AM203 Homework 8 SOLUTIONS

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① a) Linearize doubling transformation around $R=R_\infty$ (Schuster, p.48)

$Tg \equiv -\alpha g(g(-\frac{x}{\alpha}))$

Expand $f(x,R)$ about R_∞:

$$f(x,R) = f(x,R_\infty) + \Delta R\, \delta f(x) + \mathcal{O}(\Delta R^2)$$

$$\delta f \equiv \frac{\partial f(x,R)}{\partial R}\Big|_{R_\infty}, \quad \Delta R \equiv (R-R_\infty)$$

Apply doubling transformation:

$$T f(x,R) = T[f(x,R_\infty) + \Delta R\, \delta f(x)] + \mathcal{O}(\Delta R^2)$$
$$= -\alpha \{ f(m,R_\infty) + \Delta R\, \delta f(m) \} + \mathcal{O}(\Delta R^2)$$

using $m \equiv y + \Delta y$; $y \equiv f(-\frac{x}{\alpha}, R_\infty)$; $\Delta y \equiv \Delta R\, \delta f(-\frac{x}{\alpha})$

[The last step used def'n of T with $g \equiv f(x,R_\infty) + \Delta R\, \delta f(x)$]

Now expand in $\Delta y \sim \Delta R$ [with $f(x) \equiv f(x,R_\infty)$]

$$T[f + \Delta R\, \delta f] = -\alpha \{ \underbrace{f(y)}_{=Tf(x)} + \Delta y\, f'(y) + \Delta R\, \delta f(y) + \underbrace{\Delta R\, \Delta y\, \delta f'(y)}_{\mathcal{O}(\Delta R^2)} \}$$

$$= Tf(x) - \alpha \{ \Delta y\, f'(y) + \Delta R\, \delta f(y) \}$$

$$\boxed{\begin{array}{l} T f(x,R) = T f(x,R_\infty) + \Delta R\, L_{f_{R_\infty}} \delta f + \mathcal{O}(\Delta R^2), \quad \Delta R \equiv R-R_\infty \\ L_{f_{R_\infty}} \delta f \equiv -\alpha \{ \delta f(-\frac{x}{\alpha})\, f'(y) + \delta f(y) \}, \quad y \equiv f(-\frac{x}{\alpha}, R_\infty) \end{array}}$$

b) Write eigenvalue problem for L (Schuster, p. 50)

$L_g\, h(x) = \delta\, h(x)$

$L_g\, h(x) = -\alpha \{ h(-\frac{x}{\alpha})\, g'(g(-\frac{x}{\alpha})) + h(g(-\frac{x}{\alpha})) \}$

Evaluate at $x=0$

$L_g\, h(0) = -\alpha \{ h(0)\, g'(g(0)) + h(g(0)) \}$

Now expand $h(g(0))$ about $h(0)$ and keep only first term, i.e., assume $h(x)$ varies slowly enough that $h(1) \simeq h(0)$ [since $g(0)=1$]:

$L_g\, h(0) \simeq -\alpha \{ h(0)\, g'(g(0)) + h(0) \} = \delta\, h(0)$

$\hookrightarrow \boxed{-\alpha \{ g'(g(0)) + 1 \} \simeq \delta}$

① c) Find $g'(g(0)) = g'(1)$. (Schuster, p. 51)

$g(x) = Tg(x) \equiv -\alpha g[g(-\frac{x}{\alpha})]$

$g'(x) = -\alpha g'[g(-\frac{x}{\alpha})] \cdot g'(-\frac{x}{\alpha}) \cdot (-\frac{1}{\alpha}) = g'[g(-\frac{x}{\alpha})]\, g'(-\frac{x}{\alpha})$

$g''(x) = g''[g(-\frac{x}{\alpha})]\, g'(-\frac{x}{\alpha})(-\frac{1}{\alpha})\, g'(-\frac{x}{\alpha}) + g'[g(-\frac{x}{\alpha})]\, g''(-\frac{x}{\alpha})(-\frac{1}{\alpha})$

Evaluate at $x=0$

$g''(0) = (-\frac{1}{\alpha})\{ g''[g(0)]\,(g'(0))^2 + g'[g(0)]\, g''(0)\}$

The maximum is at $x=0$ so $g'(0)=0$; also $g(0)=1$

$g''(0) = -\frac{1}{\alpha}\{ g'(1)\, g''(0)\}$

↳ $\boxed{g'(1) = -\alpha}$

d) Estimate δ. (Schuster, p. 51)

From (b), $\delta \simeq -\alpha\{g'(g(0))+1\} = -\alpha\{g'(1)+1\}$

From (c), $g'(1) = -\alpha \Rightarrow \delta = -\alpha\{-\alpha+1\} \Rightarrow \boxed{\delta \simeq \alpha^2 - \alpha}$

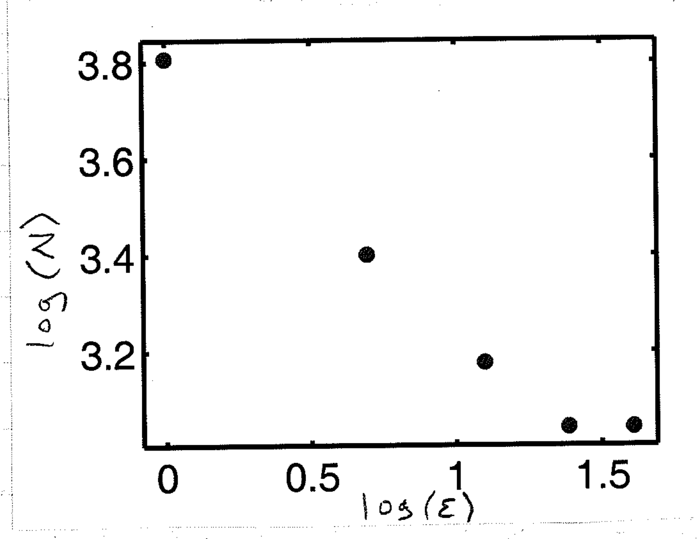
We know (e.g., Strogatz p. 383; note that α is negative in Strogatz but positive in Schuster) $\alpha = 2.5\ldots$

$\delta \simeq \alpha^2 - \alpha = \boxed{3.75}$

The correct value (e.g., Strogatz p. 355) is $\delta = 4.7\ldots$

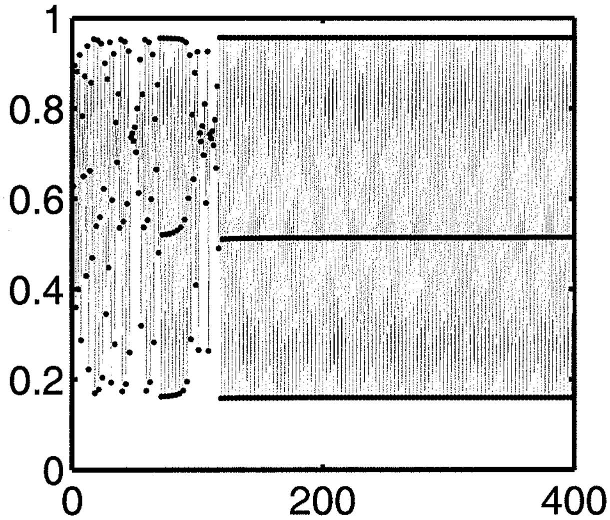
so this estimate is within 20%

(2) The period-3 window of the logistic map occurs at $r \equiv r^* = 1+\sqrt{8}$. As r is reduced from this value, the system undergoes an intermittency type I route to chaos. Here we're checking whether the laminar (periodic) bursts scale as predicted. I took 5 steps of size $5 \cdot 10^{-4}$ down from $r = r^*$ and estimated the length of the laminar intervals. The intervals varied in length, because sometimes the chaotic region seeds the ghost near the exit; I looked at the longest intervals. For these I counted how many peaks there were and estimated N as 3 times this (period-3 orbits). Defining ε such that $r = r^* - \varepsilon$, a plot of $(\log(\varepsilon), \log(N))$ for the estimates is below. The slope is near -0.5, implying $N \sim \varepsilon^{-1/2}$. The plots used to estimate N (after zooming in) are on the next page.

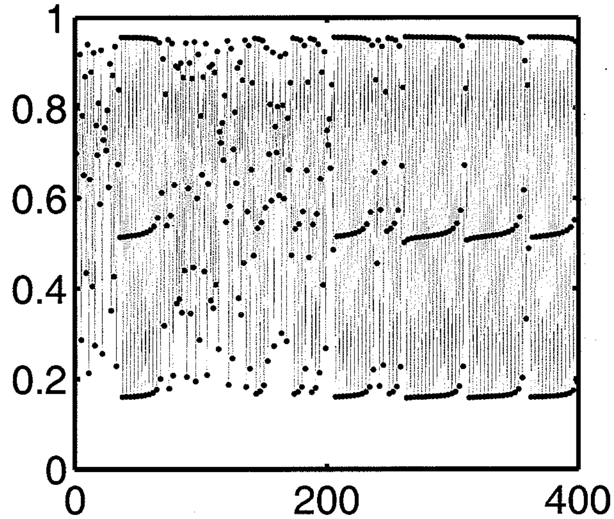


(2) cont'd

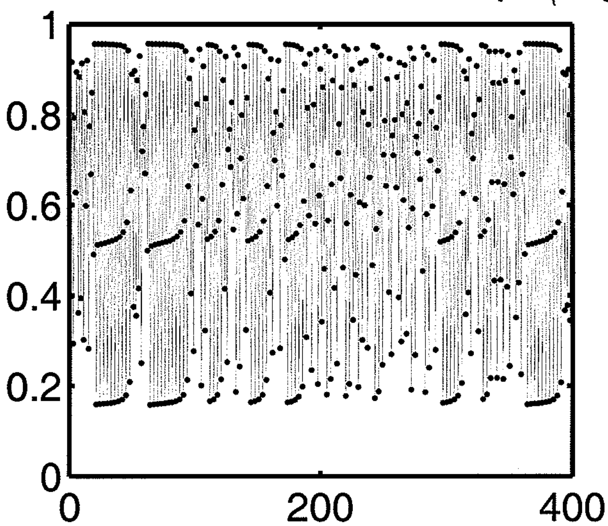
$\varepsilon = 0 * 5e{-4}$ $N \sim \infty$



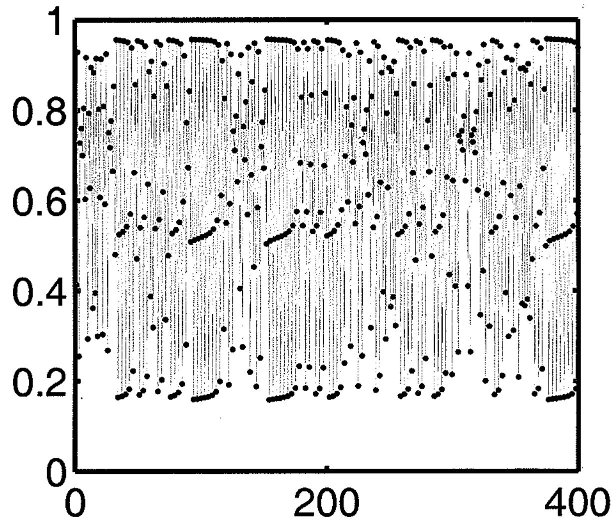
$\varepsilon = 1 * 5e{-4}$ $N \simeq 15 \cdot 3$



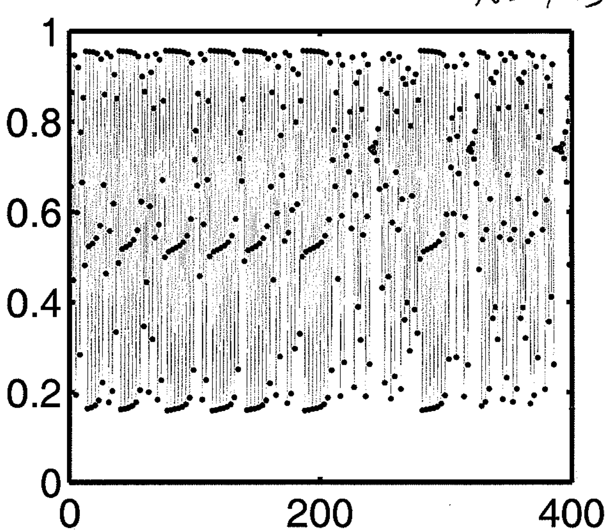
$\varepsilon = 2 * 5e{-4}$ $N \simeq 10 \cdot 3$



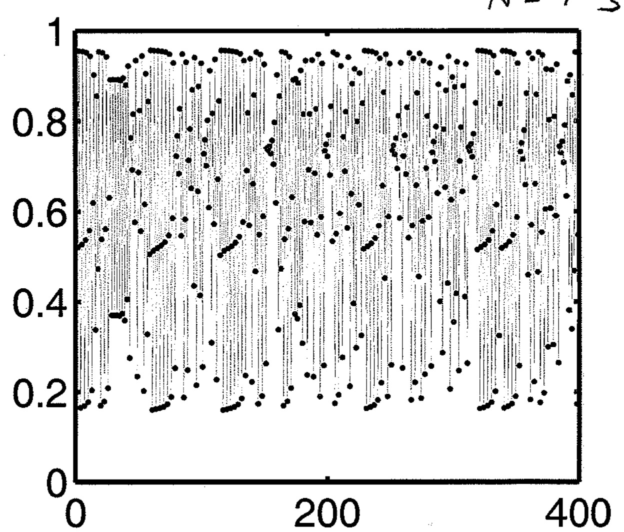
$\varepsilon = 3 * 5e{-4}$ $N \simeq 8 \cdot 3$



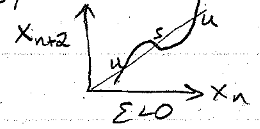
$\varepsilon = 4 * 5e{-4}$ $N \simeq 7 \cdot 3$



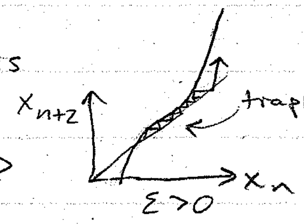
$\varepsilon = 5 * 5e{-4}$ $N \simeq 7 \cdot 3$



(3) Type III intermittency looks like this



{u, s, u indicate unstable and stable fixed points} $\Longrightarrow$



Based on the pictures, a simple map of this for small x_n and ε is (subcritical pitchfork normal form)

$\boxed{x_{n+2} = (1+\varepsilon)\, x_n + a x_n^3}$ with $a > 0$

Letting $\varepsilon \equiv 2\varepsilon'$, $a = (2+4\varepsilon)$, and dropping terms $\mathcal{O}(x_n^4, \varepsilon^2)$, this is equivalent to

$x_{n+1} = -(1+\varepsilon')\, x_n - x_n^3$ (example from Schuster, p. 98, Table 7)

Note that x_n oscillates with growing amplitude in the trapping region. We can approximate the growth of the envelope as $\frac{dx}{dn} \simeq \frac{x_{n+2} - x_n}{2}$

$\frac{dx}{dn} \simeq \frac{1}{2}\left(\varepsilon x + a x^3\right)$

Now we could estimate the period as

$N \simeq \int_{x_0}^{\infty} \left(\frac{dx}{dn}\right)^{-1} dx$, where $x_0 \gtrsim 0$ is the value the chaotic region seeds the map at, but there are issues with choosing x_0. One option is to side step this by scaling ε out of the differential equation. Define u and k s.t.

$x = \sqrt{\frac{\varepsilon}{a}}\, u, \quad n = \frac{2}{\varepsilon} k$

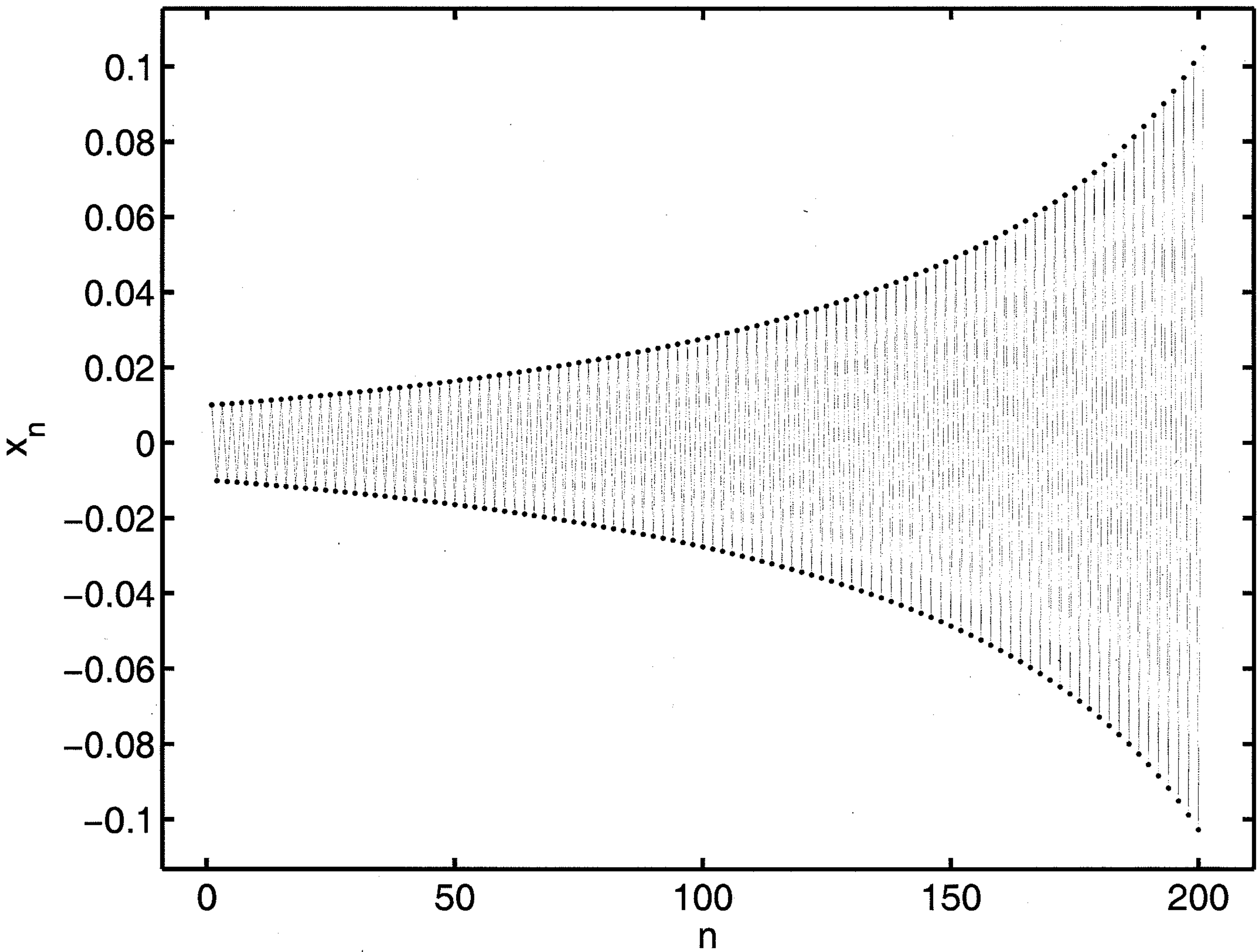
$\frac{du}{dk} = u(1+u^2)$

Since the trajectory in (k, u) is independent of ε, we can conclude from $n = \frac{2}{\varepsilon} k$ that $\boxed{N \sim \frac{1}{\varepsilon}}$

Note that this map approximates behavior in the trapping region (laminar phase of a type III intermittency route to chaos). Outside this region the map blows up rather than becoming chaotic.

(3) cont'd

Plot of x_{n+1} vs. n for $\varepsilon = 0.02$.



Question 4: Driven damped pendulum

The equation for the pendulum (scaling $\frac{g}{L} = 1$) is

$$\ddot{\theta} + \gamma\dot{\theta} + \sin\theta = f\cos(\omega t)$$

When $f = 0$ and $\gamma > 0$, the pendulum is damped but is not forced, and the solution will decay to $\theta = 0$. When $f > 0, \gamma > 0$ (forced and damped), we expect a phase locked solution. The pendulum can reduce its period by swinging higher, so we expect it to be able to mode lock to the forcing period if the forcing frequency is slower than the pendulum's linear frequency (see Figure A). If the forcing frequency is faster than the pendulum's linear frequency, the pendulum can still mode lock by swinging in opposite phase to the forcing. This is analogous to the scenario where a mass is driven back and forth with no spring (e.g., periodically knocking a book back and forth on a table). I doubt one can find a chaotic regime when the forcing period is much shorter than the pendulum's linear period (it's shortest possible natural period).

When f is increased farther and the forcing period is long enough, it will irregularly knock the pendulum over the top, and the motion will be chaotic (see Figure C). The transition from the phase locked solution to the chaotic one is analogous to the circle map, although we only observe mode locked behavior in the left-most Arnold tongue.

What about quasiperiodicity (when the pendulum's motion is a superposition of sine waves at frequencies with an irrational ratio)? Analogous to our treatment of the Lorentz system, we can check how the volume in phase space evolves. Writing

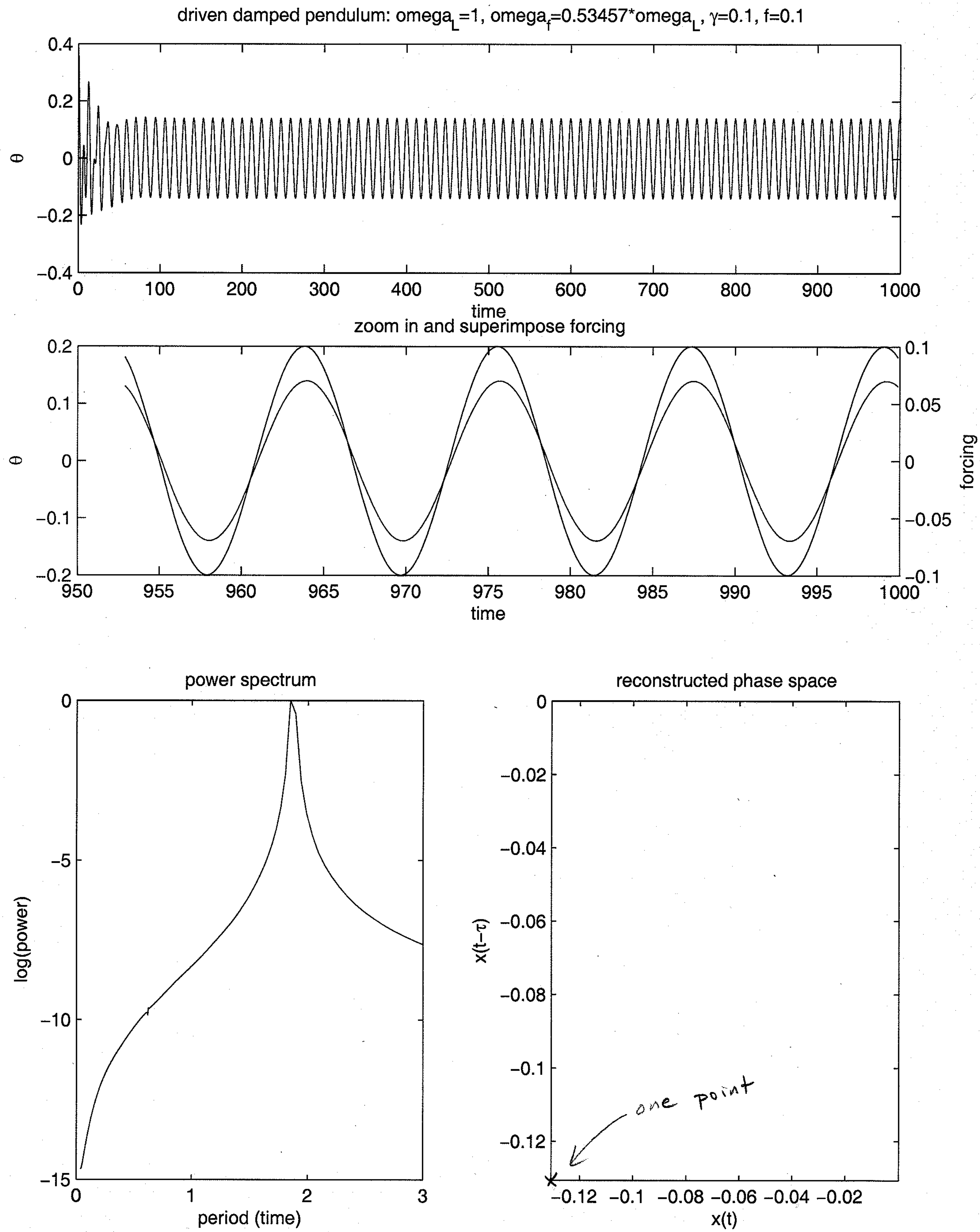
$$\dot{\mathbf{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} y \\ -\gamma y - \sin x + f\cos(\omega z) \\ 1 \end{pmatrix}$$

we see that $\nabla \bullet \mathbf{x} = -\gamma$, so when $\gamma > 0$ volume contracts in the phase space and the motion can not be on a torus (quasi-periodicity is not possible). When $\gamma = 0$, however, we can have a quasiperiodic solution with one period being slightly larger than the pendulum's linear frequency (since the oscillations are finite amplitude) and the other period being that of the forcing (see Figure B).

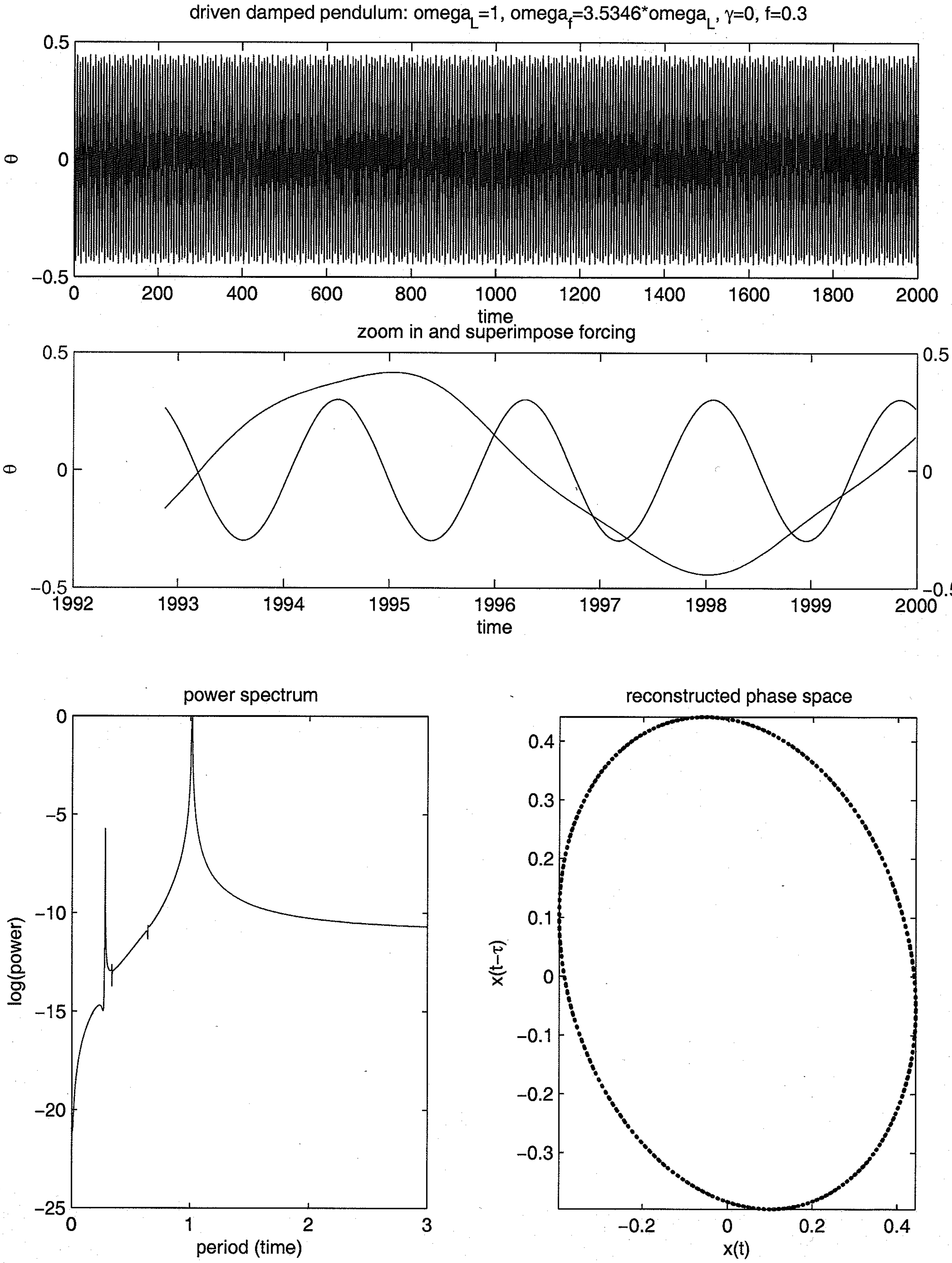
To summarize, the regimes encountered by the driven damped pendulum during the transition to chaos are as follows:

- **Stable** solution when the $f = 0$ and $\gamma > 0$.
- **Mode-locked** solution with the frequency of the forcing when f is increased (Figure A).
- **Quasi-periodic** solution when $\gamma = 0$, with the frequency of the forcing and a frequency near the pendulum's linear frequency (Figure B).
- **Chaotic** solution when $\gamma > 0$ and f is increased enough for the pendulum to sometimes flip over the top (Figure C).

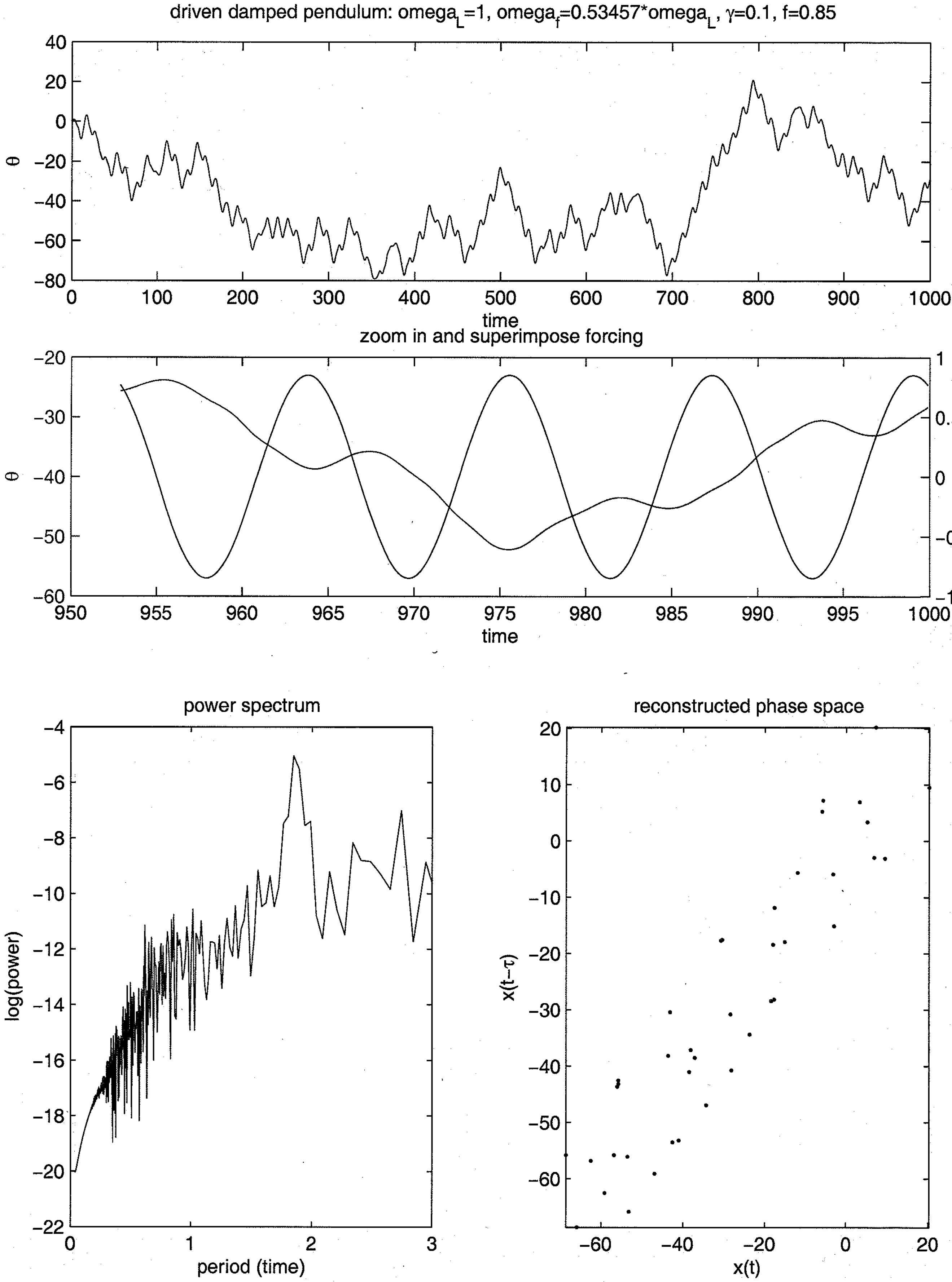
A. PERIODIC SOLUTION



B. QUASI PERIODIC SOLUTION



C. CHAOTIC SOLUTION



(5) a) Note that you can speed up matlab computing time significantly by iterating all your values of r at once: see code below (runs in a few seconds).

b) starting from random initial conditions and plotting iterates right away makes an uglier plot. Note that instead using $x_0 = \frac{1}{2}$ and plotting the first 10 iterates illustrates the "tantalizing patterns" (marked c).

c) $f(x) = rx(1-x)$ is horizontal at $x = \frac{1}{2}$, so it's not very "sensitive" to x. When x is fairly close to $\frac{1}{2}$, $f(x)$ will be very nearly $f(\frac{1}{2}) = \frac{r}{4}$. Hence there is a high density of points in the orbit diagram near $x = \frac{r}{4}$. Similar arguments apply for $f^n(x)$. This causes the tantalizing patterns.

$f(\frac{1}{2}) \equiv \boxed{F_1 = \frac{r}{4}}$; $\quad f^2(\frac{1}{2}) \equiv \boxed{F_2 = \frac{1}{4}\left(1 - \frac{r}{4}\right)r^2}$

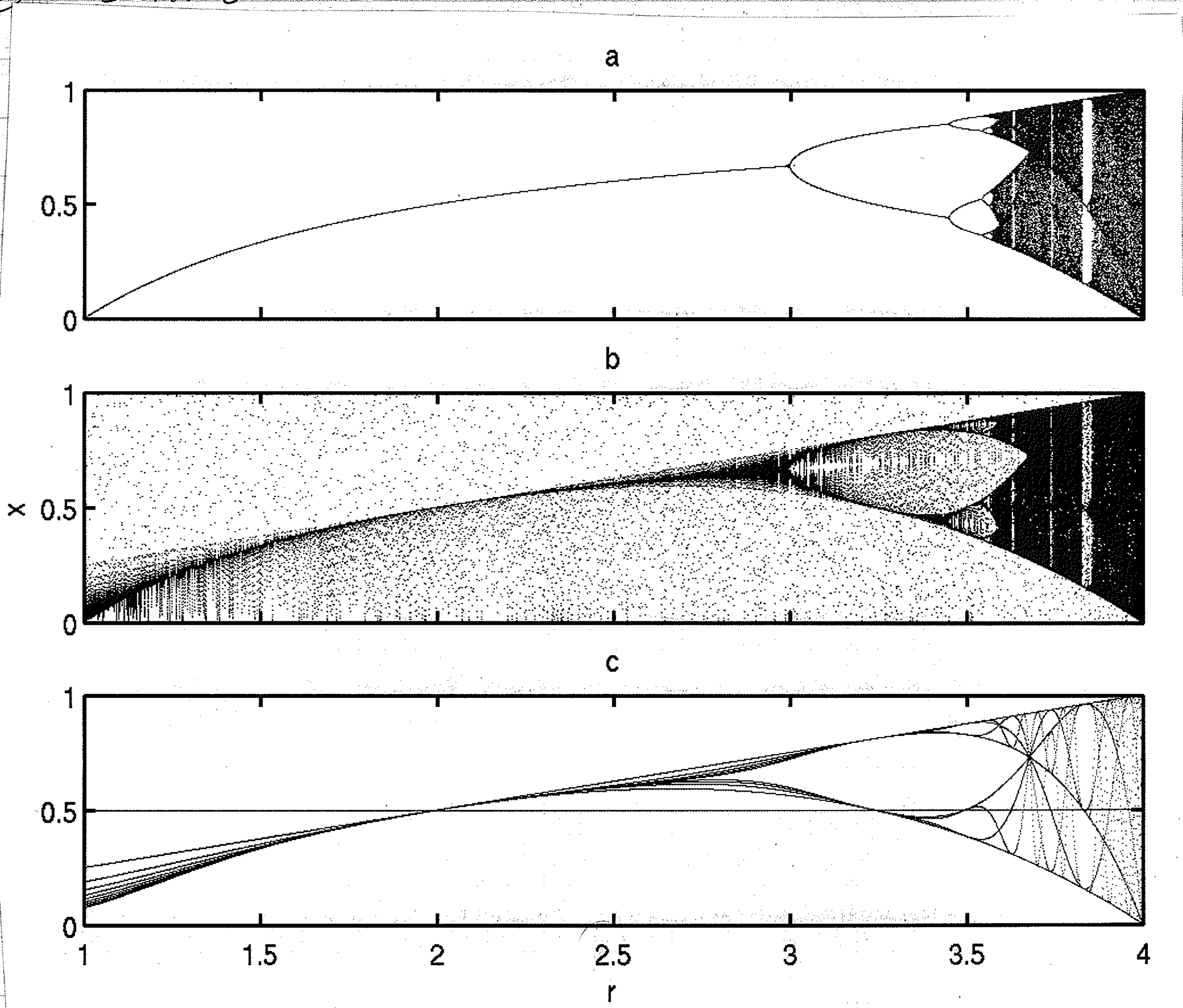
$f^3(\frac{1}{2}) \equiv \boxed{F_3 = \frac{1}{4}\left(1 - \frac{r}{4}\right)r^3\left(1 - \frac{1}{4}\left(1 - \frac{r}{4}\right)r^2\right)}$

$f^4(\frac{1}{2}) \equiv \boxed{F_4 = \frac{1}{4}(1-\frac{r}{4})r^4\left(1-\frac{1}{4}(1-\frac{r}{4})r^2\right)\left(1-\frac{1}{4}(1-\frac{r}{4})r^3\left(1-\frac{1}{4}(1-\frac{r}{4})r^2\right)\right)}$

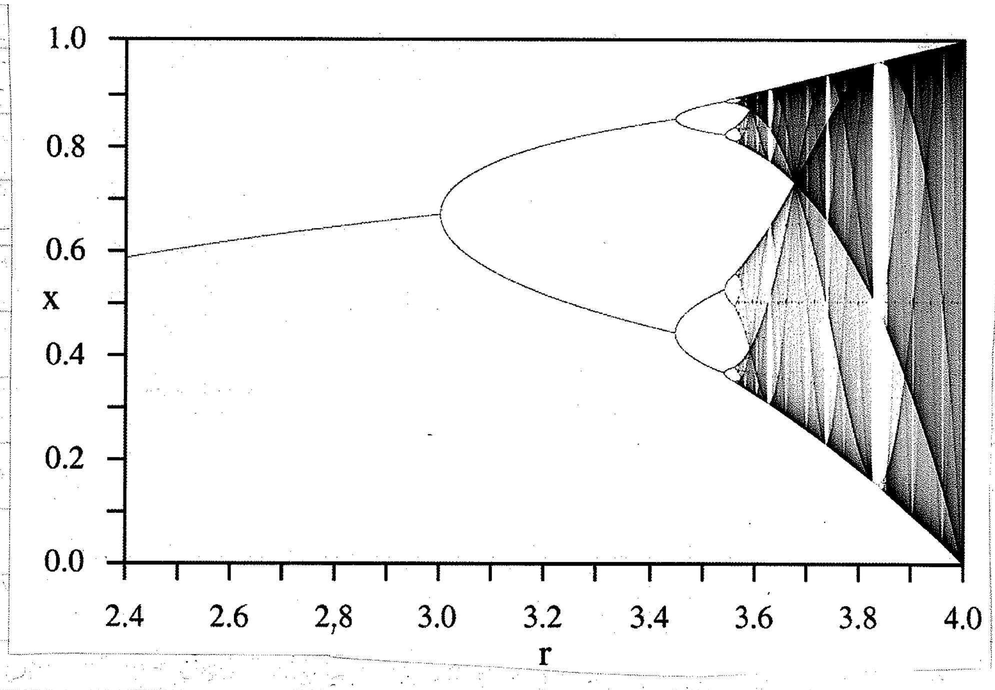
The corner of the big wedge is where F_3 and F_4 meet. Numerically solving $F_3 = F_4$ for r gives $\boxed{r = 3.679}$.

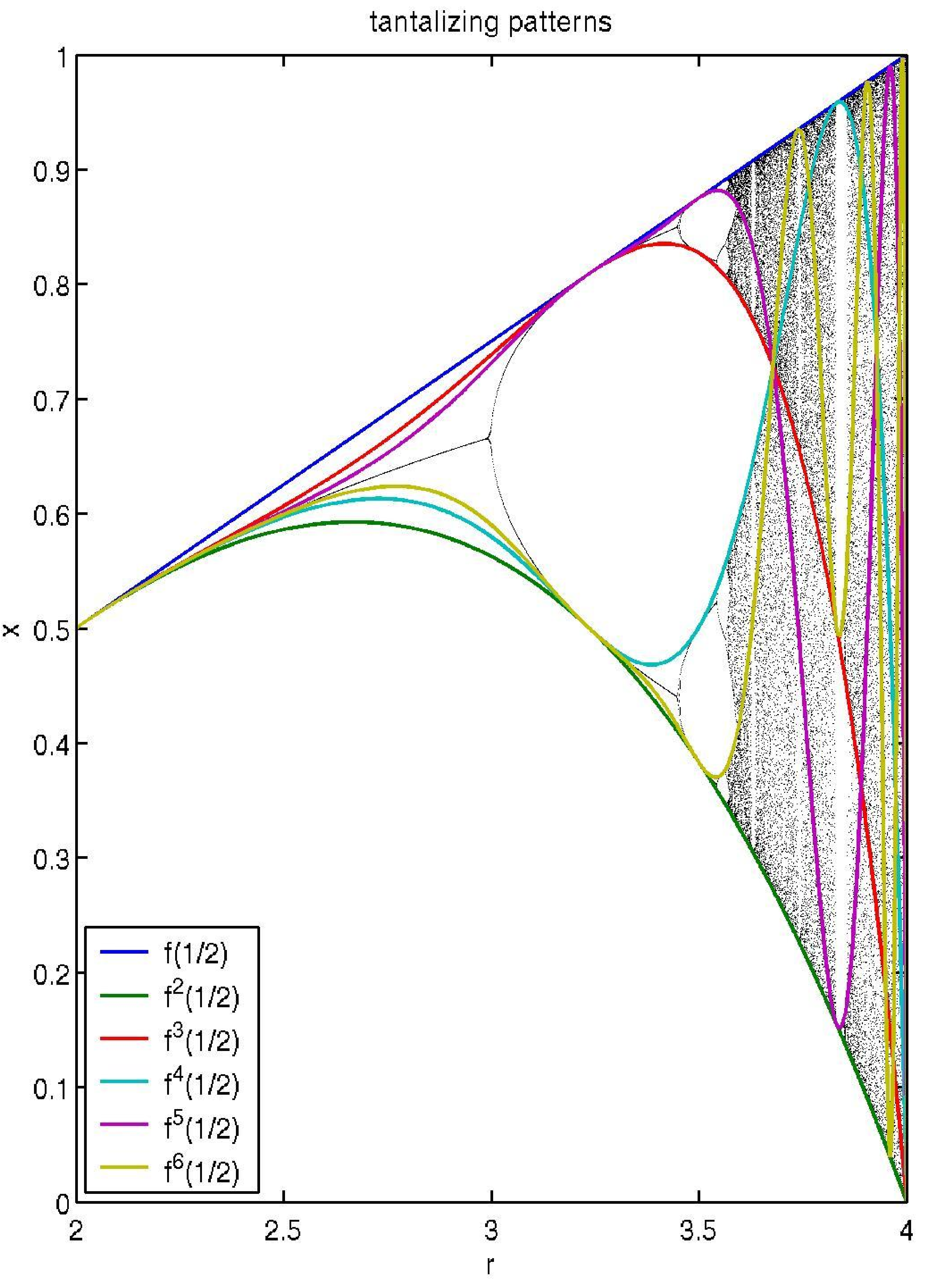
```
% plot orbit diagram for logistic map
% ian, dec 2005

N=400; % iterate map this many times
N0=300; % plot iterates after this much time for spinup
r=1:0.001:4;
x=zeros(N,length(r));
x(1,:)=rand(1,length(r));
for n=1:N-1
    x(n+1,:)=r.*x(n,:).*(1-x(n,:));
end

R=repmat(r,N,1);
plot(R(N0:end,:),x(N0:end,:),'b.','markersize',0.1)
```

(5) cont'd



Nicer grayscale figure based on density of points, from wikipedia logistic map article:




tantalizing patterns
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
x
2
2.5
3
3.5
4
r
f(1/2)
f2(1/2)
f3(1/2)
f4(1/2)
f5(1/2)
f6(1/2)