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APM203 section 3 notes

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Overview

1 Reversible systems

A system ($\dot{x} = f(x,y)$, $\dot{y} = g(x,y)$) is reversible if it is invariant (i.e., equalities still hold) under ($t \to -t$, $y \to -y$). Implications: Basic graphical or heuristic arguments relying on this symmetry often show for reversible systems that (a) linear centers are true centers (and not spirals) for the full nonlinear system, or (b) homoclinic orbits exist.

2 Existence of closed orbits

(Note that a **limit cycle** is an isolated closed orbit.) A system has no closed orbits if it has a Liapunov function or can be written as a gradient system.

2.1 Gradient systems

A gradient system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is one which can be written $\dot{\mathbf{x}} = -\nabla V$ for some function $V(\mathbf{x})$.

2.2 Liapunov functions

A Liapunov function is any function $V(\mathbf{x})$ satisfying (a) $V(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{x}^*$ and $V(\mathbf{x}^*) = 0$, and (b) $\dot{V} < 0$ for all $\mathbf{x} \neq \mathbf{x}^*$. A common guess is $V(\mathbf{x}) = x^2 + ay^2$ (e.g., Strogatz Ex. 7.2.3).

3 Separatrices, manifolds

3.1 Separatrix

A separatrix is a trajectory in phase space that separates two regions with qualitatively different trajectories. It is typically a trajectory that intersects a saddle point.

3.2 Stable and unstable invariant manifolds

The curve traced out by any trajectory in 2D phase space is an invariant manifold, because a trajectory that starts on this curve stays on it forever. An invariant manifold is a space that trajectories don't leave.

The stable manifold of a fixed point (or limit cycle) is the set of initial conditions whose trajectories will eventually approach the fixed point (or limit cycle). The unstable manifold is the set of initial conditions which will eventually approach the fixed point (or limit cycle) when evolved *backward* in time; in other words, it is the set of points that can be reached by trajectories starting infinitessimally close to the fixed point (or limit cycle). For a 2D saddle point: there are 2 curves that pass through the saddle point on which trajectories lie, and these are the stable and unstable manifold.

4 Diagonal form and Jordan form

If an $n \times n$ matrix $\mathbf{A}$ has n eigenvectors and n distinct eigenvalues, it can be diagonalized as $\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = \mathbf{D}$, where $\mathbf{D}$ is a diagonal matrix (has only zeros except along the main diagonal). $\mathbf{Q}$ can be made up of columns which are the eigenvectors of $\mathbf{A}$, and the elements of $\mathbf{D}$ will be the corresponding eigenvalues.

Jordan form is a generalization of diagonal form. A matrix in Jordan form is upper triangular, i.e., it has only zeros off the main diagonal except that it can have 1's immediately above some of the diagonal elements. A matrix $\mathbf{A}$ can be put in Jordan form as $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{J}$, where the columns of $\mathbf{P}$ are the *generalized* eigenvectors of $\mathbf{A}$: they form a basis guaranteed to span the vector space. An eigenvector v_i of $\mathbf{A}$ satisfies $\mathbf{A}v_i = \lambda_i v_i$, so $(\mathbf{A} - \lambda_i \mathbf{I})v_i = 0$. The generalized eigenvector v_{i+1} corresponding to eigenvalue λ_i satisfies $\mathbf{A}v_{i+1} = \lambda_i v_{i+1} + v_i$, so $(\mathbf{A} - \lambda_i \mathbf{I})^2 v_{i+1} = (\mathbf{A} - \lambda_i \mathbf{I})v_i = 0$.