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10/17/05

Introduction to Physical Oceanography
Homework 3 - Solutions

1. Volume transport in the Gulf Stream and Antarctic Circumpolar current (ACC):

   (a) Looking on the web you can find a lot of maps of the sea surface height for the Gulf Stream and the Drake Passage from models, observations ... Figure 1 shows two examples for the sea surface height in the area near the Gulf Stream.

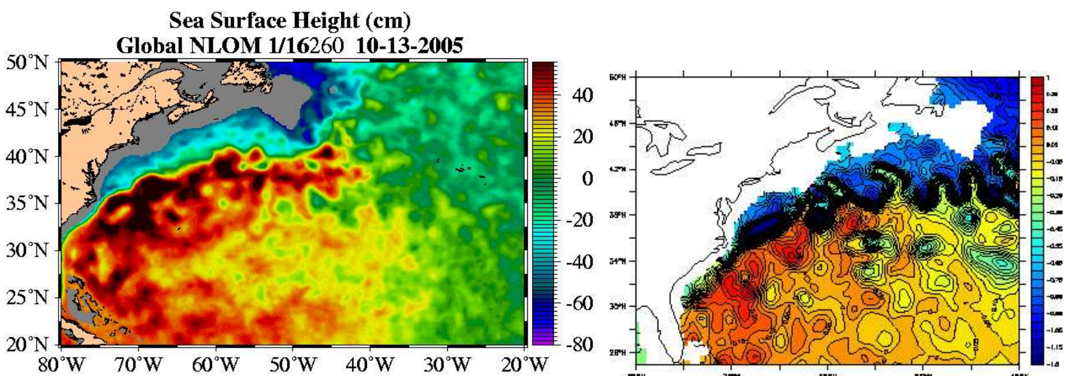


Figure 1: Map of the sea surface height near the Gulf Stream

   Figure 2 shows the sea surface height in the region of the Antarctic circumpolar current including the Drake Passage (between the southern tip of South America and Antarctica).

   (b) From figure 1 and 2 as well as other sources, we can roughly estimate the sea surface height difference (ΔSSH) across the Gulf Stream and the ACC. For the Gulf Stream over a width of approximatively $70\,km$, $\Delta SSH_{GS} \approx 1m$, while for the ACC the width is about $700\,km$ and $\Delta SSH_{ACC} \approx 1m$ (note: ΔSSH for both the Gulf Stream and the ACC highly vary in space and time).

   (c) Assuming that the ocean is in hydrostatic balance (the vertical pressure gradient balances gravity) and that the density is constant and equal to ρ_0, we can write

$$\frac{\partial p}{\partial z} = -\rho_0 g \Rightarrow \partial p = -\rho_0 g \partial z \tag{1}$$

   and then integrating from a depth z up to the surface $z = h(x)$

$$\int_{p(z)}^{p(z=h(x))} \partial p = -\rho_0 g \int_{z}^{h(x)} \partial z \Rightarrow p_{atm} - p(z) = -\rho_0 g\,(h(x) - z) \tag{2}$$

   The pressure at the surface $p(z = h(x))$ is equal to the atmospheric pressure p_{atm}. Since $patm$ is negligible compared to $p(z)$, the pressure in the ocean is given by

$$p(z) = \rho_0 g\,(h(x) - z) \tag{3}$$

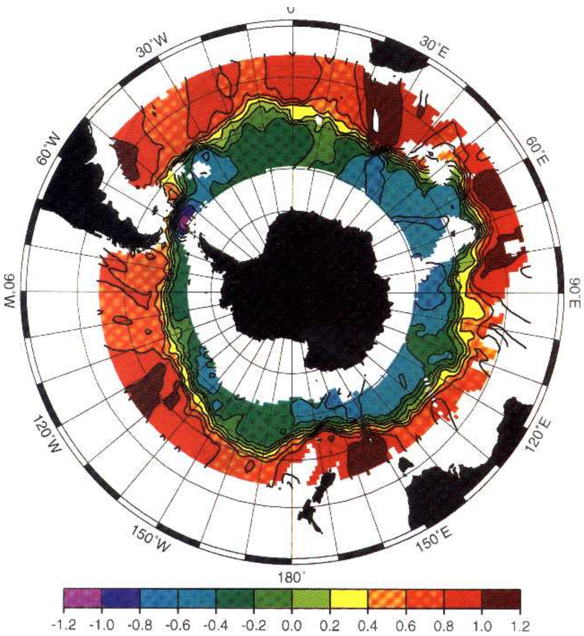


Plate 1. Objective mapped reconstructed mean SSH, contoured at 0.2-m intervals. The 0-m height contour separates the SAF (positive heights) from the PF (negative heights). Regions with no data or height errors greater than 0.05 m are white, and contours are not shown when the error exceeds 0.1 m. Typical errors are between 0.04 and 0.05 m.

Figure 2: Map of the sea surface height in the Southern Ocean

(d) Transport: we will first look for the geostrophy velocities using the geostrophic approximation and then from the velocities evaluate the transport in both the ACC and the Gulf Stream.

Gulf Stream: we will assume that the Gulf Stream velocities are primarily northward and thus use only the x-momentum equation where

$$fv = \frac{1}{\rho_0}\frac{\partial p}{\partial x}. \tag{4}$$

Taking $\frac{\partial}{\partial x}$ of Eq. 3 for the pressure in the ocean, we obtain that the velocity is proportional to the sea surface height

$$fv = \frac{1}{\rho_0}\frac{\partial}{\partial x}\left(\rho_0 g\left(h(x)-z\right)\right) = g\frac{\partial h}{\partial x}, \tag{5}$$

such that the Gulf Stream velocity is simply

$$v = \frac{g}{f}\frac{\partial h}{\partial x} \tag{6}$$

where $\partial h/\partial x = \Delta SSH_{GS}/\Delta x = 1m/70km$ (the sea surface increases eastward). Using $f = 2\Omega sin(30^\circ) = 7.3\cdot 10^{-5}s^{-1}$ and $g = 9.81m\cdot s^{-2}$, the value for the velocity v is found to be $v \approx 1.92m\cdot s^{-1}$. This value seems pretty reasonable and is close to peak velocities measured in the Gulf Stream. Assuming that the velocity is independent of z, the transport V is then given by the velocity times the cross-section area or

$$V = v\cdot\Delta x\cdot H = 1.92m\cdot s^{-1}\cdot 70km\cdot 2km \approx 269\cdot 10^6 m^3\cdot s^{-1}, \tag{7}$$

We can convert our result to Sverdrup (Sv) where $1Sv = 10^6 m^3 \cdot s^{-1}$, so that the transport found for the Gulf Stream is $269Sv$... this is a very large value! The transport of water in the Gulf Stream is assumed to vary between $30Sv$ in the Florida Current and $150Sv$ downstream of Cape Hatteras (large value due to the recirculation in the gyre, we will talk about it in our lectures about western boundary currents). Our value for the transport is so different than the observations mostly because we assumed the velocity to be independent of z and x (you can look at Figure 7.4 p141 in Knauss for a plot of the velocity as function of depth and longitude in the Gulf Stream).

For the ACC (flowing eastward) we will assume that the flow is only zonal (in the East-West direction), meaning that we will consider only the y-momentum equation, such that

$$fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \tag{8}$$

We will repeat the same exercise done previously for the Gulf Stream, assuming in this case that h is a function of y only and find that the velocity is given by

$$u = -\frac{g}{f}\frac{\partial h}{\partial y} \tag{9}$$

For the ACC, $\partial h/\partial y = \Delta SSH_{ACC}/\Delta y = 1m/700km$ (the sea surface height increases northward), and $f = 2\Omega sin(-60°) = -1.26 \cdot 10^{-4} s^{-1}$, the velocity u is found to be $u \approx 0.11 m \cdot s^{-1}$ which is close to the observed values (note that the velocity is positive in agreement with the fact that the ACC is flowing eastward). Assuming that the velocity is independent of depth, the water transport U by the ACC is given by

$$U = u \cdot \Delta y \cdot H = 0.11 m \cdot s^{-1} \cdot 700km \cdot 4km \approx 310 \cdot 10^6 m^3 \cdot s^{-1}, \tag{10}$$

The volume transport near the Drake Passage is believed to be around $150Sv$. It is again a very large value for the volume transport and it is due to the fact that we assumed the velocity to be independent of depth.

2. Why the centrifugal force is negligible compared to Coriolis?

Consider the case where a fluid parcel is at rest in a reference frame rotating with the Earth. The parcel feels a centrifugal force equal to $\Omega^2\vec{r}$, where Ω is the angular velocity of the Earth and $\vec{r}$ is the position vector from the axis of rotation. The centrifugal force directed radially outward partially balances the gravitational force directed toward the center of the Earth. We can then combine the gravity $\vec{g}$ and the centrifugal force $\Omega^2\vec{r}$, as a result the gravitational force every fluid parcel feels is changed due to the centrifugal acceleration. The Earth also changes his shape to compensate this centrifugal force, the Earth is not a perfect sphere but more a spheroid with a bulge at the equator. The changes due to the centrifugal force are very small compared to g as shown in class where $|\Omega^2 r| \approx 0.0341 << g = 9.81$, in addition they do not depend on the velocity of the fluid at all but only on the position vector and therefore has only small influence on the dynamics of the flow.

Consider now a fluid parcel moving with respect to the rotating reference frame, as seen in class an additional force appears: the Coriolis force, perpendicular to the velocity ($2\Omega \times \vec{u}$). For simplicity let assume that the fluid parcel is moving eastward only. Due to the eastward motion, the centrifugal force acting on the parcel will be increased such that the total centrifugal force is given by

$$\left(\Omega + \frac{u}{r}\right)^2 \vec{r} = \Omega^2 \vec{r} + 2\frac{\Omega u \vec{r}}{r} + \frac{u^2 \vec{r}}{r^2} \tag{11}$$

$$= \Omega^2 \vec{r} + 2\Omega u \hat{r} + \frac{u^2 \hat{r}}{r} \tag{12}$$

where u is the velocity of the parcel relative to the ground. As described previously, the term $\Omega^2\vec{r}$ is the centrifugal force due to the rotation of the Earth which was already included in gravity and assumed to have only small influence on the dynamics. The 2nd term is the Coriolis force, and the 3rd term is the centrifugal acceleration due to the motion of the flow. Comparing the 2nd and the 3rd term for large scale flows, we find that $2\Omega u = O(10^3) >> u^2/r^{-1} = O(10^{-9})$. We can then neglect the centrifugal acceleration compared to the Coriolis force for the motion of large scale flows.

3. Challenge problem: The tank of water rotates around its vertical axis with an angular velocity Ω. We assume that no external forces are acting on the system. The momentum equation in a rotating reference frame in vector form is given by

$$\frac{d\vec{u}}{dt} + 2\vec{\Omega} \times \vec{u} + \vec{\Omega} \times \vec{\Omega} \times \vec{r} = -\frac{1}{\rho}\nabla p - g\hat{z} \tag{13}$$

The water is assumed to rotate with the tank, equivalent to a solid body rotation where the velocity of the water is 0 relative to the tank (i.e, the fluid parcels have zero velocity in the rotating frame) $\Rightarrow \vec{u} = 0$. The momentum equation is then reduced to

$$\vec{\Omega} \times \vec{\Omega} \times \vec{r} = -\frac{1}{\rho}\nabla p - g\hat{z} \tag{14}$$

The centrifugal force $-\vec{\Omega} \times \vec{\Omega} \times \vec{r}$ is directed radially outward and can be expressed as the gradient of a potential such that

$$-\vec{\Omega} \times \vec{\Omega} \times \vec{r} = \nabla\left(\frac{\Omega^2 r^2}{2}\right) \tag{15}$$

We can similarly write the gravity as the gradient of a potential

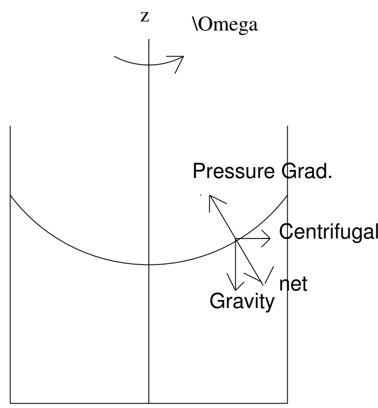
$$\nabla(gz) = g\hat{z} \tag{16}$$

and the shape of the water at the surface is given by

$$\frac{1}{\rho}\nabla p + \nabla\left(gz - \frac{\Omega^2 r^2}{2}\right) = 0 \tag{17}$$

The previous equation is similar to the hydrostatic balance (vertical pressure gradient balances gravity) but applies to a rotating system (adding to the centrifugal forces). Assuming that the density ρ is constant, the previous equation is satisfied if everywhere in the fluid we have

$$\frac{p}{\rho} + gz - \frac{\Omega^2 r^2}{2} = C \tag{18}$$



where C is a constant. The pressure $p(r,z)$ is given by

$$p(r,z) = -\rho g z + \rho \frac{\Omega^2 r^2}{2} + C \tag{19}$$

Using the boundary condition $p = 0$ at the surface of the water $z = h(x,y) = h(r)$,

$$0 = -\rho g h(r) + \rho \frac{\Omega^2 r^2}{2} + C \tag{20}$$

such that

$$h(r) = \frac{\Omega^2 r^2}{2g} + \frac{C}{\rho g} = h(0) + \frac{\Omega^2 r^2}{2g} \tag{21}$$

where $h(0) = \frac{C}{\rho g}$ is the height of the fluid in the middle of the bucket.

The equation for the surface of the fluid tells us that the surface of the water has a parabolic shape with a minimum at the center of the tank. We can see that the vector $-\vec{\Omega} \times \vec{\Omega} \times \vec{r} - g\hat{z}$ is always be perpendicular to the surface. The centrifugal force and the gravity act together in order to produce a force normal to the surface of the fluid and this net force will balance the pressure gradient at the surface of the water.