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APM203 section 5 notes

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1 Lorentz Equations

$$\dot{x} = \sigma(y-x) \tag{1}$$
$$\dot{y} = rx-y-xz \tag{2}$$
$$\dot{z} = xy-bz \tag{3}$$

With parameters $\sigma > 0$, $r > 0$, $b > 0$. At $r < 1$, the origin is stable, but it becomes unstable and 2 stable fixed points appear in a supercritical pitchfork bifurcation at $r = 1$. At $r > r_H = \frac{\sigma(\sigma+b+3)}{\sigma-b-1}$, both the limit cycles undergo subcritical Hopf bifurcations and become unstable. Note that we often take $\sigma = 10$, $b = 8/3$, and vary r.

2 Time-integration by finite difference

Consider the differential equation

$$\frac{\partial C(x,t)}{\partial t} = F(C(x,t)) \tag{4}$$

Where $F(C)$ is used to represent the right hand side, which might include spatial derivates of $C(x,t)$. There are a few ways to approximate the slope of a function $f(x)$ which is known only at points $x_n = n\,\Delta x$. Writing $f_n \equiv f(n\,\Delta x)$, we can approximate the slope at x_n as $\frac{1}{\Delta x}(f_{n+1}-f_n)$ (forward difference), $\frac{1}{\Delta x}(f_n - f_{n-1})$ (backward difference), or $\frac{1}{2\Delta x}(f_{n+1}-f_{n-1})$ (centered difference). One can show that errors in forward and backward difference methods go like Δx whereas in centered difference they go like Δx^2, so centered difference is often preferable.

2.1 Improved Euler's method

Since we typically know the initial value, $C(x,t=0)$, and want to find C at future times, it can be simplest to just use forward difference in time, $\frac{\partial C(x,t)}{\partial t} = \frac{1}{\Delta t}(C_{n+1} - C_n) = F(C_n)$, where $C_n \equiv C(x, n\Delta t)$, or

$$C_{n+1} = C_n + F(C_n)\Delta t \tag{5}$$

This is called Euler's method.

A more accurate version of this is the improved Euler method, which uses something similar to centered difference: The derivative is approximated as the average between the derivative at C_n and at the "trial step" $\tilde{C}_{n+1}$. First we evaluate the trial step using the standard Euler method,

$$\tilde{C}_{n+1} = C_n + F(C_n)\Delta t \tag{6}$$

Then we evaluate the actual step using the average slope

$$C_{n+1} = C_n + \frac{1}{2}[F(C_n) + F(\tilde{C}_{n+1})]\Delta t \tag{7}$$

2.2 Leapfrog method

When you use centered difference in time, it's called the leapfrog method:

$$C_{n+1} = C_{n-1} + F(C_n)2\Delta t \tag{8}$$

You need to know the value at the first 2 time steps for this method, rather than just the initial condition, so when using the Leapfrog method one often integrates the first step or two using the improved Euler method.

We showed in Homework 2 that the leapfrog method can be unstable. The Robert filter is often used to smooth the solution and avoid numerical instabilities. It is similar to adding a diffusive term in time. Note that the second time derivative is approximated using centered difference as $\frac{\partial^2 C(x,t)}{\partial t^2} = \frac{1}{\Delta t^2}(C_{n+1} - 2C_n + C_{n-1})$. The Robert filter is added to the leapfrog scheme as

$$C_{n+1} = C_{n-1} + F(C_n)2\Delta t + R_F(C_n - 2C_{n-1} + C_{n-2}) \tag{9}$$

Where R_F is the Robert filter coefficient. R_F is typically chosen to be the smallest vlaue that gives smooth evolution of C.