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APM203 section 2 notes

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Overview

1 2D linearized stability theory

(Strogatz, p. 123-138.) A linear 2D system has the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{1}$$

with

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \tag{2}$$

This linear system necessary has just one fixed point, $\mathbf{x}^* = (\mathbf{0}, \mathbf{0})$.

We'll define

$$\tau \equiv \text{trace}(\mathbf{A}) = a + d \tag{3}$$

$$\Delta \equiv \det(\mathbf{A}) = ad - bc \tag{4}$$

The eigenvalues are

$$\lambda_{1,2} = \frac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right) \tag{5}$$

To understand the following table, it helps to also consider that $\Delta = \lambda_1 \lambda_2$ and $\tau = \lambda_1 + \lambda_2$.

The table below is a companion to the diagram in Strogatz (p. 137, Figure 5.2.8).

Fixed point classification

$\Delta > 0$	$\tau > 2\sqrt{\Delta}$	unstable nodes
$\Delta > 0$	$\tau = \pm 2\sqrt{\Delta}$	stars*, degenerate nodes*
$\Delta > 0$	$0 < \tau < 2\sqrt{\Delta}$	unstable spirals
$\Delta > 0$	$\tau = 0$	centers*
$\Delta > 0$	$0 > \tau > -2\sqrt{\Delta}$	stable spirals
$\Delta > 0$	$\tau < -2\sqrt{\Delta}$	stable nodes
$\Delta = 0$	any τ	non-isolated fixed points*
$\Delta < 0$	any τ	saddle points

*borderline case

Along the line with stars and degenerate nodes, it is the former if there are 2 unique eigenvectors and the latter if there is only one.

Linearization is robust, except for borderline cases where it can give the wrong result for the stability of a nonlinear system (correct result may be on the borderline or in the adjacent regions immediately to either side of the line in Figure 5.2.8). There is no one recipe for how to approach borderline cases, but we'll come across several techniques that are sometimes applicable.

2 2D nonlinear systems

Here

$$\dot{\mathbf{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} f(x) \\ g(y) \end{pmatrix} \tag{6}$$

Fixed points (x^*, y^*) satisfy $\begin{pmatrix} f(x^*) \\ g(y^*) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

To linearize around a fixed point, use the Jacobian

$$\mathbf{A} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \tag{7}$$

evaluated at (x^*, y^*).

2.1 Plotting in phase space

In Matlab, use `quiver(x,y,u,v)` to plot a vector field.

Sometimes it's more instructive to plot a "direction field" instead of a vector field. Just divide your flow vectors $(u, v) = (f(x,y), g(x,y))$ by their amplitude ($\sqrt{u^2 + v^2}$).

The solution to (1) is

$$\mathbf{x(t)} = \mathbf{x_0} e^{\mathbf{At}} \tag{8}$$

where $e^{\mathbf{At}}$ represents the matrix exponential. Analytically, the matrix exponential can be calculated by diagonalizing **A**, but in Matlab you can just use `expm(A)`. This allows you to plot trajectories of an initial condition in phase space.

For nonlinear systems, it's often extremely useful to look at nullclines: lines along which $\dot{x} = 0$ or $\dot{y} = 0$. Draw the line and then draw horizontal or vertical arrows along it to show the flow direction.

A useful program to plot trajectories of nonlinear ODEs in Matlab can be downloaded from http://math.rice.edu/ dfield/

3 Circle map

$$\Theta_{n+1} = \Theta_n + \Omega - \frac{K}{2\pi} sin(2\pi\Theta_n) \text{ mod } 1 \tag{9}$$

The winding number (p/q) is found by dropping the mod 1 restriction from (9) and evaluating

$$\frac{p}{q} = lim_{n\to\infty} \frac{\Theta_n - \Theta_o}{n} \tag{10}$$