EPS 131 Problem set 5 SOLUTIONS

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1\. i. The particle goes from $\vec{x}$ to $\vec{x}+\delta\vec{x}$ in δt, so its velocity is $\vec{u} = \frac{\delta \vec{x}}{\delta t}$ and its rate of change of temperature is

$$\frac{dT}{dt} = \frac{T(\vec{x}+\delta\vec{x}, t+\delta t) - T(\vec{x},t)}{\delta t} \overset{\text{Taylor expand}}{=} \frac{T(\vec{x},t) + \delta\vec{x}\cdot\nabla T(\vec{x},t) + \delta t\, T_t(\vec{x},t) - T(\vec{x},t)}{\delta t}$$

$$= \boxed{T_t(\vec{x},t) + \vec{u}\cdot\nabla T(\vec{x},t)}$$

ii. $\frac{dx}{dt} = u = ax \Rightarrow x = x_0 e^{at}$ fluid parcel trajectory

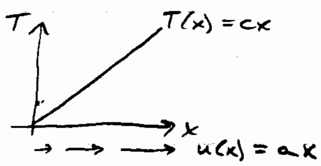
$T(x,t) = cx$ [Eulerian]

$\boxed{T(x_0,t) = c x_0 e^{at}}$ [Lagrangian]

$\boxed{\frac{dT}{dt} = a c x_0 e^{at}}$ [Lagrangian]

iii. $\frac{dT}{dt} = T_t + \vec{u}\cdot\nabla T = T_t + u\frac{\partial T}{\partial x} = 0 + ax\cdot c = \boxed{acx}$ [Eulerian]

Since $x = x_0 e^{at}$, this is the same result.

iv. 

It is colder to the left and warmer to the right. The fluid, flowing to the right, advects the cold temperature into the warm region. But at the same time, fluid parcels are warming as $T(x_0,t) = c x_0 e^{at}$. In this problem, fluid parcel warming exactly cancels advective cooling at every (Eulerian) location.

2\. i. For the horizontal momentum, we can write the u and v equations from the "midterm review" (3/23) as

$$\frac{d\vec{u}_H}{dt} + f(-v,u) = -\frac{1}{\rho_0}\nabla_H p + \left(A_H \nabla_H^2 + A_v \frac{\partial^2}{\partial z^2} - r\right)\vec{u}_H$$

with $\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{u}_H \cdot \nabla_H + w\frac{\partial}{\partial z}$, $\vec{u}_H = (u,v)$, $\nabla_H = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$

For the coriolis term, we can use

$f(-v,u) = f\hat{z} \times \vec{u}_H$, which is no surprise since

$\vec{\Omega} \times \vec{u}_H = |\vec{\Omega}| \sin\theta\, \hat{z} \times \vec{u}_H = f\hat{z} \times \vec{u}_H$. This gives

$$\boxed{\underbrace{\frac{\partial \vec{u}_H}{\partial t}}_{\text{rate of velocity change}} + \underbrace{(\vec{u}_H \cdot \nabla)\vec{u}_H + w\frac{\partial}{\partial z}\vec{u}_H}_{\text{momentum advection}} + \underbrace{f\hat{z} \times \vec{u}_H}_{\text{coriolis}} = \underbrace{-\frac{1}{\rho_0}\nabla_H p}_{\text{pressure gradient}} + \left(\underbrace{A_H \nabla_H^2 + A_v \frac{\partial^2}{\partial z^2}}_{\text{diffusive friction}} - \underbrace{r}_{\text{bottom friction}}\right)\vec{u}_H}$$

The vertical momentum equation is similar:

$$\boxed{\underbrace{\frac{\partial w}{\partial t} + (\vec{u}_H \cdot \nabla) w + w\frac{\partial w}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + \left(A_H \nabla_H^2 + A_v \frac{\partial^2}{\partial z^2}\right) w}_{\text{" same as } \vec{u}_H \text{ "}} - \underbrace{g\frac{\rho}{\rho_0}}_{\text{gravity}}}$$

Note that all 3 momentum equations can be written neatly as

$$\frac{d\vec{u}}{dt} + f\hat{z} \times \vec{u} = -\frac{1}{\rho}\nabla p - g\frac{\rho}{\rho_0}\hat{z} + \left(A_H \nabla_H^2 + A_v \frac{\partial^2}{\partial z^2}\right)\vec{u} - r\vec{u}_H$$

with $\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{u} \cdot \nabla$, $f\hat{z} \times \vec{u} = f(-v,u,0)$

ii. $\nabla \cdot \vec{u} = \boxed{\underbrace{\nabla_H \cdot \vec{u}_H}_{\text{horizontal mass divergence}} + \underbrace{\frac{\partial w}{\partial z}}_{\text{vertical mass divergence}} = 0}$ Mass cons'n (incompressible)

$\frac{dT}{dt} = \boxed{\underbrace{\frac{\partial T}{\partial t}}_{\text{temp rate of change}} + \underbrace{\vec{u}_H \cdot \nabla_H T + w\frac{\partial T}{\partial z}}_{\text{temp advection}} = \underbrace{K\left(\nabla_H^2 + \frac{\partial^2}{\partial z^2}\right) T}_{\text{temp diffusion}}}$ Temperature equation

iii. Geostrophy: $f\hat{z} \times \vec{u}_H = -\frac{1}{\rho_0}\nabla_H p$

Damped inertial oscillations: $\frac{\partial \vec{u}_H}{\partial t} + f\hat{z} \times \vec{u}_H = -r\vec{u}_H$

Ekman drift: $f\hat{z} \times \vec{u}_H = A_v \frac{\partial^2}{\partial z^2}\vec{u}_H$

iv. 5 scalar equations. 6 scalar unknowns: u, v, w, p, ρ, T.

For closure (closed set), need $\rho(T)$. In atm, $\rho = \frac{p}{kT}$

In ocean, use $\rho = -\alpha(T - T_0) + \beta(S - S_0)$. Salinity S satisfies $\frac{dS}{dt} = K\nabla^2 S$.

3. $x = r\cos\theta$,

$u = \dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta = \dot{r}\frac{x}{r} - \dot{\theta}y$

$\boxed{u = \dot{r}\frac{x}{\sqrt{x^2+y^2}} - \dot{\theta}y}$

$y = r\sin\theta$

$v = \dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta = \dot{r}\frac{y}{r} + \dot{\theta}x$

$\boxed{v = \dot{r}\frac{y}{\sqrt{x^2+y^2}} + \dot{\theta}x}$

Solid body rotation: $u_r = \dot{r} = 0$, $u_\theta = r\dot{\theta} = \omega r \Rightarrow \dot{\theta} = \omega$

$\boxed{u = -\omega y, \quad v = \omega x}$ velocity field

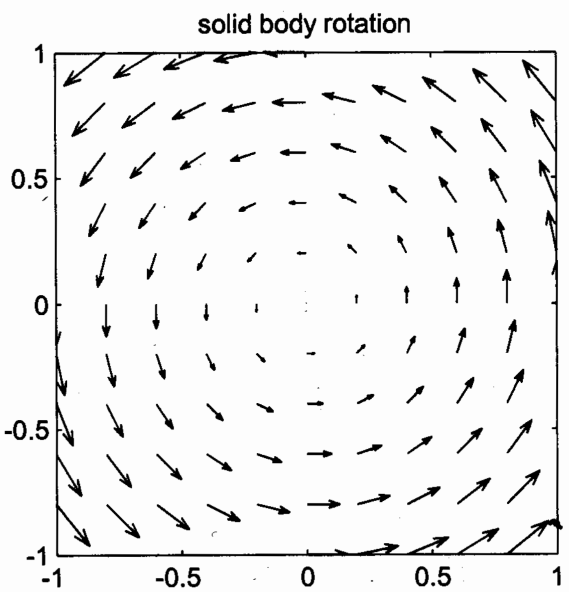
$\text{curl}\,\vec{u} = \hat{z}\cdot\nabla\times\vec{u} = v_x - u_y = \omega - (-\omega) = \boxed{2\omega}$ curl

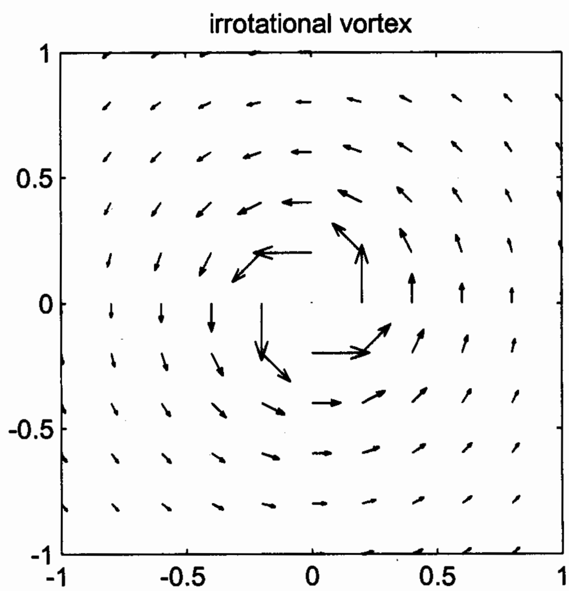
Irrotational vortex: $u_r = \dot{r} = 0$, $u_\theta = r\dot{\theta} = \frac{\lambda}{r} \Rightarrow \dot{\theta} = \frac{\lambda}{r^2} = \frac{\lambda}{x^2+y^2}$

$\boxed{u = \frac{-\lambda y}{x^2+y^2}, \quad v = \frac{\lambda x}{x^2+y^2}}$ velocity field

$\text{curl}\,\vec{u} = v_x - u_y = \left(\frac{\lambda(x^2+y^2) - 2x(\lambda x)}{(x^2+y^2)^2} - \frac{-\lambda(x^2+y^2) - 2y(-\lambda y)}{(x^2+y^2)^2}\right) = \frac{2\lambda(x^2+y^2) - 2\lambda x^2 - 2\lambda y^2}{(x^2+y^2)^2} = \boxed{0}$

Both are same results as what was found in class using cylindrical coordinates.





4. See Cushman-Roisin (course reading list), p. 62-66, for a more detailed description of this problem.

Given (boundary conditions):

$$\vec{u}(z \to \infty) = (U, 0, 0) \quad \text{[interior]} \quad (4.1)$$

$$\vec{u}(z=0) = (0,0,0) \quad \text{[bottom]} \quad (4.2)$$

Near the bottom, assume balance between Coriolis and friction:

$$\tilde{u} = u - U, \quad \tilde{v} = v \quad \text{[velocity near bottom]}$$

$$-f\tilde{v} = A_v \frac{\partial^2 \tilde{u}}{\partial z^2}, \quad f\tilde{u} = A_v \frac{\partial^2 \tilde{v}}{\partial z^2} \qquad \text{solve for } v:$$

$$\hookrightarrow -f\tilde{v} = A_v \frac{\partial^2}{\partial z^2}\left(\frac{A_v}{f}\frac{\partial^2 \tilde{v}}{\partial z^2}\right) = \frac{A_v^2}{f}\frac{\partial^4 \tilde{v}}{\partial z^4}$$

$$\left(f^2 + A_v^2 \frac{\partial^4}{\partial z^4}\right)\tilde{v} = 0$$

Try $\tilde{v} = a e^{\lambda z}$

$$f^2 + A_v^2 \lambda^4 = 0 \Rightarrow \boxed{\lambda = \pm(1 \pm i)\frac{1}{d}}, \quad d \equiv \sqrt{\frac{2A_v}{f}} \quad [d = \text{"Ekman depth"}]$$

Ruling out exponentially growing sol'ns to satisfy (4.1), this gives

$$v = \tilde{v} = e^{-z/d}\left(a_1 \cos\frac{z}{d} + a_2 \sin\frac{z}{d}\right)$$

Since $u = U + \tilde{u} = U + \frac{A_v}{f}\frac{\partial^2 \tilde{v}}{\partial z^2} = U \pm \frac{A_v}{f}(-1 \mp i)^2 \frac{1}{d^2} v = U \pm i v$

$$u = U + e^{-z/d}\left(a_1 \sin\frac{z}{d} - a_2 \cos\frac{z}{d}\right)$$

We have satisfied (4.1) with this (u,v). To satisfy (4.2), we need to let $a_1 = 0$, $a_2 = U$. This gives the solution

$$\boxed{\begin{aligned} u &= U\left(1 - e^{-z/d}\cos\frac{z}{d}\right) \\ v &= U e^{-z/d} \sin\frac{z}{d} \end{aligned}}$$

Figure: Ekman bottom spiral