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APM203 section 7 notes

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1 Routes to chaos in dissipative systems

We've studies three routes to chaos in dissipative systems. The following outline is adapted from Schuster, Table 12.

- **Period-doubling** (Feigenbaum, 1978)
  - i. Pitchfork bifurcation
  - ii. Infinite cascade of period doublings, universal scaling parameters
  - iii. Logistic map: $x_{n+1} = rx_n(1-x_n)$
- **Intermittency** (Pomeau and Manneville, 1979)
  - i. Saddle-node (Type I), Hopf (Type II), or inverse period-doubling (Type III) bifurcation.
  - ii. Signal randomly alternates between nearly periodic and irregular behavior. Duration of periodic bursts scales as $\varepsilon^{-1/2}$ (Type I) or ε^{-1} (Type II, III).
  - iii. See Schuster, Table 7 (p. 98) for examples of Poincare map equations for all 3 types.
- **Quasi-periodicity** (Ruelle, Takens, and Newhouse, 1978)
  - i. Hopf bifurcation
  - ii. *Continuous system:* Stationary solution → periodic motion → quasi-periodic motion (2 spectral peaks, irrational ratio of frequencies) → 3-torus is "typically" unstable, so chaos after third Hopf bifurcation (or possibly after slightly larger parameter value). *Discrete map:* For small nonlinearity (K), winding number is rational (i.e., mode-locked solution, resonance) in Arnold's tongues and irrational outside the tongues; when nonlinearity is increased, tongues overlap, and solution jumps irregularly between resonances.
  - iii. Periodically forced pendulum: $\ddot{\theta} + \gamma\dot{\theta} + \sin\theta = A\cos(\omega t) + B$
    Circle map: $\Theta_{n+1} = \Theta_n + \Omega - \frac{K}{2\pi} sin(2\pi\Theta_n)$ mod 1