

EPS131, Introduction to Physical Oceanography and Climate

Section 2: Temperature and Salinity

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1 Energy balance and the greenhouse effect

Estimating Earth's surface temperature based on an energy balance stating that the energy absorbed from the sun equals that which escapes as heat to outer space. Start with the relevant physical constants.

- $S_0 = 1388$ watts/m²: top of the atmosphere insolation at the equator.
- $\sigma = 5.66961 \times 10^{-8}$ W m⁻²K⁻⁴ Stefan-Boltzmann coefficient
- T globally averaged temperature of Earth
- $\alpha = 0.25$ albedo
- $\epsilon = 0.75$ the emissivity of the atmosphere

Incoming radiation equals outgoing radiation,

$$\frac{S_0}{4}(1 - \alpha) = \sigma T^4$$

gives

$$T = \left(\frac{(S_0/4)(1 - \alpha)}{\sigma} \right)^{1/4} = 260K \equiv T_0.$$

This is too cold! The reason is that we neglected the effect of the atmosphere.

Add the effect of the atmosphere, which absorbs all “long-wave” radiation and then re-emits it at the atmospheric temperature θ both up and down.

$$\begin{aligned} \frac{S_0}{4}(1 - \alpha) + \sigma\theta^4 &= \sigma T^4 \\ \sigma T^4 &= 2\sigma\theta^4. \end{aligned}$$

Substitute the second eqn in the first one,

$$\frac{S_0}{4}(1 - \alpha) + \frac{1}{2}\sigma T^4 = \sigma T^4$$

to find,

$$T = \left(\frac{(S_0/4)(1 - \alpha)}{\sigma/2} \right)^{1/4} = T_0 2^{1/4} = 309K,$$

which is a bit better, although it's too warm now.

Finally, the atmosphere is not a perfect black body and only absorbs some (most) of the radiation from the surface, so the Stephan-Boltzmann constant needs to be multiplied by

an “emissivity” ϵ which is smaller than but close to one. The emissivity, [also equal to the absorptivity](#), is a function of the CO₂ concentration among other things and implies that not all the radiation emitted from the ground is absorbed by the atmosphere, but only a fraction proportional to ϵ . The energy balances for the surface and atmosphere are now,

$$\begin{aligned}\frac{S_0}{4}(1 - \alpha) + \epsilon\sigma T^4 &= \sigma T^4 \\ \epsilon\sigma T^4 &= 2\epsilon\sigma\theta^4.\end{aligned}$$

Substitute the second eqn into the first one,

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$$\frac{S_0}{4}(1 - \alpha) + \frac{1}{2}\epsilon\sigma T^4 = \sigma T^4$$

so that, using $\epsilon = 0.75$

$$T = \left(\frac{(S_0/4)(1 - \alpha)}{\sigma(1 - \epsilon/2)} \right)^{1/4} = T_0(1 - \epsilon/2)^{-1/4} = 289K,$$

which is reasonably close to the observed global mean surface temperature.

Anthropogenic warming. The mechanism of warming due to the addition of anthropogenic greenhouse gases is somewhat different from the natural greenhouse effect discussed above. As shown in the slides, anthropogenic greenhouse gas raises the emission level (level of last absorption) so that the black body radiation comes from a colder temperature due to the decrease in temperature with height (lapse rate). We can calculate the change in emission height given the lapse rate and anthropogenic radiative forcing as follows.

First, calculate the temperature decrease required to lower the outgoing LW radiation by ΔF W/m². For this purpose, calculate the emission height temperature before the addition of anthropogenic CO₂ using a balance of outgoing long-wave radiation from the emission height and the incoming solar radiation minus the reflected part, or $(S_0/4) \times (1 - \alpha) = \sigma T^4$. This leads to $T = ((S_0/4) \times (1 - \alpha)/\sigma)^{1/4}$. Next, as the emission height increases by Δh , the temperature goes down by ΔT . The value of ΔT that leads to a ΔF W/m² decrease in LW is obtained using $\sigma T^4 - \Delta F = \sigma(T - \Delta T)^4$, or $\Delta T = ((\sigma T^4 - \Delta F)/\sigma)^{1/4} - T$. Finally, assuming a lapse rate of 6.5 K/km, the change in emission height (m) implied by ΔT is $\Delta h = (\Delta T/6.5) \times 1000$.

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For $\Delta F = 4$ watts/m², a typical value for the increase in emission height is $\Delta h = 150$ m.

2 A calculation of the expected sea-level rise due to ocean warming

Given that the mass of ocean water is not changed by the warming, we can write an equation stating that the mass (per unit area) before warming is equal to that after the warming. The mass before the warming is equal to the depth of water being warmed (d_0) times the density at temperature T_0 before the warming occurred, $\rho(T_0)$. The mass after the warming is the product of the depth of the expanded water body (d) times the new density of the warmer water, $\rho(T_0 + \Delta T)$. Equating the mass before and after the warming, we have,

$$m = d\rho(T_0 + \Delta T) = d_0\rho(T_0)$$

which implies that $d = d_0\rho(T_0)/\rho(T_0 + \Delta T)$, or

$$\begin{aligned} \Delta d &= d - d_0 = d_0 \left(\frac{\rho(T_0)}{\rho(T_0 + \Delta T)} - 1 \right) \\ &= d_0 \frac{\rho(T_0) - \rho(T_0 + \Delta T)}{\rho(T_0 + \Delta T)} = d_0 \frac{-\Delta\rho}{\rho(T_0 + \Delta T)} \approx -d_0 \frac{\Delta\rho}{\rho_0}, \end{aligned}$$

so that sea level rise is proportional to the depth of penetration of the warming. Here, ρ_0 is a constant reference density, say 1025 kg/m^3 . Note that while the warming ΔT is positive, the change in density, $\Delta\rho$, is negative.

More generally, rather than taking the values of the density as a function of temperature, we could use the appropriate “equation of state” relating density to temperature. A good first approximation is that the density changes linearly with temperature according to

$$\rho(T) = \rho_0(1 - \alpha(T - T_0)), \quad \alpha(T_0) = - \left. \frac{1}{\rho_0} \frac{\partial \rho}{\partial T} \right|_{T=T_0},$$

where α is the “thermal expansion coefficient”, itself a function of temperature, T_0 is the temperature before the warming and T the temperature after the warming. This is a linearization of the equation of state using Taylor expansion. This allows us to calculate the density change in terms of the temperature change as $\Delta\rho = -\rho_0\alpha\Delta T$, so the sea level rise becomes $\Delta d = d_0\alpha\Delta T$.

Example: $d_0 = 500 \text{ m}$, $\Delta T = 3^\circ$, $T_0 = 16^\circ$, $\rho(16^\circ\text{C}, 35 \text{ ppt}) = 1025.75 \text{ kg/m}^3$, $\rho(19^\circ\text{C}, 35 \text{ ppt}) = 1025.022 \text{ kg/m}^3$, and we find $\Delta d = 35 \text{ cm}$, which is not far from the IPCC estimate of $\sim 50 \text{ cm}$.

Next, for an estimate of global sea level rise, let $\Delta T(x, y, z) = T - T_0$ be the warming as a function of location and depth, divide the ocean into layers of thickness $d = dz$ and apply the above mass balance to each vertical layer dz in the ocean. Integrating over all such layers from the ocean bottom at $z = -H$ to the surface at $z = 0$, and over all horizontal locations

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(x, y) , we find that the increase in ocean volume is,

$$\Delta V = \int dx \int dy \int_{-H}^0 -\frac{1}{\rho_0} \Delta \rho(x, y, z) dz.$$

The change to GMSL is this change in volume divided by the ocean area A . Using the fact that $\Delta \rho = \rho(T + \Delta T) - \rho(T) \approx -\rho_0 \alpha \Delta T(x, y, z)$, we find,

$$\Delta \text{GMSL} = \frac{1}{A} \int dx \int dy \int_{-H}^0 -\frac{1}{\rho_0} \Delta \rho(x, y, z) dz = \frac{1}{A} \int dx \int dy \int_{-H}^0 \alpha \Delta T(x, y, z) dz,$$

where that $\alpha = \alpha(T(x, y, z))$ is a function of the local temperature, for example, $\alpha(2^\circ \text{C}) = 0.78 \cdot 10^{-4} \text{ }^\circ\text{C}^{-1}$, while $\alpha(10^\circ \text{C}) = 1.6 \cdot 10^{-4} \text{ }^\circ\text{C}^{-1}$, reflecting the sensitivity of the density to the temperature: $\rho(2^\circ \text{C}) - \rho(3^\circ \text{C}) = 0.09 \text{ kg/m}^3$, while $\rho(10^\circ \text{C}) - \rho(11^\circ \text{C}) = 0.18 \text{ kg/m}^3$.

3 On the exponential stratification of the deep ocean: abyssal recipes

(based on Munk, 1966).

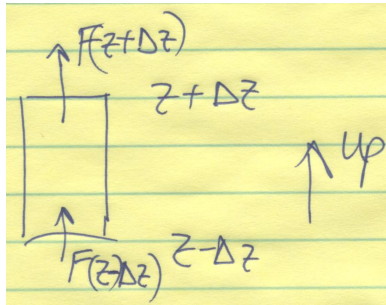
Consider a water parcel in the deep ocean, below 1 km depth, which is affected by diffusion of heat from above and upwelling of cold water from below. The heat equation is, schematically,

$$\frac{\partial}{\partial t} \text{heat in parcel} = \text{net flux due to upwelling} + \text{net flux due to diffusion.} \quad (1)$$

The time rate of change of heat inside the parcel is

$$\frac{\partial}{\partial t} (c_p \rho T \Delta x \Delta y 2 \Delta z).$$

The diffusive heat flux at the bottom and top of the parcel are given by



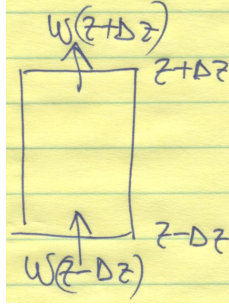
$$F(z - \Delta z) = -c_p \rho \kappa \left. \frac{\partial T}{\partial z} \right|_{z - \Delta z} \Delta x \Delta y$$

$$F(z + \Delta z) = -c_p \rho \kappa \left. \frac{\partial T}{\partial z} \right|_{z + \Delta z} \Delta x \Delta y$$

The net diffusive flux into the parcel is the difference,

$$\begin{aligned} F(z - \Delta z) - F(z + \Delta z) &= - \left. \frac{\partial F}{\partial z} \right|_z 2 \Delta z = - \frac{\partial}{\partial z} (-c_p \rho \kappa \frac{\partial T}{\partial z} \Delta x \Delta y) 2 \Delta z \\ &= c_p \rho \kappa \frac{\partial^2 T}{\partial z^2} \Delta x \Delta y 2 \Delta z. \end{aligned}$$

Next, consider the heat flux due to the upwelling velocity, w ,



In this case, the fluxes at the top and bottom of the parcel are

$$F_{upwelling}(z - \Delta z) = c_p \rho T w(z - \Delta z) \Delta x \Delta y$$

$$F_{upwelling}(z + \Delta z) = c_p \rho T w(z + \Delta z) \Delta x \Delta y,$$

where $w(z - \Delta z)$ is the upwelling velocity at the bottom of the parcel being considered. The net flux is,

$$F_{upwelling}(z - \Delta z) - F_{upwelling}(z + \Delta z) = -c_p \rho \frac{\partial(wT)}{\partial z} \Delta x \Delta y 2\Delta z$$

writing the final heat budget equation (1) using the terms we have developed, dropping the $c_p \rho \Delta x \Delta y 2\Delta z$ factor that appears in all terms, leads to,

$$\frac{\partial T}{\partial t} + \frac{\partial(wT)}{\partial z} = \kappa \frac{\partial^2 T}{\partial z^2}.$$

Next, in a steady state, the first term on the left vanishes, and assuming for simplicity that the vertical velocity is constant, we find

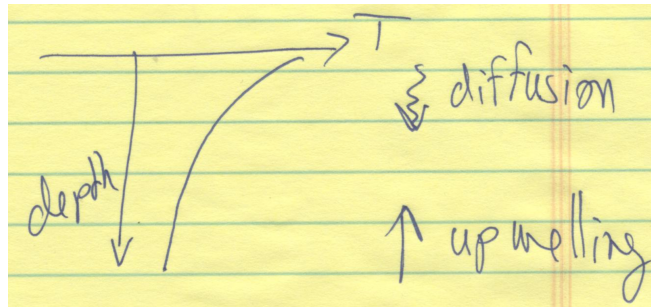
$$w \frac{\partial T}{\partial z} = \kappa \frac{\partial^2 T}{\partial z^2}. \quad (2)$$

The solution is exponential, $T = T_0 + ae^{bz}$, and substituting it into the equation, we find

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$$T = T_0 + ae^{\frac{w}{\kappa}z},$$

where $b = w/\kappa$. Substituting typical numbers, $\kappa = 10^{-4} \text{ m}^2/\text{s}$, $w = 10^{-7} \text{ m/s}$, so that $\kappa/w = 1 \text{ km}$. The balance represented by (2) is described schematically in the following figure,



That is, a water parcel at mid-depth is being heated from above via diffusion of heat from the warmer upper ocean and is being cooled due to the upwelling of cold water from below.

4 Evaporation–precipitation and ocean salinity changes

Consider how salinity changes due to evaporation minus precipitation and the concept of a “virtual salt flux”. For this purpose, consider a bucket of salt water of salinity S (defined as the mass fraction of salt in a unit mass of water), volume V , and surface area A . Mass conservation for the bucket states that,

$$\frac{dV}{dt} = -EA,$$

where E is the net evaporation rate per unit surface area and A is the surface area of the bucket. Salt conservation is,

$$\frac{d}{dt}(SV) = 0,$$

which we can expand, using the mass conservation into

$$V \frac{dS}{dt} = -S \frac{dV}{dt} = SEA \approx S_0 EA.$$

That is, salinity changes are proportional both to the evaporation rate and to the salinity of the water body undergoing the evaporation. This makes sense, as freshwater ($S = 0$) undergoing evaporation will not have its salinity increase. We term $S_0 E$ the “virtual salt flux” per unit area, as it represents a salinity increase equivalent to that which would have occurred due to the injection of salt into the bucket.

5 Density and the equation of state

Density is a function of both temperature and salinity, and at a constant pressure, this may be approximately expressed as

$$\rho(T, S) = \rho_0 (1 - \alpha(T - T_0) + \beta(S - S_0)).$$

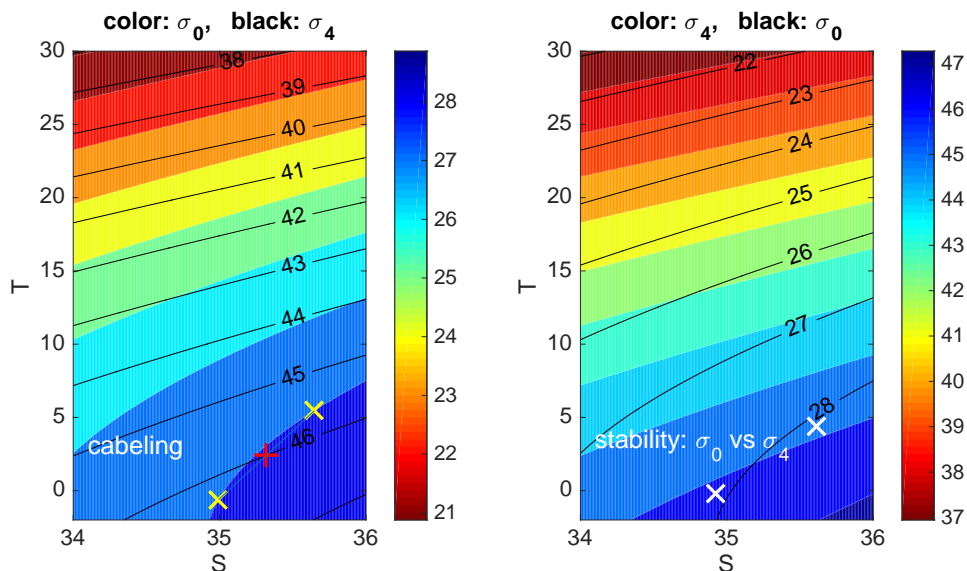
α is a strong function of temperature, varying (at surface pressure, $p = 0$) from $254E-7$ at $T = -2^\circ\text{C}$ to $3413E-7$ at $T = 31^\circ$. That's a factor of more than 10 in the effect of temperature on density... $\beta = \frac{1}{\rho_0} \frac{\partial \rho}{\partial S}$ is pretty much constant, with $\partial \rho / \partial S$ around 0.8 for all temperatures.

A typical range of densities in the ocean is $\rho = 1024$ to 1045 kg/m^3 , and it is therefore convenient to use: $\sigma = \rho - 1000$.

A more accurate approximation needs to include nonlinearities (e.g., Mamaev, 1964; Friedrich and Levitus, 1972),

$$\rho(T, S) = \rho_0 (1 - \alpha(T - T_0) + \beta(S - S_0) + \gamma(T - T_0)^2 + \delta(T - T_0)(S - S_0)).$$

In addition, all coefficients ($\alpha, \beta, \gamma, \delta$) are a function of pressure. The nonlinearities and different pressure dependence of the temperature and salinity coefficients lead to interesting phenomena:



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1. Consider σ_θ , the density of a parcel brought adiabatically to the surface (a typical range of 24–26). Note that deep North Atlantic ocean regions that seem to be unstably stratified at depth based on σ_θ , are stably stratified. The seeming unstable stratification is due to the nonlinearity involved in pressure dependence of the equation of state (see `density_demo.m`). To address this, one may use, instead, σ_4 , the density of a parcel brought adiabatically to a depth of 4 km, as a better measure of deep stability and

stratification.

2. Cabbeling: two water masses of different T and S yet of equal densities at a given pressure or depth mix and lead to a water mass that has a higher density. This would not happen with a linear equation of state: assuming two water masses mix at equal ratios, the density of the averaged temperature and salinity is equal to the average of the densities of the two parcels. See `density_demo.m`.

References

- Friedrich, H. and Levitus, S. (1972). An approximation to the equation of state for sea water, suitable for numerical ocean models. *Journal of Physical Oceanography*, 2(4):514–517.
- Mamaev, O. (1964). A simplified relationship between density, temperature and salinity of sea water. *Izv. Akad. Nauk SSSR, Ser. Geofiz*, 2:309–311.
- Munk, W. H. (1966). Abyssal recipes. *Deep Sea Research and Oceanographic Abstracts*, 13(4):707–730.