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APM203 section 1 notes

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Overview

1 Continuous systems (ODEs) and discrete systems (maps)

In this course we are looking mostly at first order ODEs,

$$\frac{dx}{dt} \equiv \dot{x} = f(x) \tag{1}$$

(note that x and f may be vectors) and trying to describe the solution $x(t)$. We are also looking at maps. If we discretize the ODE (1) as $\frac{x_{n+1}-x_n}{\Delta t} \approx \dot{x} = f(x_n)$, we can write (1) approximately as a map, $x_{n+1} = (x_n + f(x_n))\Delta t$. In general, we write maps as

$$x_{n+1} = g(x_n) \tag{2}$$

and try to describe the solution x_n (plot x_n vs. n).

1.1 ODEs vs. PDEs

We'll be looking mostly at ODEs, equations involving x and temporal derivatives of x (e.g., dx/dt, d^2x/dt^2) that we solve for $x(t)$. An example of a PDE would be an equation involving y, dy/dx, and dy/dt that we would solve for $y(x,t)$.

1.2 Higher time derivatives

A 1-dimensional ODE with a 2nd derivative in time (equation with x, dx/dt, d^2x/dt^2) can be written equivalently as a 2-dimensional ODE with only 1st derivatives in time. If $\ddot{x} = f(x)$, we can write $x_1 = x$ and $x_2 = \dot{x}$, so

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} x_2 \\ f(x_1, x_2) \end{pmatrix} \tag{3}$$

We've recast the differential equation $\ddot{x} = f(x)$ as $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$.

2 Dimensional analysis: The Buckingham-Pi Theorem

Nondimensionalization typically comes up in one of two related types of problems. In one type (a), you have an equation (usually a differential equation rather than a solution) and you want to simplify it. You can reduce the number of parameters by nondimensionalizing it. In the other type (b), you don't have an equation, but you want to guess at the form of the solution based on the quantities that you think should be relevant to the problem. The Buckingham-Pi Theorem gives an approach to both types of problems. The examples below illustrate its use in both situations.

The following discussion is adapted from Howard Stone's ES220 course notes.

The Buckingham-Pi Theorem states that given n variables expressible in terms of r independent dimensions, there are a total of $n-r$ independent dimensionless groups (i.e., combinations of physical variables).

Example (a): Simplifying the equation for damped mass on a spring. A mass-spring oscillator with linear damping has a displacement from equilibrium $x(t)$ that is described by the equations

$$m\frac{d^2x}{dt^2} + \zeta\frac{dx}{dt} + kx = 0 \tag{4}$$

$$x(0) = x_0,\ \frac{dx}{dt}(0) = 0 \tag{5}$$

The solution should be of the form $x(t; m, \zeta, k, x_0)$, which involves 6 variables (x, t, m, ζ, k, x_0 and 3 dimensions (mass, length, time). According to Buckingham-Pi Theorem, the solution should require knowledge of $6-3=3$ dimensionless groups. We can pick them to be nondimensionalized space X, nondimensionalized time T, and one parameter Λ involving physical parameters of the problem. We could find these dimensionless groups with the matrix method we used in class, but it isn't necessary with this few variables. We can scale x with x_0 and t with $\sqrt{m/k}$. A dimensionless combination of the parameters m, k, and ζ is $\zeta/\sqrt{mk}$. With a little algebra, we can write (4)-(5) as

$$\frac{d^2X}{dT^2} + \Lambda\frac{dX}{dT} + X = 0 \tag{6}$$

$$X(0) = 1,\ \frac{dX}{dT}(0) = 0 \tag{7}$$

with

$$X \equiv \frac{x}{x_0},\ T \equiv \frac{t}{\sqrt{m/k}},\ \Lambda \equiv \frac{\zeta}{\sqrt{mk}} \tag{8}$$

The point is that we've re-written (4)-(5) which had 4 tunable parameters (m, k, ζ, and x_0) as (6)-(7) which has only 1 parameter (Λ) and hence is far easier to analyze. Physically, Λ relates to how much the oscillator is damped each cycle. Three relevant physical quantities - the period, amplitude, and mass (or momentum amplitude) - have been scaled out.

Example (b): Guessing at the form of the solution for projectile motion. Consider a projectile of mass m fired horizontally at speed v_0 from an initial vertical height h. We seek the horizontal displacement x_f when the projectile hits the ground. Based on physical intuition, we can guess that it will be

$$x_f = f(m, h, v_0, g) \tag{9}$$

Again, we could make a matrix of variables and dimensions, but it's not necessary here. There are 5 variables (x_f, m, h, v_0, g) and 3 dimensions (length, mass, and time), so Buckingham-Pi tells us there will be 2 dimensionless groups. In this problem, only one variable (m) involves mass, so it can't be in the answer and we could equivalently say we have 4 variables and 2 dimensions. We can write the dimensionless groups as $X = \frac{x_f}{h}$ and $\Omega = \frac{gh}{v_0^2}$, so the solution we are looking for takes the form

$$X = F(\Omega) \tag{10}$$

with

$$X \equiv \frac{x_f}{h}, \ \ \Omega \equiv \frac{gh}{v_0^2}, \tag{11}$$

Using this type of approach, we can not know the function F, i.e., the functional relation between X and Ω. (The actual answer here is $F(\Omega) = \sqrt{\frac{2}{\Omega}}$.) To approximate F, we'd need to make observations or do an experiment. The problem is significantly simplified because we only need to figure out the functional dependence on a single variable, Ω, rather than on m, h, v_0, and g, so far less observations are necessary than a glance at the dimensional problem (9) might imply.

3 Fixed points and their stability

The first thing to look at when examining a dynamic system is fixed points and their stability. We can do it analytically or graphically.

Graphically: For ODEs plot $\dot{x}$ vs. x and examine zero crossings (fixed points) and the slope at these crossings (stable if it's negative). For maps make a cobweb plot.

Analytically, we proceed as follows:

	ODE $\dot{x} = f(x)$	map $x_{n+1} = G(x_n)$
fixed point ($x*$)	$f(x*) = 0$	$G(x*) = x*$
stable if	$f'(x*) < 0$	$\lvert G'(x*)\rvert < 1$

Note that in many problems, the analytical approach will fail (fixed points can't be solved for, $f'(x*) = 0$, or $|G'(x*)| = 1$) but the graphical approach will yield readily accessible results.

In higher dimensional systems, the stability of ODEs depends on the eigenvalues of $\frac{\partial f_i(x*)}{\partial x_j}$ being less than zero; the stability of maps depends on the absolute value of the eigenvalues of $\frac{\partial G_i(x*)}{\partial x_j}$ being less than 1.

4 Bifurcations

(a) Saddle-node bifurcation: $\dot{x} = \mu - x^2$ *(a stable/unstable pair of fixed points is created or destroyed)*

(b) Transcritical bifurcation: $\dot{x} = \mu x - x^2$ *(a fixed point changes its stability; a nearby fixed point with opposite stability is created so that the flow far from the bifurcation point remains unchanged)*

(c) Supercritical pitchfork bifurcation: $\dot{x} = \mu x - x^3$ *(in a problem with right-left symmetry, a stable fixed point becomes unstable with stable fixed points created on either side of it)*

(c) Subcritical pitchfork bifurcation: $\dot{x} = \mu x + x^3 - x^5$ *(Initially, there is one stable fixed point at the origin. Saddle-node bifurcations occur on both sides of it, creating a situation with three stable fixed points. At the actual subcritical pitchfork bifurcation, the two unstable fixed points disappear into the origin and the stable fixed point at the origin becomes unstable; the stable fixed points on either side of the origin remain.)*