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Introduction to Physical Oceanography
Homework 5 - Solutions

1. Inertial oscillations with bottom friction (non-selective scale): The governing equations for this problem are

$$\begin{aligned}\frac{\partial u}{\partial t} - fv &= -Ju \\ \frac{\partial v}{\partial t} + fu &= -Jv\end{aligned}$$

This system can be written as

$$\frac{\partial \vec{u}}{\partial t} = \mathbf{A}\vec{u} \tag{1}$$

where $\vec{u} = (u, v)$ and

$$\mathbf{A} = \begin{pmatrix} -J & f \\ -f & -J \end{pmatrix}.$$

I will solve the system a little bit more rigorously than what was done in class, but we should all get the same answer at the end! The general solution to this equation is given by

$$\vec{u} = \alpha \vec{a}_1 e^{\lambda_1 t} + \beta \vec{a}_2 e^{\lambda_2 t} \tag{2}$$

We can find the eigenvalues $\lambda_{1,2}$ of the system by calculating the determinant of the matrix $\mathbf{A} - \lambda \mathbf{I}$. This leads to

$$\lambda^2 + 2\lambda J + (J^2 + f^2) = 0 \Rightarrow \lambda_{1,2} = -J \pm if \tag{3}$$

(This is equivalent to the coefficient a found in class). We can look for the eigenvectors $\vec{a}_1$ and $\vec{a}_2$ corresponding to the eigenvalues $\lambda_1 = -J + if$ and $\lambda_2 = -J - if$ respectively. The eigenvectors satisfies $A\vec{a}_i = \lambda_i \vec{a}_i$. I found $\vec{a}_1 = (1, i)$ and $\vec{a}_2 = (1, -i)$. We can write the velocity components as

$$\begin{aligned}u &= \alpha e^{(-J+if)t} + \beta e^{(-J-if)t} \\ v &= i\alpha e^{(-J+if)t} - i\beta e^{(-J-if)t}\end{aligned}$$

The coefficients α and β can be found from the initial conditions, I will assume that $u(t = 0) = u_0$ and $v(t = 0) = 0$ leading to

$$\begin{aligned}u_0 &= \alpha + \beta \\ 0 &= i\alpha - i\beta\end{aligned}$$

such that $\alpha = \beta = u_0/2$.

$$\begin{aligned}u &= \alpha e^{(-J+if)t} + \beta e^{(-J-if)t} \Rightarrow u = u_0 e^{-Jt} cos(ft) \\ v &= i\alpha e^{(-J+if)t} - i\beta e^{(-J-if)t} \Rightarrow v = -u_0 e^{-Jt} sin(ft)\end{aligned}$$

(You recognize the u velocity found in class). The velocity is oscillating with a frequency equal to the Coriolis parameter as expected from inertial oscillations but also decays exponentially in time due to the addition of bottom friction. The decay of the velocity depends only on the friction coefficient (non-selective scale friction). Figure 1(a) and (c) show that the velocity oscillates and decays as function of time. As $t \to \infty$, $u, v \to 0$.

Now that we have found the velocities, we can look for the Lagrangian trajectories $x(t;x_0)$ and $y(t;y_0)$ of a fluid parcel which was at the location (x_0, y_0) at $t = 0$. We need to integrate the velocity:

$$\begin{aligned}
\int_{x_0}^{x} dx &= \int_{t=0}^{t} u_0 e^{-Jt} cos(ft) dt \\
&\Rightarrow \; x = x_0 + \frac{u_0}{f^2+J^2}\left[J + e^{-Jt}\left(-Jcos(ft) + fsin(ft)\right)\right] \\
\int_{y_0}^{y} dy &= -\int_{t=0}^{t} u_0 e^{-Jt} sin(ft) \\
&\Rightarrow \; y = y_0 + \frac{u_0}{f^2+J^2}\left[-f + e^{-Jt}\left(fcos(ft) + Jsin(ft)\right)\right]
\end{aligned}$$

For the trajectories of a fluid parcel, we see in figure 1(b) and (d) that the trajectories oscillate and decay too. But as $t \to \infty$, we have

$$\begin{aligned}
x &\to x_0 + \frac{u_0 J}{J^2 + f^2} \\
y &\to y_0 - \frac{u_0 f}{J^2 + f^2}
\end{aligned}$$

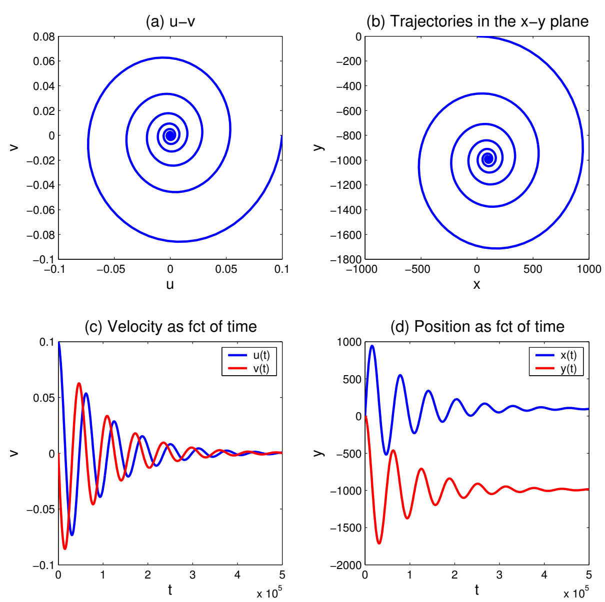

Figure 1: (a) u(t) as function of v(t), (b) the trajectory of a fluid parcel in the x-y plane, (c) u and v as function of time, (d) x and y as function of time.

2. Coastal upwelling: Strong seasonal winds are displacing warm surface water away from the coast leaving a space that is filled in by water coming from the deep ocean. This rise of water is called upwelling. The water upwelling is cold, rich in nutrient and comes from below the thermocline.

   Example 1: Californian coast (see figure 2). In a strip near the coast, we see that the sea surface temperature is colder than away from the coast. We conclude that this region is an upwelling zone. If we look at satellite pictures, we will see that this region is characterized by greenish color due to the chlorophyll. We expect the wind to have a strong southward component, causing a net Ekman transport to the right of the wind (in the Northern Hemisphere), such that the water transport is westward (away from the coast) and as a result we have upwelling of cold water rich in nutrient.

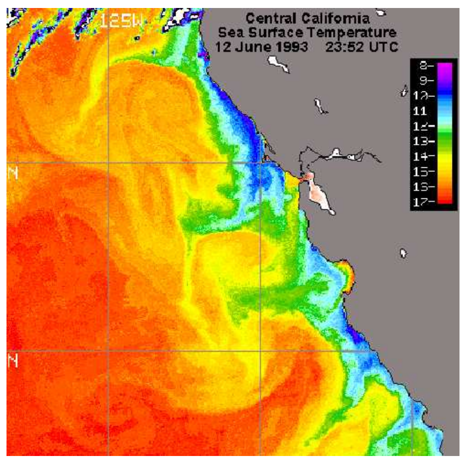

Figure 2: Sea surface temperature along the central Californian coast

   Example 2: Coast of Senegal (see figure 2). The upwelling mechanism is the same than the one described above.

3. Scale selective friction: The velocity at $t = 0$ is given by

$$u(t=0,x,y) = U\cos(kx+ly); \qquad v(t=0,x,y) = -U\cos(lx+ky).$$

   and the governing equations are simply

$$u_t = K_h(u_{xx}+u_{yy}); \qquad v_t = K_h(v_{xx}+v_{yy})$$

   (a) The balance in these equations is represented by the linearized acceleration (where $Du/Dt \approx \partial u/\partial t$, here we have neglected the nonlinear advective terms such that the acceleration is now equal to the local rate of change) and a scale selective friction.

   (b) The velocity field at $t = 0$ is shown in figure 4. I have plotted the contours of each of velocity component at $t = 0$ and the vector field using quiver.

   (c) We wish to find u and v as function of time. Because of friction, we expect the amplitude of the velocity to decrease exponentially with time while the oscillations in the

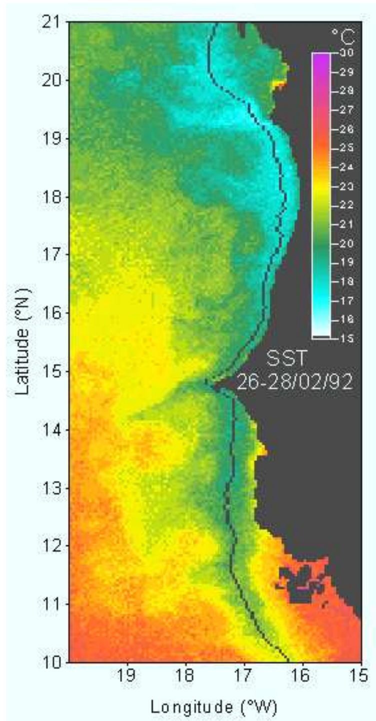

Figure 3: Sea surface temperature along the coast of Senegal

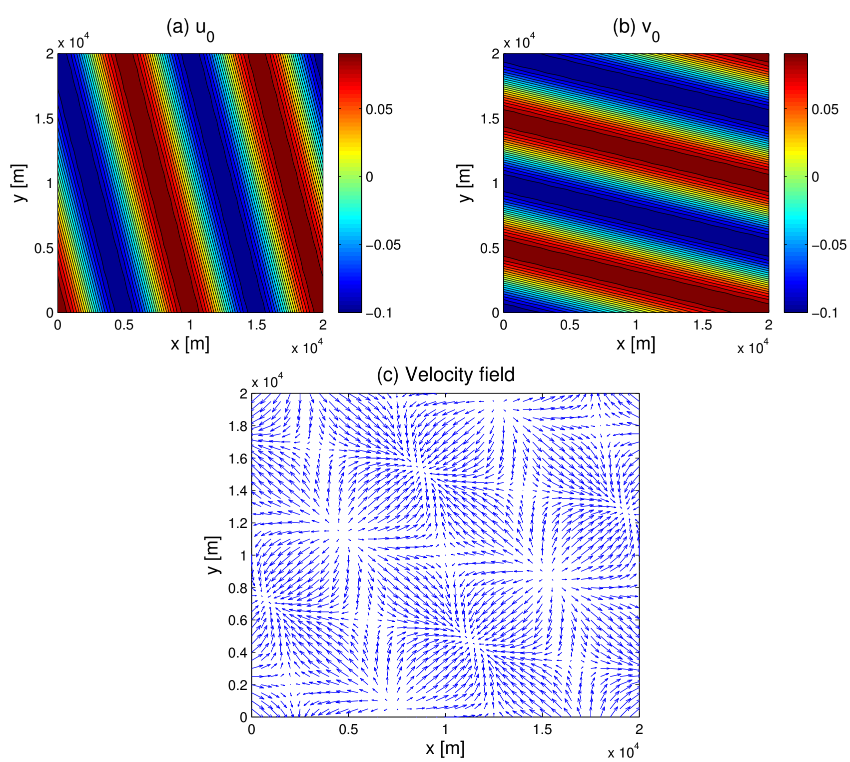

Figure 4: Contours of u_0 and v_0, and velocity field.

$x-y$ plane are not affected. We can guess solutions of the form $u = Ae^{\gamma_1 t}cos(kx+ly)$ and $v = Be^{\gamma_2 t}cos(lx+ky)$.

$$\begin{aligned} u_t &= K_h(u_{xx}+u_{yy}) \Rightarrow \gamma_1 = K_h(-k^2-l^2) \Rightarrow \gamma_1 = -K_h(k^2+l^2) \\ v_t &= K_h(v_{xx}+v_{yy}) \Rightarrow \gamma_2 = K_h(-l^2-k^2) \Rightarrow \gamma_2 = -K_h(l^2+k^2) \end{aligned}$$

We have $\gamma_1 = \gamma_2 = -K_h(k^2+l^2) = \gamma$, such that

$$\begin{aligned} u &= Ae^{-K_h(k^2+l^2)t}cos(kx+ly) \\ v &= Be^{-K_h(k^2+l^2)t}cos(lx+ky) \end{aligned}$$

Using the initial conditions, we can find A and B such that

$$\begin{aligned} u(t=0) &= Acos(kx+ly) = Ucos(kx+ly) \Rightarrow A = U \\ v(t=0) &= Bcos(lx+ky) = -Ucos(lx+ky) \Rightarrow B = -U \end{aligned}$$

Therefore the velocities as function of time are given by

$$\begin{aligned} u &= Ue^{-K_h(k^2+l^2)t}cos(kx+ly) \\ v &= -Ue^{-K_h(k^2+l^2)t}cos(lx+ky) \end{aligned}$$

(d) The parameters k and l are the wavenumber. The wavenumber is proportional to the number of peaks per unit distance, and then inversely proportional to the wavelength (the wavelength is basically it is the distance between 2 crests or 2 troughs of the wave in a given direction).

The corresponding wavelength for k and l will be

$$\begin{aligned} \lambda_k &= \frac{2\pi}{k} \\ \lambda_l &= \frac{2\pi}{l} \end{aligned}$$

For this specific problem, we have $\lambda_k = 10km$ and $\lambda_l = 40km$. For the x-component of the velocity u, λ_k is the wavelength in the x-direction while λ_l is the wavelength in the y-direction. For the y-component of the velocity v, the situation is opposite, λ_k is the wavelength in the y-direction and λ_l is the wavelength in the x-direction.

(e) Since we found that the solution decays exponentially, where $\gamma = -K_h(k^2+l^2)$, the decay time scale is given by

$$\tau = \left|\frac{1}{\gamma}\right| = \frac{1}{K_h(k^2+l^2)} \tag{4}$$

In this case, the decay of the velocity depends on the friction coefficient K_h and on the scale of the problem (given by k and l). Increasing the friction coefficient and the wavenumber (equivalent to decreasing the wavelength) results in increasing the decay time scale (the decay is faster).

4. Challenge problem: Inertial oscillations in presence of bottom friction and pressure gradient. Consider the following equations

$$\begin{aligned} u_t - fv &= -\frac{p_x}{\rho_0} - Ju \\ v_t + fu &= -Jv \end{aligned}$$

assuming that the pressure gradient p_x is constant in space and time. For simplicity, I will define $C = -\frac{p_x}{\rho_0}$ since it is just a constant.

Using the x-momentum equation, we can write v as function of u, such that

$$v = -\frac{C}{f} + \frac{J}{f}u + \frac{1}{f}u_t \quad (5)$$

and plugging v into the y-momentum equation, we obtain a 2nd order ODE:

$$u_{tt} + 2Ju_t + (f^2 + J^2)u = CJ \quad (6)$$

The solution to this equation is the homogeneous solution (sine and cosine) plus a particular solution ($u_p = CJ/(J^2 + f^2)$) and is given by

$$u = \frac{CJ}{J^2 + f^2} + C_1 e^{-Jt} sin(ft) + C_2 e^{-Jt} cos(ft) \quad (7)$$

The coefficients C_1 and C_2 depend on the initial conditions. By using $u(t = 0) = u_0$, we get

$$u = \frac{CJ}{J^2 + f^2} + C_1 e^{-Jt} sin(ft) + \left(u_0 - \frac{CJ}{J^2 + f^2}\right) e^{-Jt} cos(ft) \quad (8)$$

Now that we have u, we can find v using Eq. 5, I found

$$v = -\frac{Cf}{J^2 + f^2} + \left(\frac{CJ}{J^2 + f^2} - u_0\right) e^{-Jt} sin(ft) + C_1 e^{-Jt} cos(ft) \quad (9)$$

Using the initial condition for v where $v(t = 0) = 0$, I found that $C_1 = Cf/(J^2 + f^2)$, such that our solutions are

$$\begin{aligned} u &= \frac{CJ}{J^2 + f^2} + \frac{Cf}{J^2 + f^2} e^{-Jt} sin(ft) + \left(u_0 - \frac{CJ}{J^2 + f^2}\right) e^{-Jt} cos(ft) \\ v &= -\frac{Cf}{J^2 + f^2} + \left(\frac{CJ}{J^2 + f^2} - u_0\right) e^{-Jt} sin(ft) + \frac{Cf}{J^2 + f^2} e^{-Jt} cos(ft) \end{aligned}$$

Recall that $C = -p_x/\rho_0$, if we set $C = 0$, we recover the limit of inertial oscillations in presence of bottom friction from question 1.

The velocity is oscillating and decaying as in question 1 but due to the presence of a constant pressure gradient in the x-direction, the steady state achieved by the velocity is not zero! (see figure 5). As $t \to \infty$, the velocity tends to

$$\begin{aligned} u &= -\frac{J p_x}{\rho_0 (J^2 + f^2)} \\ v &= \frac{f p_x}{\rho_0 (J^2 + f^2)} \end{aligned}$$

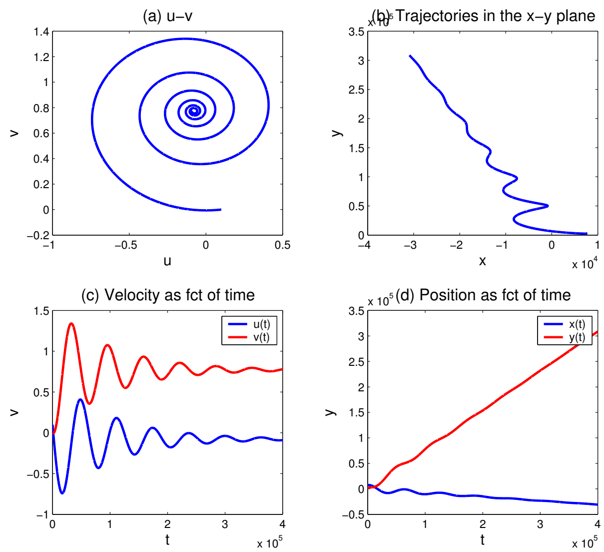

Figure 5: (a) u(t) as function of v(t), (b) the trajectory of a fluid parcel in the x-y plane, (c) u and v as function of time, (d) x and y as function of time.

To better understand this problem, I have also calculated the Lagrangian trajectories and found that

$$
\begin{aligned}
x &= \frac{-p_x}{\rho_0(J^2+f^2)}\left[Jt+\left(\frac{J^2-f^2}{J^2+f^2}+\frac{u_0 J\rho_0}{p_x}\right)e^{-Jt}cos(ft)+\left(\frac{-2Jf}{J^2+f^2}-\frac{u_0 f\rho_0}{p_x}\right)e^{-Jt}sin(ft)\right] \\
y &= \frac{-p_x}{\rho_0(J^2+f^2)}\left[-ft+\left(\frac{-J^2+f^2}{J^2+f^2}-\frac{u_0 J\rho_0}{p_x}\right)e^{-Jt}sin(ft)-\left(\frac{u_0 f p_x}{\rho_0}+\frac{-2Jf}{J^2+f^2}\right)e^{-Jt}cos(ft)\right]
\end{aligned}
$$

Due to the constant pressure gradient, in addition to a decaying oscillation, the trajectories also have a linear trend. Physically, we can compare the constant pressure gradient to the slope of an inclined plane. If a ball starts moving on this inclined planed, due to Coriolis it will be deflected to the right. Due to friction, the ball will never reach back to the top of the inclined plane. It will continue to oscillate but the amplitude of the oscillation will decrease with time due to friction (see figure 5).