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APM203 section 4 notes

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Overview

1 Multiple time scales: averaging method

The method of multiple scales is used frequently to find approximate solutions to physical problems. Here we use multiple time scales for systems which are nearly simple harmonic oscillators, $\ddot{x}+x+\varepsilon h(x,\dot{x})=0$. If $\varepsilon=0$, this would describe a simple harmonic oscillator with frequency $\omega=1$, but we are concerned with $0<\varepsilon\ll 1$. The solution to this is an oscillation in which the amplitude and phase slowly wander.

To apply the two-timing method, assume there are two time scales, $\tau=t$ and $T=\varepsilon t$. Use $x=x_0(\tau,T)+\varepsilon x_1(\tau,T)+O(\varepsilon^2)$ and expand the derivatives ($\partial/\partial t$, $\partial^2/\partial^2 t$) in T and τ. Solving the $O(\varepsilon^0)$ terms, you get $x_0(\tau,T)=A(T)\cos(\tau)+B(T)\sin(\tau)$, or equivalently a sine with amplitude and phase varying in T or two complex exponentials. Next, write down the $O(\varepsilon^1)$ terms and. The key step is here: eliminate the "secular" terms. The solution to $\ddot{x}_1+x_1=\cos(t)$ with $x_1(0)=\dot{x}_1(0)=0$ is $x(t)=\frac{1}{2}\,t\sin(t)$. This grows without bound; the solution to the original equation should be nearly a simple harmonic oscillator and should be bounded. We don't actually find x_1; rather, we just make sure that it doesn't grow without bound by solving for $A(T)$ and $B(T)$ that eliminate any secular terms in x_1. This can be done by using sine and cosine identities to find the $\cos(t)$ and $\sin(t)$ terms, or, more generally, by finding the coefficients of the $\cos(t)$ and $\sin(t)$ terms in the Fourier expansion of $RHS=-2\partial_{\tau T}x_0-h$, which are the average over a cycle of $\frac{1}{2}RHS\cos(t)$ and $\frac{1}{2}RHS\cos(t)$.

In the "method of averaging", the same equations for $(A(t),B(t))$ [or equivalently $(r(t),\phi(t))$] are derived slightly differently, using a method which doesn't explicitly involve the removal of secular terms. Rather, the only approximation is equating the amplitudes (A, B) with their average values over one cycle of $\sin(t)$. Hence we can see that the two-timing method should work well for a wide range of values of ε: as long as we can approximate the varying amplitude and phase of an oscillatory solution as constant during a cycle of the oscillation, the two-timing solution will hold. For example, if $x(t)$ approaches a roughly sinusoidal limit cycle, the two-timing solution will always be a good approximation to the amplitude of the limit cycle, even if it is sometimes inaccurate in describing the approach from an initial condition to the limit cycle.

2 Hopf bifurcations

A Hopf bifurcation is similar to a pitchfork bifurcation, except that it involves a limit cycle rather than just fixed points. A Hopf bifurcation describes the creation and destruction of a limit cycle.

- **Supercritical Hopf:** At first there is just a stable fixed point at the origin ($\mu < 0$ in Strogatz) and all perturbations decay to zero. As a parameter is varied, the fixed point at the origin becomes unstable and a stable limit cycle appears around it; the amplitude of the limit cycle increases as the parameter is further varied.
- **Subcritical Hopf:** At first there is an unstable fixed point at the origin ($\mu > 0$ is Strogatz), perhaps with a stable limit cycle encircling it beyond the region where the bifurcation occurs. Varying a parameter causes the origin to become stable with an unstable limit cycle immediately around it.