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Introduction to Physical Oceanography
Homework 1 - Solutions

1. Given the trajectories of fluid elements

$$
\begin{aligned}
x &= x_0 e^{-\alpha t} \\
y &= y_0 e^{\alpha t} \\
z &= z_0
\end{aligned}
$$

(a) the Lagrangian velocities are found by taking the time derivatives of $x(t,x_0)$, $y(t,y_0)$ and $z(t,z_0)$

$$
\begin{aligned}
u(x_0,t) = \frac{\partial x(t,x_0)}{\partial t} &= -\alpha x_0 e^{-\alpha t} \\
v(y_0,t) = \frac{\partial y(t,y_0)}{\partial t} &= \alpha y_0 e^{\alpha t} \\
w(z_0,t) = \frac{\partial z(t,z_0)}{\partial t} &= 0
\end{aligned}
$$

Remember that to find the Lagrangian velocity, we are following a fluid parcel which was at the location (x_0,y_0,z_0) at time $t = t_0$.

In order to find the Eulerian velocity field, we need to express the Lagrangian velocity as function of t and x,y,z. Using the trajectories of the fluid elements, we obtained that

$$
\begin{aligned}
u(x,t) &= -\alpha x \\
v(y,t) &= \alpha y \\
w(z,t) &= 0
\end{aligned}
$$

For the Eulerian velocity, we are asking what is the local velocity at a fixed location (x,y,z) at time t.

(b) The 3rd component of the velocity, w, is zero, we can therefore consider the motion as being 2D. In order to find the streamlines (curves tangential to the velocity field at a given time), we can use the following expression derived in class

$$\frac{dy}{dx} = \frac{v}{u} \tag{1}$$

Using the results from (a), we obtained a differential equation for the streamlines:

$$\frac{dy}{dx} = -\frac{y}{x} \tag{2}$$

An equation for the streamlines $y = y(x)$ is obtained by integrating the previous ODE using the initial conditions (x_0,y_0) such that

$$\int^y \frac{dy}{y} = -\int^x \frac{dx}{x} \Rightarrow yx = c \Rightarrow y = \frac{y_0 x_0}{x} \tag{3}$$

The streamlines are then given by

$$y = \frac{x_0 y_0}{x} \tag{4}$$

A plot of few streamlines is given in Fig. 3.

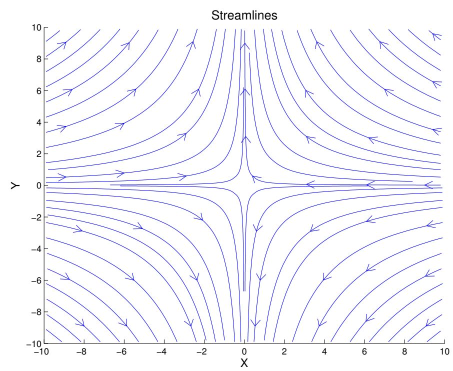

Figure 1: Streamlines plotted in the $x-y$ plane where the arrows are determined by the Eulerian velocity field.

2. Challenge problem: In this problem, the velocity field is given and we are left with finding the trajectories of the fluid elements. The simplest way to solve this coupled system of 1st order linear ODEs will be linear algebra.

   The velocities are defined by $u = \frac{\partial x}{\partial t}$, $v = \frac{\partial y}{\partial t}$ and $w = \frac{\partial z}{\partial t}$, such that the system is given by

$$\begin{aligned} \frac{\partial x}{\partial t} &= -\mu x - \Omega y \\ \frac{\partial y}{\partial t} &= \mu y + \Omega x \\ \frac{\partial z}{\partial t} &= 0 \end{aligned}$$

   The solution to the equation $\partial z/\partial t = 0$ using the initial conditions is simply given by $z = z_0$. Since the 3rd equation which was uncoupled has been solved, we can rewrite the 2D system in matrix form such that $\frac{\partial}{\partial t}\vec{x} = \mathbf{A}\vec{x}$, where the matrix $\mathbf{A}$ is such that

$$\mathbf{A} = \begin{pmatrix} -\mu & -\Omega \\ \Omega & \mu \end{pmatrix}.$$

   and the vector $\vec{x}$ is given by $\vec{x} = (x, y)$.

   To solve the system, one needs to look for the eigenvalues λ and corresponding eigenvectors $\vec{a}$ of the system. The general solution is expressed as

$$\vec{x} = \alpha \vec{a}_1 e^{\lambda_1 t} + \beta \vec{a}_2 e^{\lambda_2 t} \tag{5}$$

   By solving the characteristic polynomial $det(\mathbf{A} - \lambda \mathbf{I}) = 0$, we found that the eigenvalues are given $\lambda = \pm\sqrt{\mu^2 - \Omega^2}$.

The eigenvectors, by definition, satisfy $A\vec{a} = \lambda\vec{a}$. Some algebra leads to

$$\vec{a}_1 = \left(-\frac{\sqrt{\mu^2 - \Omega^2} + \mu}{\Omega}, 1\right) \tag{6}$$

and

$$\vec{a}_2 = \left(\frac{\sqrt{\mu^2 - \Omega^2} - \mu}{\Omega}, 1\right) \tag{7}$$

The trajectories $x(t)$ and $y(t)$ are given by

$$\begin{aligned} x &= \alpha\frac{-\sqrt{\mu^2 - \Omega^2} - \mu}{\Omega}e^{-\sqrt{\mu^2-\Omega^2}t} + \beta\frac{\sqrt{\mu^2 - \Omega^2} - \mu}{\Omega}e^{\sqrt{\mu^2-\Omega^2}t} \\ y &= \alpha e^{-\sqrt{\mu^2-\Omega^2}t} + \beta e^{\sqrt{\mu^2-\Omega^2}t} \end{aligned}$$

Using the initial conditions, we obtain the coefficients α and β such that

$$\begin{aligned} \alpha &= \frac{-\Omega x_0 + y_0\left(\sqrt{\mu^2 - \Omega^2} - \mu\right)}{2\sqrt{\mu^2 - \Omega^2}} \\ \beta &= \frac{\Omega x_0 + y_0\left(\sqrt{\mu^2 - \Omega^2} + \mu\right)}{2\sqrt{\mu^2 - \Omega^2}} \end{aligned}$$

Two different behaviors of the trajectories will emerge depending on the relative values of Ω and μ.

(a) Case 1: $\Omega > \mu$, the eigenvalues are purely imaginary, $\lambda = \pm i\sqrt{\Omega^2 - \mu^2}$. The solutions for x and y are then oscillatory. The frequency of the oscillations in both x and y is given by $\omega = \sqrt{\Omega^2 - \mu^2}$.

(b) Case 2: $\Omega < \mu$, the eigenvalues are real, $\lambda = \pm\sqrt{\mu^2 - \Omega^2}$. The solutions are growing exponentially.

Some plots below are showing the trajectories x and y as function of time for case 1 and 2. In addition, I added a plot of the streamlines for both cases. In case 1, the streamlines are ellipses (since the frequency of oscillations in both directions are equal) while in case 2 they are hyperbolas (due to exponent form of the solution).

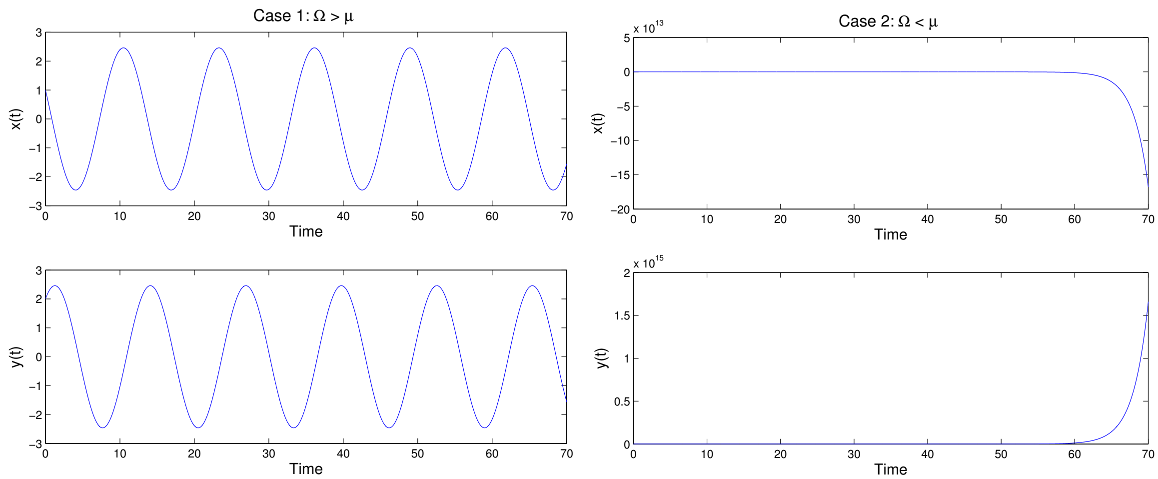

Figure 2: Fluid trajectories x and y as function of time for the two different cases $\Omega > \mu$ and $\Omega < \mu$.

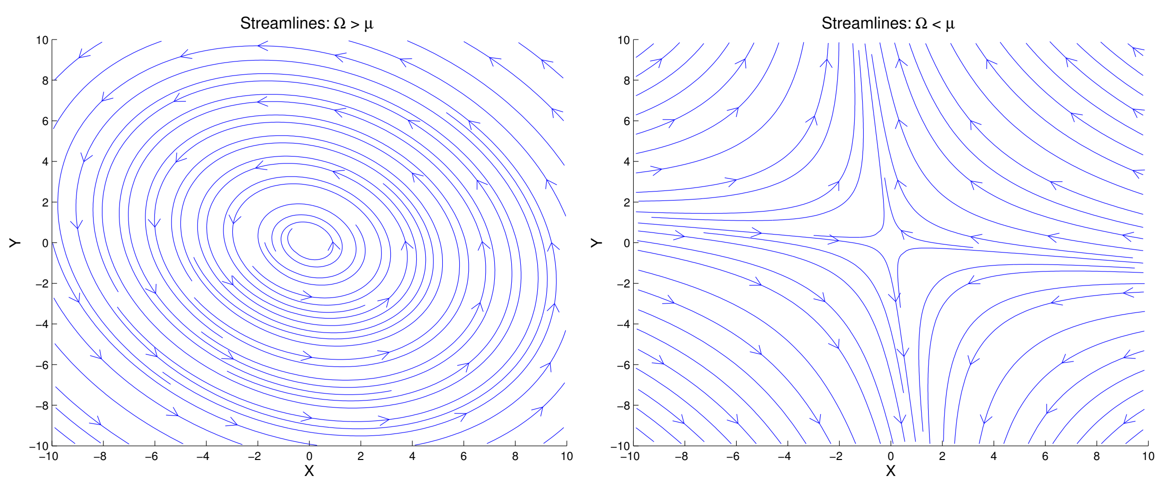

Figure 3: Streamlines plotted in the $x - y$ plane for the two different cases where $\Omega > \mu$ and $\Omega < \mu$.