relative vorticity is negligible relative to the planetary vorticity away from strong currents such as the Gulf Stream, the *planetary* vorticity needs to decrease. This implies a southward motion of the fluid columns toward the equator. In the *subpolar* gyres in the Northern Hemisphere, the Ekman pumping is upward, and the resulting interior velocity is northward.

In the Southern Hemisphere, a downward (negative) Ekman pumping drives a northward (positive, northward) flow V as can be seen in (6) given that f is negative there. The factor f in front of the Ekman pumping indicates that the column has some planetary vorticity f that is stretched or compressed by the Ekman pumping. In the southern hemisphere, the planetary vorticity is negative. The compression of fluid columns leads to a decrease of this negative vorticity and, therefore, to an increase in planetary vorticity. This requires a northward motion.

In the subtropical gyre, the broad southward interior flow should return northward somewhere, which naively could occur in a narrow and fast boundary current either in the east or west,



The next section considers the vorticity balance near the boundaries to determine which of these two options makes sense.

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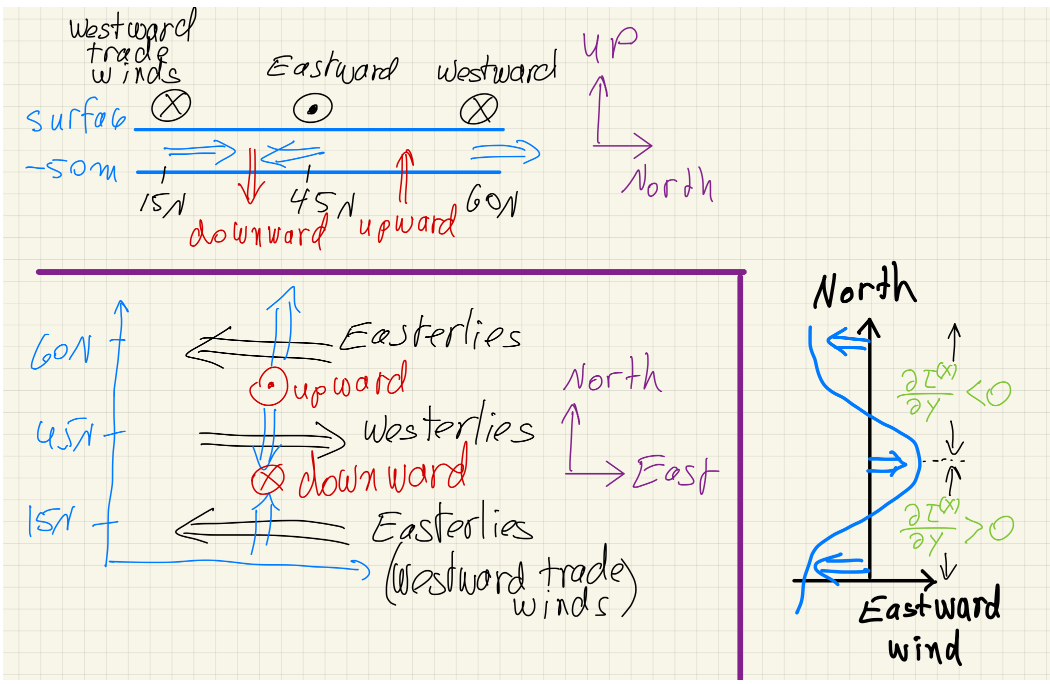
Orders of magnitude for Ekman transport vs Sverdrup transport. Suppose the wind stress $\tau^{(x)}(y)$ at 40N is westward at an amplitude 0.1 $\mathrm{N/m^2}$, reducing to zero further north over a distance of 1000 km. Let the basin width be $L = 2500$ km. The maximum northward Ekman transport over the entire east-west width of the basin is then $LM^{(y)} = L\tau^{(x)}/(\rho f) = 2500 \times 10^3 \times 0.1/(1025 \times 10^{-4}) = 2.4 \times 10^6\ \mathrm{m^3/s} = 2.4$ Sv. The curl is $\mathrm{curl}(\tau) \approx 0.1/(10^6\ \mathrm{m})$ so that the Sverdrup transport integrated over the width of the basin is $L\,\mathrm{curl}(\tau)/(\rho\beta) = (2500 \times 10^3) \times (0.1/10^6)/(1025 \times 10^{-11}) = 24$ Sv. Clearly, the Sverdrup transport driven by vortex stretching is much larger than the Ekman transport driven directly by the wind.

Why are there both a subtropical gyre and a subpolar gyre in the North Atlantic? The North Atlantic is characterized by a *subtropical gyre* involving a northward flow in the Gulf Stream balanced by a southward flow in the interior (roughly 15–45N). There is also a *subpolar gyre* with a southward flow in the Labrador current, again compensated by a return northward flow in the interior in the northern part of the North Atlantic (north of 45N). To see why there are two gyres in the North Atlantic, consider the following schematic of the winds in the North Atlantic (similarly in the North Pacific, etc.), where we approximate the wind as being in the zonal directly only, $\vec{\tau} = (\tau^{(x)}, 0)$.



Note that in the subtropical gyre, roughly 15N–45N, the wind increases with latitude so that the wind curl $-\partial\tau^{(x)}/\partial y$ is negative, implying a downward Ekman pumping. The wind over the subpolar gyre, 45N-60N, decreases with latitude, implying an upward Ekman pumping. The downward subtropical Ekman pumping leads to vortex compression and, therefore, to a southward movement of fluid elements. Mathematically, the curl of the wind is balanced by βv, and the velocity v must be negative or southward. The Ekman pumping drives a northward interior Sverdrup flow in the subpolar gyre. Each of the two gyres is closed by a western boundary current, as discussed next.

8 Wind driven circulation: western boundary currents

As seen in the last figure, V must be large near the boundary regardless of whether it is in the east or west. As a result, we expect βV and the $-r\partial V/\partial x$ part of $-r\zeta$ are expected to dominate other terms, giving the approximate vorticity balance,

$$\beta V = -r\frac{\partial V}{\partial x}.$$

This is a balance between changes in planetary vorticity due to the meridional movement and friction. The solution is, therefore,

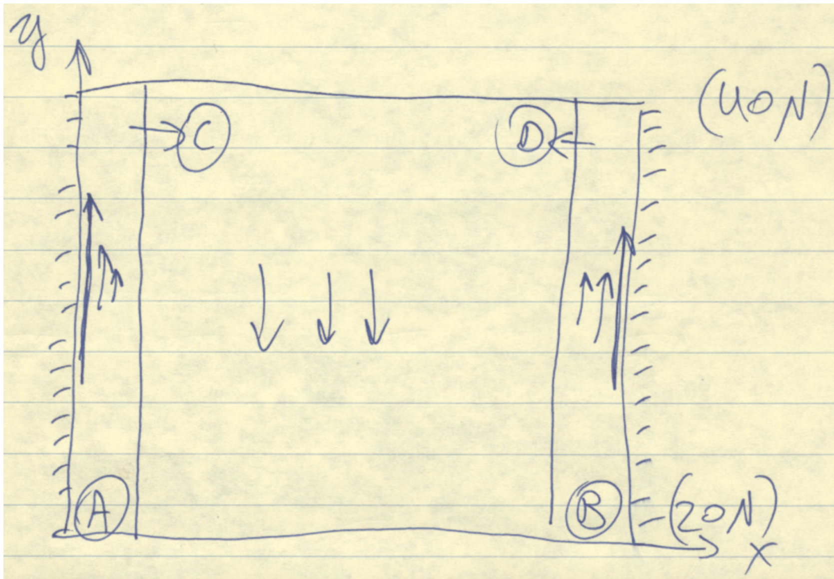
$$V = V_0 e^{-\frac{\beta}{r}x}.$$

This solution decays with increasing x. Taking $x = 0$ at the western boundary thus leads to an exponential decay of the current away from the boundary, which is a Gulf-Stream-like acceptable solution. Alternatively, consider placing the boundary current at the eastern boundary, choosing $x = 0$ there. In this case, the current grows exponentially eastward, away from the boundary, because x becomes more strongly negative, and the exponential part in the solution increases. Such exponential growth is not an acceptable solution, leaving the western boundary as the only possible way to close the circulation. This explains why the Gulf Stream, Kuroshio, and other such strong boundary currents are on the western side of their basins.

8.1 Heuristic explanation of western boundary currents

Consider two different heuristic explanations for the observation that narrow and fast boundary currents are located on the west side of ocean basins. The first is based on Rossby waves. We found previously that long Rossby waves have a westward group (energy) propagation, while short Rossby waves have an eastward group propagation. The wind-forced motions within ocean basins are large-scale, as do the weather systems that produce these winds (thousands of km) and thus excite westward propagating waves. When these waves reach the western boundary region and are reflected back eastward, they are characterized by short scales because only short Rossby waves have eastward energy propagation. We also found previously, when discussing scale-selective friction, that short waves are dissipated faster than long waves. So when the short waves are reflected, they are dissipated before getting far from the western boundary. One, therefore, expects to see short scales near the western boundary and longer scales in the rest of the basin, consistent with the existence of the narrow Gulf Stream there.

For the second explanation, consider the following two scenarios where the western boundary current (WBC) of a sub-tropical gyre is in the east (A) or west (B),



we have $-r\zeta \approx -r\partial v/\partial x$. And then,

In (A) $-r\partial v/\partial x > 0$ so that the fluid gains vorticity (or loses negative vorticity) as it flows northward.

In (B) $-r\partial v/\partial x < 0$ so that the fluid loses vorticity as it flows northward.

The only relevant vorticity in the interior is the planetary one, as the relative vorticity is small. Fluid moves southward in the interior and, therefore, loses planetary vorticity. In scenario (A), the boundary current allows the fluid to gain vorticity as it travels northward. Thus, the fluid can re-join the interior in the north with appropriately high vorticity to match the value of the planetary vorticity there. In scenario (B), the fluid loses vorticity as it flows south in the interior and loses some more as it travels north in the boundary current, so there can be no steady state.

9 Abyssal circulation, Stommel-Arons

Following Stommel and Arons (1960), consider the circulation of the abyssal ocean — say the lower 2–3 km, below a depth of 1–2 km or so. It is driven by vortex stretching due to the upward velocity at the upper part of the abyssal ocean, which is assumed uniform in space and which compensates for bottom water formation, say some 20 Sv. Divide by ocean area to find that the vertical velocity driving the abyssal circulation is of the order of $w_{\text{abyssal}} \approx 20\ \text{Sv}/\left((2/3)4\pi R_E^2\right) \approx 2\ \text{m/yr}$.

We found above (3) that taking the curl of the momentum equation, one finds

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) - f\frac{\partial w}{\partial z} + \beta v = -r\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right),$$

where $f = 2\Omega \sin\theta_0$ and $\beta = 2\Omega \cos\theta_0 / R_E$, with θ_0 being the center latitude of the basin. Assuming a steady state and neglecting friction to be small away from western boundary currents,

$$\beta v = f \frac{\partial w}{\partial z},$$

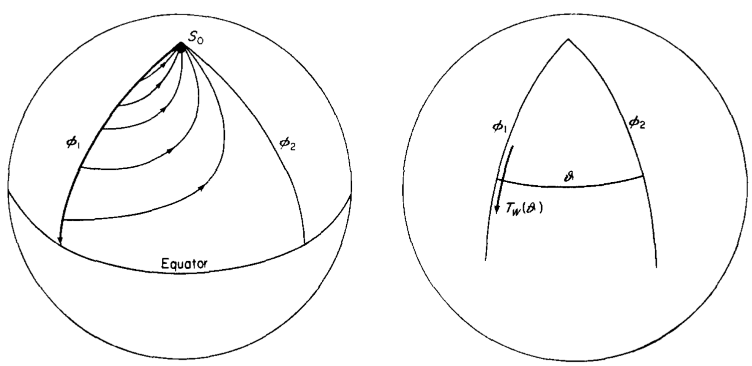
representing a balance between vortex stretching and meridional advection of planetary vorticity. In spherical coordinates, this takes the form,

$$\frac{2\Omega}{R_E} \cos\theta v = 2\Omega \sin\theta \frac{\partial w}{\partial z}.$$

Integrating from the ocean bottom to the top of the abyssal ocean, this gives

$$V = R_E \tan\theta w_{\text{abyssal}}.$$

Note that this velocity is always poleward because the vertical velocity is positive (upward). Now consider the schematic on the left of the following figure from Stommel and Arons (1960), representing a northern hemisphere ocean basin where the deep-water formation occurs in the north (North Atlantic, although note that the NADW is a deep rather than an abyssal water source...). The schematic on the right shows the component in the mass balance of an ocean sector.



The basin is bounded by two meridians, at longitudes ϕ_1 and ϕ_2, and extends from the equator to the north pole. The mass source for the abyssal water is assumed at the north corner of the domain, yet as we have seen, the interior transport per unit east-west distance, V, is poleward everywhere, toward the source rather than away from it. This suggests that there must be a western boundary current that carries the source water equatorward, where it feeds the interior flow and upwells to the upper ocean. We can use a mass balance calculation for the abyssal layer to find the transport of the implied western boundary current. The area of the basin is $R_E^2(\phi_2 - \phi_1)$, and the source volume rate is related to the uniform upwelling as $S_0 = w_{\text{abyssal}} R_E^2(\phi_2 - \phi_1)$. The interior meridional transport may, therefore, be written as

$$V = \frac{S_0}{R_E(\phi_2 - \phi_1)} \tan\theta.$$

Consider the mass budget of the equatorward part of the basin, from $\theta = 0$ to θ. The surface area of this part is,

$$A(y) = \int_{\phi_1}^{\phi_2} d\phi \int_0^{\theta} R_E^2 \cos\theta' \, d\theta' = R_E^2 \sin\theta(\phi_2 - \phi_1),$$

while the longitudinal extent of its northward boundary is given by $L_x(\theta) = R_E \cos\theta(\phi_2 - \phi_1)$. Letting the transport of the deep boundary current be $Q(y)$ (positive southward), the mass balance of the basin balances the entering WBC transport with the exiting interior flow and upward velocity,

$$\begin{aligned}
Q(y) &= w_{\text{abyssal}} A(\theta) + V L_x(\theta) \\
&= \frac{S_0}{R_E^2(\phi_2 - \phi_1)} R_E^2 \sin\theta(\phi_2 - \phi_1) + \left(\frac{S_0}{R_E^2(\phi_2 - \phi_1)} \tan\theta \right) (R_E \cos\theta(\phi_2 - \phi_1)) \\
&= 2 S_0 \sin\theta.
\end{aligned}$$

This suggests that the deep WBC transport vanishes at the equator and is twice the value of the source at the pole. The factor of two on the RHS reflects the need for the WBC to balance both the upwelling and the northward interior flow.

References

H. Stommel and A. B. Arons. On the abyssal circulation of the world ocean-i. stationary planetary flow patterns on a sphere. *Deep-Sea Research*, 6:140–154, 1960.