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Nonlinear Dynamics and Chaos

Review of some of the topics covered in homework problems, based on section notes.

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1 One dimensional continuous systems

1.1 Fixed points and their stability

The first thing to look at when examining a dynamic system is fixed points and their stability. We can do it analytically or graphically.

Graphically: For ODEs plot $\dot{x}$ vs. x and examine zero crossings (fixed points) and the slope at these crossings (stable if it's negative). For maps make a cobweb plot.

Analytically, we proceed as follows:

	ODE $\dot{x} = f(x)$	map $x_{n+1} = G(x_n)$
fixed point $(x*)$	$f(x*) = 0$	$G(x*) = x*$
stable if	$f'(x*) < 0$	$\lvert G'(x*)\rvert < 1$

Note that in many problems, the analytical approach will fail (fixed points can't be solved for, $f'(x*) = 0$, or $|G'(x*)| = 1$) but the graphical approach will yield readily accessible results.

In higher dimensional systems, the stability of ODEs depends on the eigenvalues of $\frac{\partial f_i(x*)}{\partial x_j}$ being less than zero; the stability of maps depends on the absolute value of the eigenvalues of $\frac{\partial G_i(x*)}{\partial x_j}$ being less than 1.

1.2 Bifurcations

(a) Saddle-node bifurcation: $\dot{x} = \mu - x^2$ *(a stable/unstable pair of fixed points is created or destroyed)*

(b) Transcritical bifurcation: $\dot{x} = \mu x - x^2$ *(a fixed point changes its stability; a nearby fixed point with opposite stability is created so that the flow far from the bifurcation point remains unchanged)*

(c) Supercritical pitchfork bifurcation: $\dot{x} = \mu x - x^3$ *(in a problem with right-left symmetry, a stable fixed point becomes unstable with stable fixed points created on either side of it)*

(c) Subcritical pitchfork bifurcation: $\dot{x} = \mu x + x^3 - x^5$ *(Initially, there is one stable fixed point at the origin. Saddle-node bifurcations occur on both sides of it, creating a situation with three stable fixed points. At the actual subcritical pitchfork bifurcation, the two unstable fixed* *points disappear into the origin and the stable fixed point at the origin becomes unstable; the stable fixed points on either side of the origin remain.)*

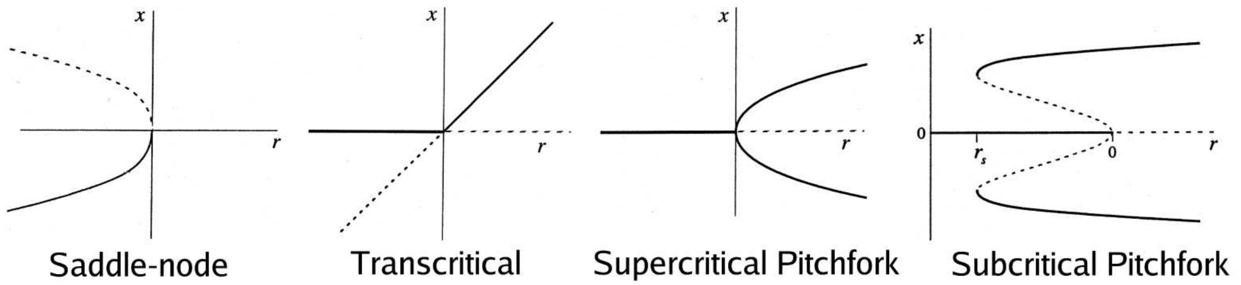

Figure 1: 1D bifurcations *(Strogatz, Figures 3.1.4, 3.2.2, 3.4.2, 3.4.7)*.

2 Two dimensional continuous systems

2.1 Linearized stability theory

(Strogatz, p. 123-138.) A linear 2D system has the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{1}$$

with

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \tag{2}$$

This linear system necessary has just one fixed point, $\mathbf{x}^* = (\mathbf{0}, \mathbf{0})$.

For a nonlinear 2D system, let

$$\dot{\mathbf{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} f(x) \\ g(y) \end{pmatrix} \tag{3}$$

Fixed points (x^*, y^*) satisfy $\begin{pmatrix} f(x^*) \\ g(y^*) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

To linearize around a fixed point such that $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + O(\mathbf{x}^2)$, use the Jacobian

$$\mathbf{A} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \tag{4}$$

evaluated at (x^*, y^*).

We'll define

$$\tau \equiv \mathrm{trace}(\mathbf{A}) = a + d \tag{5}$$

$$\Delta \equiv \det(\mathbf{A}) = ad - bc \tag{6}$$

The eigenvalues are

$$\lambda_{1,2} = \frac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right) \tag{7}$$

To understand the following diagram and table, it helps to also consider that $\Delta = \lambda_1\lambda_2$ and $\tau = \lambda_1 + \lambda_2$.

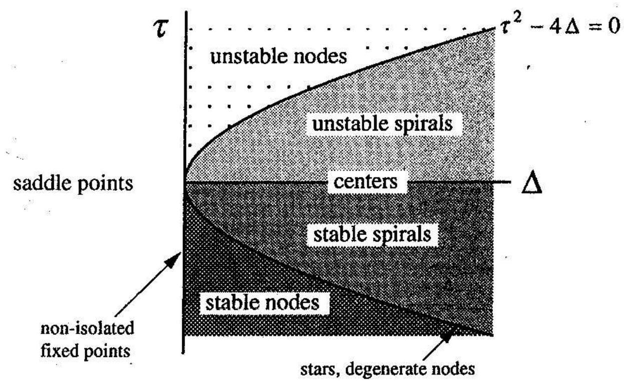

Figure 2: Fixed point classification *(Strogatz Figure 5.2.8)*.

$\Delta > 0$	$\tau > 2\sqrt{\Delta}$	unstable nodes
$\Delta > 0$	$\tau = \pm 2\sqrt{\Delta}$	stars*, degenerate nodes*
$\Delta > 0$	$0 < \tau < 2\sqrt{\Delta}$	unstable spirals
$\Delta > 0$	$\tau = 0$	centers*
$\Delta > 0$	$0 > \tau > -2\sqrt{\Delta}$	stable spirals
$\Delta > 0$	$\tau < -2\sqrt{\Delta}$	stable nodes
$\Delta = 0$	any τ	non-isolated fixed points*
$\Delta < 0$	any τ	saddle points

*borderline case

Along the line with stars and degenerate nodes, it is the former if there are 2 unique eigenvectors and the latter if there is only one.

Linearization is robust, except for borderline cases where it can give the wrong result for the stability of a nonlinear system (correct result may be on the borderline or in the adjacent regions immediately to either side of the line in Figure 2). There is no one recipe for how to approach borderline cases, but we considered several techniques that are sometimes applicable. The main thing we are interested in is whether a fixed point is stable, so the most interesting borderline cases are at the edge of the stable region in Figure 2, which is the lower right quadrant. Reversibility (discussed below) can be helpful to check whether linearized centers are true centers in the full nonlinear system. When linearization implies non-isolated fixed points, arguments based on the stabilizing or destabilizing effect of the dropped nonlinear terms can be instructive (cf. Homework 3 #2b).

2.2 Plotting phase portraits

In Matlab, use `quiver(x,y,u,v)` to plot a vector field.

Sometimes it's more instructive to plot a "direction field" instead of a vector field. Just divide your flow vectors $(u,v) = (f(x,y), g(x,y))$ by their amplitude ($\sqrt{u^2+v^2}$).

The solution to the linear system (1) is

$$\mathbf{x}(\mathbf{t}) = \mathbf{x_0} e^{\mathbf{At}} \tag{8}$$

where $e^{\mathbf{At}}$ represents the matrix exponential. Analytically, the matrix exponential can be calculated by diagonalizing $\mathbf{A}$, but in Matlab you can just use `expm(A)`. This allows you to plot trajectories of an initial condition in phase space.

For nonlinear systems, it's often useful to look at nullclines: lines along which $\dot{x} = 0$ or $\dot{y} = 0$. Draw the line and then draw horizontal or vertical arrows along it to show the flow direction.

A useful program to plot trajectories of 2D nonlinear ODEs in Matlab can be downloaded from http://math.rice.edu/∼dfield/

2.3 Index of a fixed point

We can define an index for any curve in phase space. The index for a curve is the sum of the indices of the fixed points enclosed. Closed orbits must enclose fixed points whose indices sum to $+1$.

To find the index of a fixed point, draw the nullclines, draw a circle around the fixed point, and draw an arrow representing the direction of the flow at each point where the circle intersects the nullclines. Then follow the circle once around counterclockwise, pointing your arm in the direction of each arrow, and see whether you rotate your arm in a full circle. The index is the number of times you have to rotate your arm counterclockwise (i.e., the number of counterclockwise revolutions made by the vector field as $\mathbf{x}$ moves once counterclockwise around the fixed point). Note that you only need to consider nullclines, since the flow vector needs to cross a nullcline every time it rotates through horizontal or vertical directions.

The index is -1 for saddle points and $+1$ for nodes, stars, centers, and spirals.

2.4 Reversible systems

A system ($\dot{x} = f(x,y)$, $\dot{y} = g(x,y)$) is reversible if it is invariant (i.e., equalities still hold) under ($t \to -t$, $y \to -y$). Implications: Basic graphical or heuristic arguments relying on this symmetry often show for reversible systems that (a) linear centers are true centers (and not spirals) for the full nonlinear system, or (b) homoclinic orbits exist.

2.5 Existence of closed orbits

(Note that a **limit cycle** is an isolated closed orbit.) A system has no closed orbits if it has a Liapunov function or can be written as a gradient system.

2.5.1 Gradient systems

A gradient system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is one which can be written $\dot{\mathbf{x}} = -\nabla V$ for some function $V(\mathbf{x})$.

2.5.2 Liapunov functions

A Liapunov function is any function $V(\mathbf{x})$ satisfying (a) $V(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{x}^*$ and $V(\mathbf{x}^*) = 0$, and (b) $\dot{V} < 0$ for all $\mathbf{x} \neq \mathbf{x}^*$. A common guess is $V(\mathbf{x}) = x^2 + ay^2$ (e.g., Strogatz Ex. 7.2.3).

2.6 Hopf bifurcations

A Hopf bifurcation is similar to a pitchfork bifurcation, except that it involves a limit cycle rather than just fixed points. A Hopf bifurcation describes the creation and destruction of a limit cycle.

- **Supercritical Hopf:** At first there is just a stable fixed point at the origin ($m < 0$ in Strogatz) and all perturbations decay to zero. As a parameter is varied, the fixed point at the origin becomes unstable and a stable limit cycle appears around it; the amplitude of the limit cycle increases as the parameter is further varied.
- **Subcritical Hopf:** At first there is an unstable fixed point at the origin ($m > 0$ is Strogatz), perhaps with a stable limit cycle encircling it beyond the region where the bifurcation occurs. Varying a parameter causes the origin to become stable with an unstable limit cycle immediately around it.

3 Separatrices, manifolds, and center manifold theory

3.1 Separatrix

A separatrix is a trajectory in phase space that separates two regions with qualitatively different trajectories. It is typically a trajectory that intersects a saddle point.

3.2 Stable and unstable invariant manifolds

The curve traced out by any trajectory in 2D phase space is an invariant manifold, because a trajectory that starts on this curve stays on it forever. An invariant manifold is a space that trajectories don't leave.

The stable manifold of a fixed point (or limit cycle) is the set of initial conditions whose trajectories will eventually approach the fixed point (or limit cycle). The unstable manifold is the set of initial conditions which will eventually approach the fixed point (or limit cycle) when evolved *backward* in time; in other words, it is the set of points that can be reached by trajectories starting infinitessimally close to the fixed point (or limit cycle). For a 2D saddle point: there are 2 curves that pass through the saddle point on which trajectories lie, and these are the stable and unstable manifold.

3.3 Center manifold theory

Consider a 3D system where $\dot{z} = -z$ so $z = 0$ is stable. In the $x - y$ plane, a 2D bifurcation (e.g. Hopf) could occur, even though this is a higher dimensional system. Here $\hat{z}$ is the stable manifold and the $x - y$ plane is the center manifold. If, instead, we had $\dot{z} = z$, then $\hat{z}$ would be an unstable manifold. Where there's an unstable manifold, you'll be unlikely to observe a bifurcation in the center manifold, since any small perturbation from the center manifold will grow.

In general, the manifolds aren't flat. Center manifold theory gives a method to approximate the dynamics on the center manifold near a local bifurcation point, thereby allowing one to analytically examine the bifurcation behavior.

3.3.1 Diagonal form and Jordan form

If an $n \times n$ matrix $\mathbf{A}$ has n eigenvectors and n distinct eigenvalues, it can be diagonalized as $\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = \mathbf{D}$, where $\mathbf{D}$ is a diagonal matrix (has only zeros except along the main diagonal). $\mathbf{Q}$ can be made up of columns which are the eigenvectors of $\mathbf{A}$, and the elements of $\mathbf{D}$ will be the corresponding eigenvalues.

Jordan form is a generalization of diagonal form. A matrix in Jordan form is upper triangular, i.e., it has only zeros off the main diagonal except that it can have 1's immediately above some of the diagonal elements. A matrix $\mathbf{A}$ can be put in Jordan form as $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{J}$, where the columns of $\mathbf{P}$ are the *generalized* eigenvectors of $\mathbf{A}$: they form a basis guaranteed to span the vector space. An eigenvector v_i of $\mathbf{A}$ satisfies $\mathbf{A}v_i = \lambda_i v_i$, so $(\mathbf{A} - \lambda_i \mathbf{I})v_i = 0$. The generalized eigenvector v_{i+1} corresponding to eigenvalue λ_i satisfies $\mathbf{A}v_{i+1} = \lambda_i v_{i+1} + v_i$, so $(\mathbf{A} - \lambda_i \mathbf{I})^2 v_{i+1} = (\mathbf{A} - \lambda_i \mathbf{I})v_i = 0$.

4 Other tools

4.1 Multiple time scales: averaging method

The method of multiple scales is used frequently to find approximate solutions to physical problems. Here we use multiple time scales for systems which are nearly simple harmonic oscillators, $\ddot{x} + x + \varepsilon h(x, \dot{x}) = 0$. If $\varepsilon = 0$, this would describe a simple harmonic oscillator with frequency $\omega = 1$, but we are concerned with $0 < \varepsilon \ll 1$. The solution to this is an oscillation in which the amplitude and phase slowly wander.

To apply the two-timing method, assume there are two time scales, $\tau = t$ and $T = \varepsilon t$. Use $x = x_0(\tau,T) + \varepsilon x_1(\tau,T) + O(\varepsilon^2)$ and expand the derivatives ($\partial/\partial t$, $\partial^2/\partial^2 t$) in T and τ. Solving the $O(\varepsilon^0)$ terms, you get $x_0(\tau,T) = A(T)\cos(\tau) + B(T)\sin(\tau)$, or equivalently a sine with amplitude and phase varying in T or two complex exponentials. Next, write down the $O(\varepsilon^1)$ terms and. The key step is here: eliminate the "secular" terms. The solution to $\ddot{x}_1 + x_1 = \cos(t)$ with $x_1(0) = \dot{x}_1(0) = 0$ is $x(t) = \frac{1}{2}\, t\sin(t)$. This grows without bound; the solution to the original equation should be nearly a simple harmonic oscillator and should be bounded. We don't actually find x_1; rather, we just make sure that it doesn't grow without bound by solving for $A(T)$ and $B(T)$ that eliminate any secular terms in x_1. This can be done by using sine and cosine identities to find the $\cos(t)$ and $\sin(t)$ terms, or, more generally, by finding the coefficients of the $\cos(t)$ and $\sin(t)$ terms in the Fourier expansion of $RHS = -2\partial_{\tau T} x_0 - h$, which are the average over a cycle of $\frac{1}{2}RHS\,\cos(t)$ and $\frac{1}{2}RHS\,\cos(t)$.

In the "method of averaging", the same equations for ($A(t)$, $B(t)$) [or equivalently ($r(t)$, $\phi(t)$)] are derived slightly differently, using a method which doesn't explicitly involve the removal of secular terms. Rather, the only approximation is equating the amplitudes (A, B) with their average values over one cycle of $\sin(t)$. Hence we can see that the two-timing method should work well for a wide range of values of ε: as long as we can approximate the varying amplitude and phase of an oscillatory solution as constant during a cycle of the oscillation, the two-timing solution will hold. For example, if $x(t)$ approaches a roughly sinusoidal limit cycle, the two-timing solution will always be a good approximation to the amplitude of the limit cycle, even if it is sometimes inaccurate in describing the approach from an initial condition to the limit cycle.

4.2 Time-integration by finite difference

Consider the differential equation

$$\frac{\partial C(x,t)}{\partial t} = F(C(x,t)) \tag{9}$$

Where $F(C)$ is used to represent the right hand side, which might include spatial derivates of $C(x,t)$. There are a few ways to approximate the slope of a function $f(x)$ which is known only at points $x_n = n\,\Delta x$. Writing $f_n \equiv f(n\,\Delta x)$, we can approximate the slope at x_n as $\frac{1}{\Delta x}(f_{n+1} - f_n)$ (forward difference), $\frac{1}{\Delta x}(f_n - f_{n-1})$ (backward difference), or $\frac{1}{2\Delta x}(f_{n+1} - f_{n-1})$ (centered difference). One can show that errors in forward and backward difference methods go like Δx whereas in centered difference they go like Δx^2, so centered difference is often preferable.

4.2.1 Improved Euler's method

Since we typically know the initial value, $C(x,t=0)$, and want to find C at future times, it can be simplest to just use forward difference in time, $\frac{\partial C(x,t)}{\partial t} = \frac{1}{\Delta t}(C_{n+1} - C_n) = F(C_n)$, where $C_n \equiv C(x, n\Delta t)$, or

$$C_{n+1} = C_n + F(C_n)\Delta t \tag{10}$$

This is called Euler's method.

A more accurate version of this is the improved Euler method, which uses something similar to centered difference: The derivative is approximated as the average between the derivative at C_n and at the "trial step" $\tilde{C}_{n+1}$. First we evaluate the trial step using the standard Euler method,

$$\tilde{C}_{n+1} = C_n + F(C_n)\Delta t \tag{11}$$

Then we evaluate the actual step using the average slope

$$C_{n+1} = C_n + \frac{1}{2}[F(C_n) + F(\tilde{C}_{n+1})]\Delta t \tag{12}$$

4.2.2 Leapfrog method

When you use centered difference in time, it's called the leapfrog method:

$$C_{n+1} = C_{n-1} + F(C_n)2\Delta t \tag{13}$$

You need to know the value at the first 2 time steps for this method, rather than just the initial condition, so when using the Leapfrog method one often integrates the first step or two using the improved Euler method.

We showed in Homework 2 that the leapfrog method can be unstable. The Robert filter is often used to smooth the solution and avoid numerical instabilities. It is similar to adding a diffusive term in time. Note that the second time derivative is approximated using centered difference as $\frac{\partial^2 C(x,t)}{\partial t^2} = \frac{1}{\Delta t^2}(C_{n+1} - 2C_n + C_{n-1})$. The Robert filter is added to the leapfrog scheme as

$$C_{n+1} = C_{n-1} + F(C_n)2\Delta t + R_F(C_n - 2C_{n-1} + C_{n-2}) \tag{14}$$

Where R_F is the Robert filter coefficient. R_F is typically chosen to be the smallest value that gives smooth evolution of C.

5 Routes to chaos in dissipative systems

5.1 Lorentz Equations

$$\dot{x} = s(y - x) \tag{15}$$

$$\dot{y} = rx - y - xz \tag{16}$$

$$\dot{z} = xy - bz \tag{17}$$

With parameters $s > 0$, $r > 0$, $b > 0$. At $r < 1$, the origin is stable, but it becomes unstable and 2 stable fixed points appear in a supercritical pitchfork bifurcation at $r = 1$. At $r > r_H = \frac{s(s+b+3)}{s-b-1}$, both the limit cycles undergo subcritical Hopf bifurcations and become unstable. Note that we often take $s = 10$, $b = 8/3$, and vary r.

5.2 Period-doubling route to chaos

5.2.1 Logistic map

The logistic map, which can be used to approximately describe population growth, is

$$x_{n+1} = rx_n(1 - x_n) \tag{18}$$

With $0 \leq x \leq 1$ and $0 \leq r \leq 4$. When $1 < r < 3$, $x^* = 0$ is an unstable fixed point and $x^* = 1 - 1/r$ is a stable fixed point. At $r > 3$, $x^* = 1 - 1/r$ becomes unstable and a 2-cycle is born (solution jumps back and forth between two values of x). At $r \approx 3.449$, the 2-cycle becomes unstable and a 4-cycle is born. At $r > r_\infty \approx 3.570$, chaotic solutions exist, as well as windows of periodic behavior in small ranges of r.

When the fixed point $x^* = 1 - 1/r$ occurs at the maximum of the logistic map ($x_m = 1/2$), it is called **superstable** because the linearization yields $|f'(x^*)| = 0$. Small perturbations around this fixed point converge to zero as $\eta_{n+1} = (-2\eta_0)^n$; recall that for regular stable fixed points perturbations converge as $\eta_n = \lambda^n \eta_0$ with $|\lambda| < 1$. Fixed points of an n-cycle solution are similarly superstable at particular values of r.

5.2.2 Universality

A unimodal map is a map where x_{n+1} vs x_n is everywhere smooth and concave down, such that there is a single maximum.

All unimodal maps undergo quantitatively similar period-doubling roots to chaos described by the universal Feigenbaum constants δ, α:

$$\delta = lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} \approx 4.669 \tag{19}$$

where r_n is the value of r where a 2^n cycle first appears; and

$$\alpha = lim_{n\to\infty} \frac{d_n}{d_{n+1}} \approx -2.5029 \tag{20}$$

where d_n is the distance from the maximum of f (at $x_m = 1/2$ for the logistic map) to the nearest point x in a 2^n-cycle (Strogatz, p. 373).

One way to approximate α is by use of the Feigenbaum function (renormalization)

$$g(x) = \alpha g(g(x/\alpha)), \;\; g'(0) = 0, \;\; g(0) = 1 \tag{21}$$

Expand $g(x)$ around the local maximum ($x = 0$) as $g(x) \approx 1 + bx^2$ and solve for α (Strogatz p. 384, Schuster p. 47; note that Schuster defines α to be positive).

The above discussion is specific to unimodal maps. A quadratic map like the logistic map is unimodal, but a quartic map like $x_{n+1} = r - x^4$ is not unimodal since it's not concave down at the maximum.

5.2.3 Renormalization

Renormalization is the process by which $f^2(x,r)$, and ultimately $f^n(x,r)$, is rescaled to resemble $f(x,r)$ near the superstable point. At each step we scale f and x by a factor of α and shift r to the superstable value for the next cycle.

It seems strange that *every* map with a quadratic maximum would exhibit the same behavior. A hint lies in the fact that the maps all look the same infinitessimally near the hump (i.e., quadratic). The renormalization approach explains that near r values that have superstable n-cycle orbits, $f^n(x)$ sees only an x-value near the hump. This is a fixed point of $f^n(x)$; in $f(x)$, where there is an n-cycle rather than a fixed point, there are $(n-1)$ other x-values in the cycle and these can be far from the hump.

5.3 Circle map

$$Q_{n+1} = Q_n + W - \frac{K}{2p} sin(2pQ_n) \bmod 1 \tag{22}$$

The winding number (p/q) is found by dropping the mod 1 restriction from (22) and evaluating

$$\frac{p}{q} = lim_{n\to\infty} \frac{Q_n - Q_o}{n} \tag{23}$$

5.4 Summary of routes to chaos

We've studies three routes to chaos in dissipative systems. The following outline is adapted from Schuster, Table 12.

- **Period-doubling** (Feigenbaum, 1978)
  - i. Pitchfork bifurcation
  - ii. Infinite cascade of period doublings, universal scaling parameters
  - iii. Logistic map: $x_{n+1} = rx_n(1-x_n)$
- **Intermittency** (Pomeau and Manneville, 1979)
  - i. Saddle-node (Type I), Hopf (Type II), or inverse period-doubling (Type III) bifurcation.
  - ii. Signal randomly alternates between nearly periodic and irregular behavior. Duration of periodic bursts scales as $\varepsilon^{-1/2}$ (Type I) or ε^{-1} (Type II, III).
  - iii. See Schuster, Table 7 (p. 98) for examples of Poincare map equations for all 3 types.
- **Quasi-periodicity** (Ruelle, Takens, and Newhouse, 1978)
  - i. Hopf bifurcation

ii. *Continuous system:* Stationary solution → periodic motion → quasi-periodic motion (2 spectral peaks, irrational ratio of frequencies) → 3-torus is "typically" unstable, so chaos after third Hopf bifurcation (or possibly after slightly larger parameter value). *Discrete map:* For small nonlinearity (K), winding number is rational (i.e., mode-locked solution, resonance) in Arnold's tongues and irrational outside the tongues; when nonlinearity is increased, tongues overlap, and solution jumps irregularly between resonances.

iii. Periodically forced pendulum: $\ddot{\theta}+\gamma\dot{\theta}+\sin\theta = A\cos(\omega t)+B$
Circle map: $\Theta_{n+1} = \Theta_n + \Omega - \frac{K}{2\pi} sin(2\pi\Theta_n)$ mod 1

6 Fractal dimension

Fractals are, roughly, complex geometric shapes with structure at arbitrarily small scales, usually with some degree of self-similarity.

6.1 Similarity and box dimensions

- **Similarity dimension** (for self-similar fractals): If scaling down by a factor of r leads to m copies of the original set, then the similarity dimension is

$$d_{sim} = \frac{\ln m}{\ln r}$$

- **Box dimension** (one possible dimension definition for fractals that are not self-similar): If a set S in 2-dimensional [D-dimensional] space requires $N(\varepsilon)$ boxes [D-dimensional cubes] of side ε to cover it, then its box dimension is

$$d_{box} = \lim_{\varepsilon\to 0}\frac{\ln N(\varepsilon)}{\ln(1/\varepsilon)}$$

6.2 Multifractals: D_q and $f(\alpha)$

The box counting dimension counts all cubes needed to cover the attractor equally, without regard for the fact that some cubes are far more frequently visited (i.e., points are more dense). To take into account the density of points, we can define a dimension spectrum D_q, where q determines how much influence density variations have on the dimension. Note that q is continuous and can be less than zero.

$$D_q = \frac{1}{1-q}\lim_{\varepsilon\to 0}\frac{\ln I(q,\varepsilon)}{\ln(1/\varepsilon)}$$

$$I(q,\varepsilon) \equiv \Sigma_{i=1}^{N(\varepsilon)}\mu_i^q$$

with $N(\varepsilon)$ the number of boxes of size ε needed to cover the attractor. Here μ_i is the measure (i.e., some concept of density of the boxes). Often on an attractor μ_i is the frequency of visits to the box, $\mu_i = \lim_{T\to\infty} \frac{\eta(C_i,T)}{T}$ where η is the amount of time the orbit spends in C_i during $0 \le t \le T$. Note that when $q = 0$, or when all μ_i are equal, all boxes get equal weight so D_q reduces to the box counting dimension.

Any measure μ_i which is not constant is called a *multifractal* measure. If we cover an attractor with boxes of size ε, we can define the singularity index α_i such that the density of points in the box μ_i satisfies

$$\mu_i = \varepsilon^{\alpha_i}$$

The multifractal spectrum $f(\alpha)$ is, roughly, the box dimension of the set of boxes with singularity index α_i (see Ott 9.1 for more exact definition).

The multifractal spectrum $f(\alpha)$ and dimension spectrum D_q contain the same information about an attractor (or any set with a defined measure). They are related by

$$f(\alpha(q)) = q\,\alpha(q) - (q-1)D_q$$

$$\alpha(q) = \frac{d}{dq}\left[(q-1)D_q\right]$$

7 A simple example

The hope of this example is to help us gain intuition about the behavior of chaotic systems in general. (cf. Strogatz problems 10.3.7,9,10; Schuster section 2.1).

Question:

Prove that the logistic map is chaotic when $r = 4$.

Answer:

The logistic map with $r = 4$ is

$$x_{n+1} = 4x_n(1-x_n) \tag{24}$$

We'll start by transforming from x to a new space defined as

$$x \equiv \sin^2(\pi y) \tag{25}$$

Note that the mapping is one to one in the range of the logistic map. Inserting (24) into (25) leads (after a little algebra) to the "binary shift map" (or "Bernoulli shift")

$$y_{n+1} = 2y_n \mathrm{mod} 1 \tag{26}$$

This is very good luck! (26), which exactly represents the logistic map with $r = 4$, is arguably the single simplest nonlinear dynamic system. It is a piecewise linear one dimensional map, only nonlinear because of the one discontinuity at $y = 1/2$. Furthermore, it turns out that the binary shift map is even easier to analyze than might be expected.

To show that the deterministic system (26) is chaotic, first we need to demonstrate that it's sensitive to initial conditions. Consider two initial conditions, y_0 and $\tilde{y}_0 = y_0 + \delta_0$. Plugging this into (26)

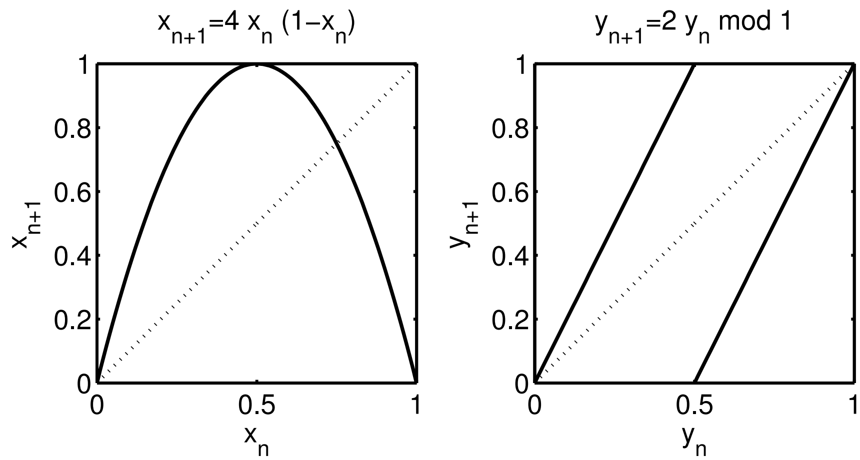

Figure 3: Logistic map and equivalent binary shift map.

shows that as long as we keep δ small enough so that it never straddles the map's discontinuity, the spacing between the trajectories will evolve as $\delta_n = 2^n \delta_0 = e^{\lambda}\delta_0$ with $\lambda = log(2) > 0$.

We've shown that a lot of systems have positive Lyapunov exponents, but we've found very few systems that we can prove are irregular (i.e., the trajectory never repeats), which is the other condition for chaos. As mentioned in class, no one has been able to prove this for the Lorentz system. As a prelude to our proof that (26) is irregular, we'll consider the decimal shift map,

$$y_{n+1} = 10 y_n \text{mod} 1 \tag{27}$$

In this map, at each iteration you multiply the previous value by 10 and then remove the integer part: every iteration just shifts the decimal point one space to the right and removes the left-most digit. Fixed points, then, are 0, $0.1\bar{1}$, $0.2\bar{2}$, ..., $0.9\bar{9}$. Initial conditions like $0.1\bar{2}$ lead to period-2 orbits.

Returning to the binary shift map (26), we see that we can treat it analogously to the decimal shift map (27) if we write y as a binary number. Every iteration just removes the leading digit of the initial condition. Hence every irrational initial condition leads to an irregular trajectory. Since the irrational numbers fill the space $[0,1]$ of initial conditions (the rationals are measure zero), nearly every trajectory of the binary shift map is irregular.

Further discussion:

Note that at every iteration the binary shift map stretches by a factor of 2 and folds at the middle, similar to the horseshoe map. Through the map's stretching, at every iteration it zooms in on the initial condition and extracts another digit of precision. The map needs to fold the space to keep the region bounded.

To think about a trajectory of the logistic map with $r = 4$, we can take the initial condition x_0, map it to y_0 via (25), and write y_0 as a binary number. The trajectory just eats through the digits of y_0. Because, in general, y_0 will be irrational, the orbit will be irregular.

This example is meant to illustrate how stretching and folding is the mechanism in chaotic systems that leads to sensitivity to initial conditions and irregular trajectories.

The key ingredient of a typical chaotic system which is missing from this discussion is a strange attractor (i.e., after a long enough time, trajectories converge on a set of points with non-integer fractal dimension). Although the logistic map does indeed have a strange attractor ($d < 1$), it's usually nicer to visualize strange attractors in 2D maps. We can make the binary shift map (26) into a 2D map by adding a term to keep track of folding,

$$y_{n+1} = \begin{cases} \frac{1}{3}x & 0 \leq y \leq \frac{1}{2} \\ \frac{1}{3}x + \frac{2}{3} & \frac{1}{2} < y \leq 1 \end{cases} \tag{28}$$

(26) and (28) form the familiar Baker map from homework 9 #1. From that homework problem, we see that the fractal structure of the attractor comes from infinitely repeated folding iterations. Note that the attractor of the Baker map is a series of vertical stripes with spaces between them that form a middle third Cantor set in the $\hat{x}$ direction.