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APM203 section 9 notes

December 13, 2005

1 Multifractals: D_q and $f(\alpha)$

The box counting dimension, $d_{box} = \lim_{\varepsilon\to 0}\frac{\ln N(\varepsilon)}{\ln(1/\varepsilon)}$, counts all cubes needed to cover the attractor equally, without regard for the fact that some cubes are far more frequently visited (i.e., points are more dense). To take into account the density of points, we can define a dimension spectrum D_q, where q determines how much influence density variations have on the dimension. Note that q is continuous and can be less than zero.

$$D_q = \frac{1}{1-q}\lim_{\varepsilon\to 0}\frac{\ln I(q,\varepsilon)}{\ln(1/\varepsilon)}$$

$$I(q,\varepsilon) \equiv \Sigma_{i=1}^{N(\varepsilon)}\mu_i^q$$

with $N(\varepsilon)$ the number of boxes of size ε needed to cover the attractor. Here μ_i is the measure (i.e., some concept of density of the boxes). Often on an attractor μ_i is the frequency of visits to the box, $\mu_i = \lim_{T\to\infty}\frac{\eta(C_i,T)}{T}$ where η is the amount of time the orbit spends in C_i during $0 \le t \le T$. Note that when $q = 0$, or when all μ_i are equal, all boxes get equal weight so D_q reduces to the box counting dimension.

Any measure μ_i which is not constant is called a *multifractal* measure. If we cover an attractor with boxes of size ε, we can define the singularity index α_i such that the density of points in the box μ_i satisfies

$$\mu_i = \varepsilon^{\alpha_i}$$

The multifractal spectrum $f(\alpha)$ is, roughly, the box dimension of the set of boxes with singularity index α_i (see Ott 9.1 for more exact definition).

The multifractal spectrum $f(\alpha)$ and dimension spectrum D_q contain the same information about an attractor (or any set with a defined measure). They are related by

$$f(\alpha(q)) = q\,\alpha(q) - (q-1)D_q$$

$$\alpha(q) = \frac{d}{dq}\left[(q-1)D_q\right]$$