EPS 131: Introduction to Physical Oceanography and Climate

Dept of Earth and Planetary Sciences, Harvard University

Homework #3

Assigned Thursday, Feb 26, 2026; due Tuesday, March 10, 2026, 10:00 am, via Gradescope

1. **Inertial oscillations:** consider inertial oscillations forced by a passing storm, where the initial velocity at $t = 0$ is given by,

$$u(t = 0) = u_0,$$
$$v(t = 0) = 0.$$

   (a) Solve for the fluid velocities at later times.

   (b) Solve for the parcel trajectories $x(t)$, $y(t)$ assuming initial conditions of $x(0) = x_0$ and $y(0) = y_0$.

   (c) Solve for the center of the circular trajectory and for the amplitude (radius) of the circular trajectory.

2. **Phase and group velocities:**

   (a) Using the dispersion relation for deep gravity waves, $\omega = \sqrt{gk}$, calculate the wavenumbers k and k' and frequencies ω and ω' of two different waves with two slightly different wave lengths $\lambda = 10$ m and $\lambda' = 11$ m.

   (b) Plot using Matlab/Python two waves of the form $\cos(kx - \omega t)$ and $\cos(k'x - \omega' t)$ at some time $t_0 = 0$ between $x = 0$ and $x = 50$, and the sum of the two waves (superimpose the three curves on the same graph).

   (c) Repeat at $t_1 = 11$ sec, show the two plots one under the other and discuss.

   (d) Calculate the group and phase velocities of these waves for a general wavenumber k. Which one is faster? Evaluate the speeds of phase and group propagation for wavelengths of 12 m and 120 m. Discuss how an individual wave crest will move relative to a wave packet. How long would it take a wave packet to cross the Atlantic Ocean from Lisbon to Boston?

   (e) ***optional extra credit. Consider $\omega = Nk/\sqrt{k^2 + m^2}$, the dispersion relation of an internal wave solution (with no rotation) of the form $\cos(kx+mz-\omega t)$. Calculate the two components of the phase velocity and the two components of the group velocity. Remembering that the phase velocity direction is $\mathbf{k} = (k, m)$, calculate the angle between the group velocity and the direction of propagation of the phase velocity, and describe what you found.

3. **Experimental wave dynamics!** Throw a stone into the Charles. Take a few pictures of the waves formed, every few seconds. Mark the propagating envelope of waves and some of the individual crests. Discuss your findings and attach the pictures to your HW solution.

4. **Solving the wave equation with given initial conditions:** Suppose the surface waves near a beach satisfy the shallow wave equation,

$$\eta_{tt} = gH\eta_{xx}.$$

Suppose also that we are given observations at time $t = 0$ of the wave height and the rate of change of the wave height,

$$\eta(x, t = 0) = 3\sin(5x)$$
$$\frac{\partial \eta}{\partial t} \equiv \eta_t(x, t = 0) = 5\sin(5x).$$

Hint: try as a solution $\eta(x,t) = (A\sin(kx - \omega t) + B\cos(kx - \omega t) + C\sin(kx + \omega t) + D\cos(kx + \omega t))$. Find A, B, C, D, k, and ω (ω is positive). Plot on the same graph your solution in Python/Matlab for $H = 12$ m, $0 < x < 8\pi k^{-1}$ and for $t = 5$ s, 15 s and 25 s.

5. ***optional extra credit. **Nonlinear wave equation and solitons!** Consider the Korteweg–de Vries (KdV) equation,

$$\partial_t\phi + \partial_x^3\phi - 6\,\phi\,\partial_x\phi = 0.$$

(a) Linearize the equation around $\phi = 0$, derive the dispersion relation, and calculate the group and phase velocities.

(b) How do you expect an initial shape such as $\phi(x, t = 0) = \exp(-x^2)$ to evolve at later times under the *linearized* equation? *Hint:* consider the initial shape as the initial state of a collection of waves $\hat{\phi}_k(k(x - ct))$, each with the amplitude given by the Fourier transform of the initial shape, $\hat{\phi}_k(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi(x, t = 0)dx$. Note that each wave travels at a velocity c corresponding to its wavenumber k.

(c) Show that the following expression is a solution of the fully nonlinear equation,

$$\phi(x,t) = -\frac{1}{2}\,c\,\mathrm{sech}^2\left[\frac{\sqrt{c}}{2}(x - c\,t)\right]$$

and plot it for, $c = 1$ and $c = 2$ in each case plotting it for both $t = 1$ and $t = 5$.

(d) Does the solution to the nonlinear equation behave as expected from the linear analysis? *Explanation:* While the linearized dynamics of this equation are dispersive (each wave number has a different velocity and thus an initial group spreads), the nonlinear solution is not dispersive, and the initial bump propagates unchanged. The reason is that each wave number in the initial bump has a different amplitude, and its velocity depends on the amplitude, which is not the case in the linear limit. Thus, the linear dispersion is exactly balanced by the nonlinear dependence of the velocity on the amplitude, and all wavenumbers composing the initial bump travel together. Such a miraculous nonlinear wave is called a **soliton**...!

[What's the point of **optional** extra credit problems? Apart from the fun of doing them, they may bring the total HW score to up to 110% therefore counting against problems you may have missed in this or other HW assignments...]