Ian Eisenman
eisenman@fas.harvard.edu
Geological Museum 101, 6-6352

APM203 section 6 notes

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Overview

1 Period-doubling route to chaos

1.1 Logistic map

The logistic map, which can be used to approximately describe population growth, is

$$x_{n+1} = rx_n(1 - x_n) \tag{1}$$

With $0 \leq x \leq 1$ and $0 \leq r \leq 4$. When $1 < r < 3$, $x^* = 0$ is an unstable fixed point and $x^* = 1 - 1/r$ is a stable fixed point. At $r > 3$, $x^* = 1 - 1/r$ becomes unstable and a 2-cycle is born (solution jumps back and forth between two values of x). At $r \approx 3.449$, the 2-cycle becomes unstable and a 4-cycle is born. At $r > r_\infty \approx 3.570$, chaotic solutions exist, as well as windows of periodic behavior in small ranges of r.

When the fixed point $x^* = 1 - 1/r$ occurs at the maximum of the logistic map ($x_m = 1/2$), it is called **superstable** because the linearization yields $|f'(x^*)| = 0$. Small perturbations around this fixed point converge to zero as $\eta_{n+1} = (-2\eta_0)^n$; recall that for regular stable fixed points perturbations converge as $\eta_n = \lambda^n \eta_0$ with $|\lambda| < 1$. Fixed points of an n-cycle solution are similarly superstable at particular values of r.

1.2 Universality

A unimodal map is a map where x_{n+1} vs x_n is everywhere smooth and concave down, such that there is a single maximum.

All unimodal maps undergo quantitatively similar period-doubling roots to chaos described by the universal Feigenbaum constants δ, α:

$$\delta = lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} \approx 4.669 \tag{2}$$

where r_n is the value of r where a 2^n cycle first appears; and

$$\alpha = lim_{n\to\infty} \frac{d_n}{d_{n+1}} \approx -2.5029 \tag{3}$$

where d_n is the distance from the maximum of f (at $x_m = 1/2$ for the logistic map) to the nearest point x in a 2^n-cycle (Strogatz, p. 373).

One way to approximate α is by use of the Feigenbaum function (renormalization)

$$g(x) = \alpha g(g(x/\alpha)), \;\; g'(0) = 0, \;\; g(0) = 1 \tag{4}$$

Expand $g(x)$ around the local maximum ($x = 0$) as $g(x) \approx 1 + bx^2$ and solve for α (Strogatz p. 384, Schuster p. 47; note that Schuster defines α to be positive).

The above discussion is specific to unimodal maps. A quadratic map like the logistic map is unimodal, but a quartic map like $x_{n+1} = r - x^4$ is not unimodal since it's not concave down at the maximum.

1.3 Renormalization

Renormalization is the process by which $f^2(x,r)$, and ultimately $f^n(x,r)$, is rescaled to resemble $f(x,r)$ near the superstable point. At each step we scale f and x by a factor of α and shift r to the superstable value for the next cycle.

It seems strange that *every* map with a quadratic maximum would exhibit the same behavior. A hint lies in the fact that the maps all look the same infinitessimally near the hump (i.e., quadratic). The renormalization approach explains that near r values that have superstable n-cycle orbits, $f^n(x)$ sees only an x-value near the hump. This is a fixed point of $f^n(x)$; in $f(x)$, where there is an n-cycle rather than a fixed point, there are $(n-1)$ other x-values in the cycle and these can be far from the hump.